TRANSFORMATION ELASTICITY AND ANELASTICITY

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TRANSFORMATION ELASTICITY AND ANELASTICITY

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SUMMARY

We present a theoretical framework for studying a large class of elastic and anelastic problems in nonlinear solids. We specifically use the transformation properties of nonlinear and linearized elasticity in this theory. Given an anelastic deformation, a non-vanishing strain does not correspond to a non-vanishing stress. That part of strain that is related to the corresponding stress is called elastic strain. The remaining part of strain is called eigenstrain or pre-strain. Eigenstrains (or anelastic sources) such as inclusions, defects, growth, phase transformations, and nonuniform temperature changes can cause residual stresses. The relaxed (natural) configuration of a residually-stressed body is a non-Euclidean manifold that cannot be isometrically embedded in the Euclidean ambient space. Using transformation anelasticity, one can construct the Riemannian material manifold of the body. In particular, the material metric explicitly depends on the distribution of eigenstrains. In this PhD thesis we utilize transformation anelasticity to study the induced elastic fields of a circumferentially-symmetric distribution of finite eigenstrains in nonlinear elastic wedges; the stress field of a nonlinear elastic solid torus with a toroidal inclusion; nonlinear elastic inclusions in anisotropic solids as well as distributed line and point defects in nonlinear anisotropic solids.

The goal in *transformation elasticity* is to transform the nonlinear or linearized boundary-value (or initial-boundary-value) problem of an elastic body to that of another elastic body using a diffeomorphism (or a smooth mapping). The diffeomorphism, in turn, explicitly determines how the different elastic fields (and elastic parameters) of the two bodies are related. In particular, it is noted that the two boundary-value problems are not related by push-forward or pull-back under the diffeomorphism. We apply this theory to formulate the nonlinear and linearized elastodynamic transformation cloaking problem in the context of the classical elasticity, the small-on-large theory of elasticity, i.e., linearized elasticity with respect to an initially stressed configuration, and in solids with microstructure, namely gradient and (generalized) Cosserat solids. In particular, we note that a cloaking transformation is neither a spatial nor a referential change of coordinates (frame). Rather, a cloaking map transforms the boundary-value problem of an isotropic and homogeneous elastic body (virtual problem) to that

of an anisotropic and inhomogeneous elastic body with a finite hole covered by a cloak that is to be designed (physical problem). The virtual body has a desired mechanical (wave-guiding) response, whereas the physical body is designed such that the same response is mimicked outside the cloak using a cloaking transformation. Finally, starting from nonlinear shell theory, we utilize transformation elasticity to formulate the transformation cloaking problem for Kirchhoff-Love plates and for elastic plates with both the in-plane and the out-of-plane displacements.

CHAPTER 1

INTRODUCTION

In this thesis, we propose theoretical frameworks to investigate some elastic and anelastic phenomena in solids using transformation properties of nonlinear and linearized elasticity.

In continuum mechanics a strain is some measure of deformation that gives the length of an infinitesimal line element assuming that the length of this line element is known in some other (reference) configuration. A stress is usually defined to be an areal density of force. Given a pair of thermodynamically-conjugate stress and strain, e.g. the first Piola-Kirchhoff stress and the deformation gradient $(\mathbf{P}, \mathbf{F})$ or the second Piola-Kirchhoff stress and the right Cauchy-Green strain $(\mathbf{S}, \mathbf{C})$, locally a non-zero strain does not correspond to a non-zero stress. That part of strain that locally is related to the corresponding stress is called elastic strain. The remaining part is usually referred to as *eigenstrain* or *pre-strain*. The term *eigenstrain* was first used by Mura [1]. Other terms have been used in the literature for the same concept, e.g. *initial strain* [2], *inherent strain* [3], and *transformation strain* [4] (see [5] for a more detailed discussion). Eigenstrains are the anelastic part of the total strain tensor that represent referential rearrangements, changes, distortions, etc. When deformations (more precisely displacement gradients) are large many measures of strain may be considered and an eigenstrain would explicitly depend on the choice of a strain measure. Eigenstrains model many different phenomena, e.g. plasticity [6, 7], thermal strains [8, 9, 10], swelling and cavitation [11, 12, 13, 14, 15], bulk and surface growth [16, 17, 18, 19, 20, 21], and defects [22, 23]. For a detailed discussion of finite eigenstrains see [24, 25, 26, 27]. In a homogeneous body by an *inclusion* we mean a region with a distribution of eigenstrains. When the region with eignstrains and the matrix are made of different materials instead of inclusion we use *inhomogeneity with eigenstrain*.

In the setting of linear elasticity [4] computed the stress field of an ellipsoidal inclusion with uniform (infinitesimal) eigenstrains in an infinite isotropic solid. There have been a few 2D extensions of Eshleby's problem to finite elasticity for harmonic materials [28, 29, 30, 31, 32]. The classical shrink-fit problem of nonlinear elasticity [33] is the nonlinear analogue of an inclusion with pure dilatational eigenstrains. The problem of finite eigenstrains in 3D nonlinear elasticity was analytically studied by [24]. They calculated the residual stress fields induced by finite radial and circumferential eigenstrains for the case of spherical balls and (finite and infinite) circular cylindrical bars made of arbitrary incompressible and isotropic solids. The problem of finite shear eigenstrains and the twist-fit problem were investigated recently by [34]. As an example, they solved the problem of a cylindrical inhomogeneity with finite shear eigenstrains and examined the effect of torsional eigensrains on the stiffness of a circular cylindrical bar.

Indeed, anelastic effects due to different sources of eigenstrains, e.g., presence of defects and inclusions, phase transformations, biological growth and remodeling, and non-uniform temperature changes, can cause a residually stressed body to fail to find a relaxed Euclidean state. Identifying the reference configuration as a differentiable manifold allows one to construct a stress-free referential configuration for a residually stressed body even when such a configuration cannot be realized in the Euclidean physical space. This construction is done by locally transforming the material metric of a stress-free body without eigenstrains (in the Euclidean space) under a mapping that explicitly depends on the distribution of eigenstrains and finding the material metric (and thus, material manifold) for the body with eigenstrains. We refer to this construction as *transformation anelasticity*. We utilize transformation anelasticity to study the effects of a circumferentially-symmetric distribution of finite eigenstrains in nonlinear elastic wedges; the elastic fields of a nonlinear solid torus with a toroidal inclusion; nonlinear elastic inclusions in anisotropic solids, along with the induced stress fields in nonlinear anisotropic solids with distributed line and point defects.

Eigenstrains are created as a result of anelastic effects such as defects, temperature changes, bulk growth, etc., and strongly affect the overall response of solids. In Chapter 3, we investigate the residual stress and deformation fields of an incompressible, isotropic, infinite wedge due to a circumferentially-symmetric distribution of finite eigenstrains. In particular, we establish explicit exact solutions for the residual stresses and deformation of a neo-Hookean wedge containing a symmetric inclusion with finite radial and circumferential eigenstrains. In addition, we numerically solve for the residual stress field of a neo-Hookean wedge induced by a symmetric Mooney-Rivlin inhomogeneity with finite eigenstrains.

In Chapter 4, we analyze the stress field of a solid torus made of an incompressible isotropic solid with a toroidal inclusion that is concentric with the solid torus and has a uniform distribution of pure dilatational finite eigenstrains. We use a perturbation analysis and calculate the residual stresses to the first order in the *thinness* ratio (the ratio of the radius of the generating circle and the overall radius of the solid torus). In particular, we show that the stress field inside the inclusion is not uniform. This is in contrast with the corresponding results for infinitely-long and finite circular cylindrical bars and spherical balls with cylindrical and spherical inclusions, respectively. We also show that for a solid torus of any size made of an incompressible linear elastic solid with an inclusion with uniform (infinitesimal) pure dilatational eigenstrains the stress inside the inclusion cannot be uniform.

In Chapter 5, we study the stress and deformation fields generated by nonlinear inclusions with finite eigenstrains in anisotropic solids. In particular, we consider finite eigenstrains in transversely isotropic spherical balls and orthotropic cylindrical bars made of both compressible and incompressible solids. We show that the stress field in a spherical inclusion with uniform pure dilatational eigenstrain in a spherical ball made of an incompressible transversely isotropic solid such that the material preferred direction is radial at any point is uniform and hydrostatic. Similarly, the stress in a cylindrical inclusion contained in an incompressible orthotropic cylindrical bar is uniform hydrostatic if the radial and circumferential eigenstrains are equal and the axial stretch is equal to a value determined by the axial eigenstrain. We also prove that for a compressible isotropic spherical ball and a cylindrical bar containing a spherical and a cylindrical inclusion, respectively, with uniform eigenstrains the stress in the inclusion is uniform (and hydrostatic for the spherical inclusion) if the radial and circumferential eigenstrains are equal. For compressible transversely isotropic and orthotropic solids, we show that the stress field in an inclusion with uniform eigenstrain is not uniform, in general. Nevertheless, in some special cases the material can be designed in order to maintain a uniform stress field in the inclusion. As particular examples to investigate such special cases, we consider compressible Mooney-Rivlin and Blatz-Ko reinforced models and find analytical expressions for the stress field in the inclusion.

In Chapter 6, we present some analytical solutions for the stress fields of nonlinear anisotropic solids with distributed line and point defects. In particular, we determine the stress fields of i) a parallel cylindrically-symmetric distribution of screw dislocations in infinite orthotropic and monoclinic media, ii) a cylindrically-symmetric distribution of parallel wedge disclinations in an infinite orthotropic medium, iii) a distribution of edge dislocations in an orthotropic medium, and iv) a spherically-symmetric distribution of point defects in a transversely isotropic spherical ball.

In the setting of linear elasticity it has been shown that the equilibrium equations for anisotropic elastic solids can be generated from those of an isotropic material by applying a proper transformation to different elastic quantities (fields) [35, 36, 37]. We introduce a similar concept in the nonlinear setting, which we refer to as *transformation elasticity*. In transformation elasticity, one maps the (nonlinear or linearized) boundary-value (or initial-boundary-value) problem of an elastic body to that of another elastic body using a diffeomorphism. This diffeomorphism, in turn, relates the stress and deformation fields, and thus, the elastic constants of the two elastic bodies. We utilize this theory in the particular case of the cloaking problem to formulate the nonlinear and linear elastodynamic transformation cloaking problem in the context of classical elasticity, the small-on-large theory of elasticity, gradient elasticity, along with (generalized) Cosserat elasticity. Furthermore, we use transformation elasticity to properly formulate the transformation cloaking problem for Kirchhoff-Love plates as well as for elastic plates in the presence of both in-plane and out-of-plane displacements.

Cloaking of objects from waves was first introduced in the field of electromagnetism in the context of invisibility. Making objects hidden to electromagnetic waves is both theoretically and practically important and has been a subject of intense research in recent years. More specifically, the idea of cloaking for electromagnetism is as follows. Suppose one is given a body (domain) Ω with a hole $\mathcal{H}$ surrounded by a cloaking region $\mathcal{C}$ (see Fig.1.1). The hole can be of any shape and the one shown in Fig.1.1 is assumed to be circular (spherical in 3D). Suppose the physical properties in $\Omega \setminus \mathcal{C}$ are uniform and isotropic. One is interested in designing the cloaking region $\mathcal{C}$ such that an electromagnetic wave passing through Ω would not interact with $\mathcal{H}$. In other words, $\mathcal{C}$ redirects the waves such that the boundary measurements are identical to those of another body with the same outer boundary as Ω without the hole and made of the same homogeneous (and isotropic) material. Let us consider a smooth mapping $\psi_\rho : \Omega \to \Omega$ such that $\psi_\rho|_{\Omega\setminus\mathcal{C}} = id$ (ψ_ρ restricted to the outside of the cloak is the identity map) and it shrinks the hole $\mathcal{H}$ to a small circle (sphere) of radius $\rho > 0$ (see Fig.1.1(b)). Note that this map is not unique, and hence, many possibilities for a cloak $\mathcal{C}$. One then transforms the physical fields such that the two problems satisfy Maxwell's equations. This usually makes the physical properties of the cloaking region $\mathcal{C}$ both inhomogeneous and anisotropic. The perfect cloaking case (the limit $\rho \to 0$) may require singular physical properties. The goal in

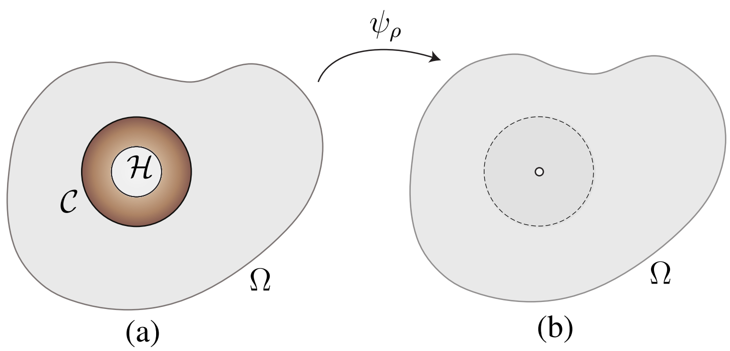


Figure 1.1: *Cloaking an object in a hole $\mathcal{H}$ by a cloak $\mathcal{C}$. The system (b) has uniform physical properties. The cloaking transformation is the identity map outside the cloaking region $\mathcal{C}$.*

elastodynamic cloaking is to make a cavity (hole) invisible to elastic waves. One idea would be to reinforce the outer boundary of the cavity by a cloak that has inhomogeneous and anisotropic elastic properties, in general. The cloak would then guide the elastic waves such that the elastic measurements made by an observer away from the cloak are identical to those when the cavity is absent. In this work we present a formulation of nonlinear and linear elastodynamic transformation cloaking for 3D elasticity as well as for elastic plates in a mathematically coherent way. In particular, we show that the invariance (covariance) of Maxwells equations is not the underlying principle that allows transformation cloaking. This, in turn, has direct implications on the way the transformation cloaking problem needs to be formulated for elastodynamics.

In Chapter 7, we formulate the problems of nonlinear and linear elastodynamic transformation cloaking in a geometric framework. In particular, it is noted that a cloaking transformation is neither a spatial nor a referential change of frame (coordinates); a cloaking transformation maps the boundary-value problem of an isotropic and homogeneous elastic body (virtual problem) to that of an anisotropic and inhomogeneous elastic body with a hole surrounded by a cloak that is to be designed (physical problem). The virtual body has a desired mechanical response while the physical body is designed to mimic the same response outside the cloak using a cloaking transformation. We show that nonlinear elastodynamic transformation cloaking is not possible while nonlinear elastostatic transformation cloaking may be possible for special deformations, e.g., radial deformations in a body with either a cylindrical or a spherical cavity. In the case of classical linear elastodynamics, in agreement with the previous observations in the literature, we show that the elastic constants in the cloak are not fully symmetric; they do not possess the minor symmetries. We prove that elastodynamic transformation cloaking is not possible regardless of the shape of the hole and the cloak. It is shown that the small-on-large theory, i.e., linearized elasticity with respect to a pre-stressed configuration, does not allow for transformation cloaking either. However, elastodynamic cloaking of a cylindrical hole is possible for in-plane deformations while it is not possible for anti-plane deformations. We next show that for a cavity of any shape elastodynamic transformation cloaking cannot be achieved for linear gradient elastic solids either; similar to classical linear elasticity the balance of angular momentum is the obstruction to transformation cloaking. We finally prove that transformation cloaking is not possible for linear elastic generalized Cosserat solids in dimension two for any shape of the hole and the cloak. In particular, in dimension two transformation cloaking cannot be achieved in linear Cosserat elasticity. We also show that transformation cloaking for a spherical cavity covered by a spherical cloak is not possible in the setting of linear elastic generalized Cosserat elasticity. We conjecture that this result is true for a cavity of any shape. It should be emphasized that in this work we do not consider the so-called metamaterials [38, 39].

In Chapter 8, we formulate the problem of elastodynamic transformation cloaking for Kirchhoff-Love plates and elastic plates with both the in-plane and out-of-plane displacements. A cloaking transformation maps the boundary-value problem of an isotropic and homogeneous elastic plate (virtual problem) to that of an anisotropic and inhomogeneous elastic plate with a hole surrounded by a cloak that is to be designed (physical problem). For Kirchhoff-Love plates, under the cloaking map the (out-of-plane) governing equations of the virtual plate is transformed to those of the physical plate up to an unknown scalar field. In doing so, one finds the initial stress and the initial tangential body force for the physical plate, along with a set of constraints on the cloaking map. These constraints involve the cloaking transformation, the unknown scalar field, and the elastic constants of the virtual plate. It is noted that the cloaking map needs to satisfy certain conditions on the outer boundary of the cloak and the inner surface of the hole. In particular, the cloaking map needs to fix the outer boundary of the cloak up to the third order. Assuming a generic radial cloaking map, we show that cloaking a circular hole in Kirchoff-Love plates is not possible; the constraints and the boundary conditions that the cloaking map needs to satisfy are the obstruction to cloaking. Next, relaxing the pure bending assumption, the transformation cloaking problem of an elastic plate in the presence of in-plane and out-of-plane displacements is formulated. In this case, there are two sets of governing equations that one needs to simultaneously transform under the cloaking map. We show that cloaking a circular hole is not possible for a general radial cloaking map in the presence of in-plane and out-of-plane displacements; similar to the case of Kirchoff-Love plates, the constraints and the boundary conditions that the cloaking map needs to satisfy obstruct transformation cloaking.

The remainder of this work is arranged as follows. In Chapter 2, we tersely review some important elements of geometric elasticity and anelasticity for isotropic and anisotropic solids. In Chapter 3, we investigate the anelasticity problem of determining the induced residual stress and deformation fields due to a circumferentially-symmetric distribution of finite eigenstrains in nonlinear elastic wedges [25]. In Chapter 4, we analyze the stress field of a nonlinear elastic solid torus with a toroidal inclusion [26]. Chapter 5 is devoted to the investigation of nonlinear elastic inclusions in anisotropic solids [27]. In Chapter 6, we present some analytical solutions for nonlinear anisotropic solids with distributed line and point defects [40]. In Chapter 7, we formulate the problems of nonlinear and linear elastodynamic transformation cloaking [41]. In Chapter 8, we carefully investigate the transformation cloaking problem in elastic plates [42]. Finally, in Chapter 9, we present some concluding remarks and briefly discuss possible paths for future research.

CHAPTER 2

ELEMENTS OF GEOMETRIC ELASTICITY AND ANELASTICITY FOR ISOTROPIC AND ANISOTROPIC SOLIDS

In this section, we briefly review some fundamental elements of the geometric theory of nonlinear elasticity and anelasticity for isotropic and anisotropic bodies. For more detailed discussions, see [43, 44, 45, 46].

2.1 Kinematics

A body $\mathcal{B}$ is assumed to be identified with a Riemannian manifold $(\mathcal{B}, \mathbf{G})$, and a configuration of $\mathcal{B}$ is a smooth embedding $\varphi : \mathcal{B} \to \mathcal{S}$, where $(\mathcal{S}, \mathbf{g})$ is also assumed to be a Riemannian manifold. An affine connection ∇ on a smooth manifold M is a linear map $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$, where $\mathcal{X}(M)$ indicates the set of all smooth vector fields on M, that has to satisfy some specific properties (see [47] for details). It turns out that there is a unique torsion-free and compatible affine connection associated with any Riemannian manifold, referred to as Riemannian connection (see for example [47, 48]). We denote the Levi-Civita connection associated with the Riemannian manifold $(\mathcal{S}, \mathbf{g})$ by $\nabla^{\mathbf{g}}$. A configuration of $\mathcal{B}$ is a smooth embedding $\varphi : \mathcal{B} \to \mathcal{S}$, where $(\mathcal{S}, \mathbf{g})$ is the Euclidean ambient space. The set of all configurations of $\mathcal{B}$ is denoted by $\mathcal{C}$. A motion is a curve $c : \mathbb{R}^+ \to \varphi_t \in \mathcal{C}$ such that φ_t assigns a spatial point $x = \varphi_t(X) = \varphi(X, t) \in \mathcal{S}$ to every material point $X \in \mathcal{B}$ at any time t. It is assumed that the body is stress-free in its reference configuration, which may have a nontrivial geometry (e.g. in the presence of eigenstrains). The deformation gradient $\mathbf{F}$ is the differential map of φ defined as

$$\mathbf{F}(X, t) = d\varphi_t(X) : T_X\mathcal{B} \to T_{\varphi_t(X)}\mathcal{S}\,. \tag{2.1}$$

The adjoint of $\mathbf{F}$ is defined as follows

$$\mathbf{F}^{\mathsf{T}}(X,t): T_{\varphi_t(X)}\mathcal{S} \to T_X\mathcal{B}\,, \qquad \mathbf{g}\left(\mathbf{F}\mathbf{V},\mathbf{v}\right) = \mathbf{G}\left(\mathbf{V},\mathbf{F}^{\mathsf{T}}\mathbf{v}\right),\ \ \forall\, \mathbf{V} \in T_X\mathcal{B},\, \mathbf{v} \in T_{\varphi_t(X)}\,. \tag{2.2}$$

The Finger deformation tensor is defined as $\mathbf{b}(x,t) = \mathbf{F}(X,t)\mathbf{F}^{\mathsf{T}}(X,t) : T_x\varphi\left(\mathcal{B}\right) \to T_x\varphi\left(\mathcal{B}\right)$. In components, $b^{ab} = F^a{}_A F^b{}_B G^{AB}$. Another measure of strain is the Lagrangian strain tensor that is defined as $\mathbf{E} = \frac{1}{2}\left(\varphi_t^*\mathbf{g} - \mathbf{G}\right)$. The Jacobian of deformation J relates the Riemannian volume elements of the material manifold $dV(X,\mathbf{G})$ and the spatial manifold $dv(\varphi_t(X),\mathbf{g})$ and is written as

$$J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}}\,\det \mathbf{F}\,, \quad dv = J\,dV\,. \tag{2.3}$$

The right Cauchy-Green deformation tensor is defined as

$$\mathbf{C}(X,t) = \mathbf{F}^{\mathsf{T}}(X,t)\mathbf{F}(X,t) : T_X\mathcal{B} \to T_X\mathcal{B}\,. \tag{2.4}$$

In the coordinate charts $\{X^A\}$ and $\{x^a\}$ for $\mathcal{B}$ and $\mathcal{S}$, respectively, in components, $\mathbf{C}$ can be written as: $C^A{}_B = G^{AL}F^a{}_L F^b{}_B g_{ab}$. The material velocity of the motion is the mapping $\mathbf{V} : \mathcal{B}\times\mathbb{R}^+ \to T\mathcal{S}$, where $\mathbf{V}(X,t) \in T_{\varphi_t(X)}\mathcal{S}$, and in components, $V^a(X,t) = \frac{\partial\varphi^a}{\partial t}(X,t)$. The spatial velocity is defined as $\mathbf{v} : \varphi_t(\mathcal{B})\times\mathbb{R}^+ \to T\mathcal{S}$ such that $\mathbf{v}_t(x) = \mathbf{V}_t \circ \varphi_t^{-1}(x) \in T_x\mathcal{S}$, where $x = \varphi_t(X)$. The material acceleration is a mapping $\mathbf{A} : \mathcal{B}\times\mathbb{R}^+ \to T\mathcal{S}$ defined as $\mathbf{A}(X,t) := D_t^{\mathbf{g}}\mathbf{V}(X,t) = \nabla^{\mathbf{g}}_{\mathbf{V}(X,t)}\mathbf{V}(X,t) \in T_{\varphi_t(X)}\mathcal{S}$, where $D_t^{\mathbf{g}}$ denotes the covariant derivative along the curve $\varphi_t(X)$ in $\mathcal{S}$. In components, $A^a = \frac{\partial V^a}{\partial t} + \gamma^a{}_{bc}V^bV^c$. The spatial acceleration is defined as $\mathbf{a} : \varphi_t(\mathcal{B})\times\mathbb{R}^+ \to T\mathcal{S}$ such that $\mathbf{a}_t(x) = \mathbf{A}_t \circ \varphi_t^{-1}(x) \in T_x\mathcal{S}$. In components, it reads $a^a = \frac{\partial v^a}{\partial t} + \frac{\partial v^a}{\partial x^b}v^b + \gamma^a{}_{bc}v^bv^c$.

2.2 Equilibrium equations

Balance of linear momentum in spatial and material forms reads

$$\begin{aligned}&\operatorname{div}_{\mathbf{g}} \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \mathbf{a}, \\ &\operatorname{Div} \mathbf{P} + \rho_0 \mathbf{B} = \rho_0 \mathbf{A},\end{aligned} \tag{2.5}$$

where $\boldsymbol{\sigma}$ and $\mathbf{P}$ are the Cauchy stress and the first Piola-Kirchhoff stress, respectively. ρ_0, $\mathbf{B}$, and $\mathbf{A}$ are the material mass density, material body force, and material acceleration, respectively, and ρ, $\mathbf{b}$, and $\mathbf{a}$ are their corresponding spatial counterparts. In terms of the Cauchy stress tensor, the localized balance of linear momentum of a body in static equilibrium and in the absence of body forces reads

$$\operatorname{div} \boldsymbol{\sigma} = \mathbf{0}\,, \tag{2.6}$$

where div denotes the spatial divergence operator. In components, the spatial divergence operator reads

$$(\operatorname{div}\boldsymbol{\sigma})^a = \sigma^{ab}{}_{|b} = \frac{\partial \sigma^{ab}}{\partial x^b} + \sigma^{ac}\gamma^b{}_{cb} + \sigma^{cb}\gamma^a{}_{cb}\,, \tag{2.7}$$

where $\gamma^a{}_{bc}$ is the Christoffel symbol of the Levi-Civita connection $\nabla^{\mathbf{g}}$ in the local chart $\{x^a\}$, defined as $\nabla^{\mathbf{g}}{}_{\partial_b}\partial_c = \gamma^a{}_{bc}\partial_a$, (similarly, for the material manifold $\nabla^{\mathbf{G}}{}_{\partial_B}\partial_C = \Gamma^A{}_{BC}\partial_A$). Note that

$$\gamma^a{}_{bc} = \frac{1}{2} g^{ak}\left(\frac{\partial g_{kb}}{\partial x^c} + \frac{\partial g_{kc}}{\partial x^b} - \frac{\partial g_{bc}}{\partial x^k}\right), \quad \Gamma^A{}_{BC} = \frac{1}{2} G^{AK}\left(\frac{\partial G_{KB}}{\partial X^C} + \frac{\partial G_{KC}}{\partial X^B} - \frac{\partial G_{BC}}{\partial X^K}\right). \tag{2.8}$$

Balance of angular momentum in local form reads $\boldsymbol{\sigma}^{\mathsf{T}} = \boldsymbol{\sigma}$ or $\mathbf{F}\mathbf{P}^{\mathsf{T}} = \mathbf{P}\mathbf{F}^{\mathsf{T}}$. Conservation of mass implies that $\rho dv = \rho_0 dV$ or $\rho J = \rho_0$, where ρ_o and ρ denote the material and spatial mass densities, respectively.

2.3 Constitutive Equations

In this chapter we restrict our calculations to compressible and incompressible isotropic, transversely isotropic, and orthotropic materials. We use structural tensors to establish a materially covariant strain energy density function corresponding to the symmetry group of the material. See [49, 50, 51, 52, 53] for detailed discussions of structural tensors and the determination of the integrity basis for the invariants of a collection of tensors.

Isotropy. For isotropic solids the energy function W depends only on the principal invariants of $\mathbf{b}$, denoted by I_1, I_2, and I_3. In the case of incompressible solids, $I_3 = 1$, and hence, $W = W(X, I_1, I_2)$. We restrict our attention to isotropic incompressible hyperelastic solids, for which the Cauchy stress has the following representation [54, 55]

$$\boldsymbol{\sigma} = (-p + 2I_2 W_{I_2})\, \mathbf{g}^\sharp + 2W_{I_1} \mathbf{b}^\sharp - 2W_{I_2} \mathbf{b}^{-1}\,, \tag{2.9}$$

where p is the Lagrange multiplier associated with the internal incompressibility condition $J = 1$, and $W_{I_1} := \frac{\partial W}{\partial I_1}$, $W_{I_2} := \frac{\partial W}{\partial I_2}$. Note that $\mathbf{b}$ and $\mathbf{C}$ have the same principal invariants, and $\mathbf{b}^\sharp = \varphi_* \left(\mathbf{G}^\sharp\right)$. Similarly, the Cauchy stress tensor in terms of the first two principal invariants of $\mathbf{C}$: $I_1 = \mathrm{tr}\boldsymbol{C}$ and $I_2 = \frac{1}{2}(\mathrm{tr}(\boldsymbol{C})^2 - \mathrm{tr}(\boldsymbol{C}^2))$ is given in components by [56]

$$\sigma^{ab} = 2F^a{}_A F^b{}_B \left[(W_{I_1} + I_1 W_{I_2}) G^{AB} - W_{I_2} C^{AB}\right] - p g^{ab}\,. \tag{2.10}$$

Transverse isotropy. Let us consider a compressible transversely isotropic solid with the unit vector $\mathbf{N}(\mathrm{X})$ identifying the material preferred direction at a point X in the reference configuration. The strain energy density function (per unit volume) is given by (see, e.g., [56, 50, 53])

$$W = W(\mathrm{X}, \mathbf{G}, \mathbf{C}^\flat, \mathbf{A})\,, \tag{2.11}$$

where $\mathbf{A} = \mathbf{N} \otimes \mathbf{N}$ is a structural tensor associated with the transverse isotropy material symmetry group. The second Piola-Kirchhoff stress tensor is written as

$$\mathbf{S} = 2\frac{\partial W}{\partial \mathbf{C}^\flat} \,. \tag{2.12}$$

The energy function W depends on five independent invariants defined as follows

$$I_1 = \operatorname{tr}\mathbf{C}\,, \quad I_2 = \det\mathbf{C}\operatorname{tr}\mathbf{C}^{-1}\,, \quad I_3 = \det\mathbf{C}\,, \quad I_4 = \mathbf{N}\cdot\mathbf{C}\cdot\mathbf{N}\,, \quad I_5 = \mathbf{N}\cdot\mathbf{C}^2\cdot\mathbf{N}\,. \tag{2.13}$$

In components, $I_1 = C^A{}_A$, $I_2 = \det(C^A{}_B)(C^{-1})^D{}_D$, $I_3 = \det(C^A{}_B)$, $I_4 = N^A N^B C_{AB}$, and $I_5 = N^A N^B C_{BQ} C^Q{}_A$. Using (6.7), one has[1]

$$\mathbf{S} = 2W_{I_n}\frac{\partial I_n}{\partial \mathbf{C}^\flat}\,, \qquad W_{I_n} := \frac{\partial W}{\partial I_n}\,, \quad n = 1,\ldots,5\,. \tag{2.14}$$

It then follows that

$$\frac{\partial I_1}{\partial \mathbf{C}^\flat} = \mathbf{G}^\sharp\,, \quad \frac{\partial I_2}{\partial \mathbf{C}^\flat} = I_2\mathbf{C}^{-1} - I_3\mathbf{C}^{-2}\,, \quad \frac{\partial I_3}{\partial \mathbf{C}^\flat} = I_3\mathbf{C}^{-1}\,, \quad \frac{\partial I_4}{\partial \mathbf{C}^\flat} = \mathbf{N}\otimes\mathbf{N}\,, \quad \frac{\partial I_5}{\partial \mathbf{C}^\flat} = \mathbf{N}\otimes\mathbf{C}\cdot\mathbf{N} + \mathbf{N}\cdot\mathbf{C}\otimes\mathbf{N}\,. \tag{2.15}$$

Therefore, (6.10) and (7.32) give the following representation for $\mathbf{S}$.

$$\mathbf{S} = 2\left\{W_{I_1}\mathbf{G}^\sharp + W_{I_2}\left(I_2\mathbf{C}^{-1} - I_3\mathbf{C}^{-2}\right) + W_{I_3}I_3\mathbf{C}^{-1} + W_{I_4}\left(\mathbf{N}\otimes\mathbf{N}\right) + W_{I_5}\left(\mathbf{N}\otimes\mathbf{C}\cdot\mathbf{N} + \mathbf{N}\cdot\mathbf{C}\otimes\mathbf{N}\right)\right\}\,. \tag{2.16}$$

In the case of incompressible materials $I_3 = 1$, and hence, $W = W(\mathrm{X}, I_1, I_2, I_4, I_5)$. Thus, from (6.12), one expresses $\mathbf{S}$ as

$$\mathbf{S} = 2\left\{W_{I_1}\mathbf{G}^\sharp + W_{I_2}\left(I_2\mathbf{C}^{-1} - \mathbf{C}^{-2}\right) + W_{I_4}\left(\mathbf{N}\otimes\mathbf{N}\right) + W_{I_5}\left(\mathbf{N}\otimes\mathbf{C}\cdot\mathbf{N} + \mathbf{N}\cdot\mathbf{C}\otimes\mathbf{N}\right)\right\} - p\mathbf{C}^{-1}\,, \tag{2.17}$$

[1]For the sake of simplicity of calculations, here we do not consider an explicit dependence of W on X, which is needed in the case of inhomogeneous bodies. Instead, we assume that the material is piece-wise homogeneous and model an inhomogeneity by using different energy functions in different regions of the body.

where p is the Lagrange multiplier associated with the incompressibility constraint $J = 1$. The Cauchy stress $\sigma^{ab} = \frac{1}{J} F^a{}_A F^b{}_B S^{AB}$ has the following representation in component form[2]

$$\sigma^{ab} = 2F^a{}_A F^b{}_B \left[(W_{I_1} + I_1 W_{I_2}) G^{AB} - W_{I_2} C^{AB} + W_{I_4} N^A N^B + W_{I_5} \left(N^Q N^A C^B{}_Q + N^P N^B C_P{}^A \right) \right] - p g^{ab}. \tag{2.19}$$

Orthotropy. We next consider a compressible orthotropic material such that $\mathbf{N}_1(\mathrm{X})$, $\mathbf{N}_2(\mathrm{X})$, and $\mathbf{N}_3(\mathrm{X})$ are three $\mathbf{G}$-orthonormal vectors specifying the orthotropic axes in the reference configuration at a point X. A choice of structural tensors for this case is given by $\mathbf{A}_1 = \mathbf{N}_1 \otimes \mathbf{N}_1$, $\mathbf{A}_2 = \mathbf{N}_2 \otimes \mathbf{N}_2$, and $\mathbf{A}_3 = \mathbf{N}_3 \otimes \mathbf{N}_3$, only two of which are independent.[3] Therefore, the energy function is written as [56, 50, 53]

$$W = W(\mathrm{X}, \mathbf{G}, \mathbf{C}^\flat, \mathbf{A}_1, \mathbf{A}_2) \,. \tag{2.20}$$

The energy function W depends on the following seven independent invariants.

$$\begin{gathered} I_1 = \operatorname{tr} \mathbf{C} \,, \quad I_2 = \det \mathbf{C} \operatorname{tr} \mathbf{C}^{-1} \,, \quad I_3 = \det \mathbf{C} \,, \quad I_4 = \mathbf{N}_1 \cdot \mathbf{C} \cdot \mathbf{N}_1 \,, \\ I_5 = \mathbf{N}_1 \cdot \mathbf{C}^2 \cdot \mathbf{N}_1 \,, \quad I_6 = \mathbf{N}_2 \cdot \mathbf{C} \cdot \mathbf{N}_2 \,, \quad I_7 = \mathbf{N}_2 \cdot \mathbf{C}^2 \cdot \mathbf{N}_2 \,. \end{gathered} \tag{2.21}$$

From (6.7), one obtains

$$\mathbf{S} = 2 W_{I_n} \frac{\partial I_n}{\partial \mathbf{C}^\flat} \,, \qquad W_{I_n} := \frac{\partial W}{\partial I_n} \,, \quad n = 1, \ldots, 7 \,. \tag{2.22}$$

[2]Note that using the Cayley-Hamilton theorem, one can write

$$\frac{\partial I_2}{\partial \mathbf{C}^\flat} = I_2 (\mathbf{C}^{-1})^\sharp - I_3 (\mathbf{C}^{-2})^\sharp = I_1 \mathbf{G}^\sharp - \mathbf{C}^\sharp. \tag{2.18}$$

[3]Note that $\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 = \mathbf{I}$.

Substituting (7.32) into (6.17), the second Piola-Kirchhoff stress tensor is written as

$$\mathbf{S}=2\Big\{W_{I_1}\mathbf{G}^\sharp+W_{I_2}\left(I_2\mathbf{C}^{-1}-I_3\mathbf{C}^{-2}\right)+W_{I_3}I_3\mathbf{C}^{-1}+W_{I_4}\left(\mathbf{N}_1\otimes\mathbf{N}_1\right)+W_{I_5}\left(\mathbf{N}_1\otimes\mathbf{C}\cdot\mathbf{N}_1+\mathbf{N}_1\cdot\mathbf{C}\otimes\mathbf{N}_1\right)$$
$$+W_{I_6}\left(\mathbf{N}_2\otimes\mathbf{N}_2\right)+W_{I_7}\left(\mathbf{N}_2\otimes\mathbf{C}\cdot\mathbf{N}_2+\mathbf{N}_2\cdot\mathbf{C}\otimes\mathbf{N}_2\right)\Big\}. \tag{2.23}$$

If the material is assumed to be incompressible, then it follows that $I_3=1$ and $W=W\left(\mathrm{X},I_1,I_2,I_4,I_5,I_6,I_7\right)$. Hence, from (6.18), one obtains the following representation for the second Piola-Kirchhoff stress tensor

$$\mathbf{S}=2\Big\{W_{I_1}\mathbf{G}^\sharp+W_{I_2}\left(I_2\mathbf{C}^{-1}-\mathbf{C}^{-2}\right)+W_{I_4}\left(\mathbf{N}_1\otimes\mathbf{N}_1\right)+W_{I_5}\left(\mathbf{N}_1\otimes\mathbf{C}\cdot\mathbf{N}_1+\mathbf{N}_1\cdot\mathbf{C}\otimes\mathbf{N}_1\right)$$
$$+W_{I_6}\left(\mathbf{N}_2\otimes\mathbf{N}_2\right)+W_{I_7}\left(\mathbf{N}_2\otimes\mathbf{C}\cdot\mathbf{N}_2+\mathbf{N}_2\cdot\mathbf{C}\otimes\mathbf{N}_2\right)\Big\}-p\mathbf{C}^{-1}. \tag{2.24}$$

The Cauchy stress tensor is given in components by

$$\sigma^{ab}=2F^a{}_AF^b{}_B\Big[(W_{I_1}+I_1W_{I_2})G^{AB}-W_{I_2}C^{AB}+W_{I_4}N_1{}^AN_1{}^B+W_{I_5}\left(N_1{}^QN_1{}^AC^B{}_Q+N_1{}^PN_1{}^BC_P{}^A\right)$$
$$+W_{I_6}N_2{}^AN_2{}^B+W_{I_7}\left(N_2{}^SN_2{}^AC^B{}_S+N_2{}^KN_2{}^BC_K{}^A\right)\Big]-pg^{ab}. \tag{2.25}$$

Transformation anelasticity. In the following, we briefly discuss how the Riemannian material manifold of bodies with a distribution of finite eigenstrains is constructed. In geometric anelasticity one starts with a stress-free body $\mathcal{B}$ without eigenstrains sitting in the Euclidean space with metric $\mathbf{G}_0$. This means that the body free of eigenstrains is a Riemannian manifold $(\mathcal{B},\mathbf{G}_0)$. Suppose now that the same body is endowed with a distribution of eigenstrains. Note that here we are not concerned with nucleation or dynamics of eigenstrains and assume that we are given a static distribution of eigenstrains. The effect of an eigenstrain distribution is to locally transform a line element $d\mathbf{X}_0$ to $d\mathbf{X}=\mathbf{K}d\mathbf{X}_0$, where $\mathbf{K}$ explicitly depends on the distribution of eigenstrains. Note that

$$\mathbf{G}_0\left(d\mathbf{X}_0,d\mathbf{X}_0\right)=\mathbf{G}\left(d\mathbf{X},d\mathbf{X}\right), \tag{2.26}$$

where $\mathbf{G} = \mathbf{K}_*\mathbf{G}_0$ is the push-forward of $\mathbf{G}_0$ by $\mathbf{K}$. Note that the linear transformation $\mathbf{K}(X) : T_X\mathcal{B} \to T_X\mathcal{B}$ is defined locally. In the manifold $(\mathcal{B}, \mathbf{G})$, the body with the distributed eigenstrains is stress-free because the distances are set to be those of the hypothetically relaxed body. Note that in components, $G_{AB} = K^\alpha{}_A K^\beta{}_B (G_0)_{\alpha\beta}$, where the coordinate charts $\{\bar{X}^\alpha\}$ and $\{X^A\}$ in the initial and distorted reference configurations, respectively, are assumed.

CHAPTER 3

FINITE EIGENSTRAINS IN NONLINEAR ELASTIC WEDGES

3.1 Introduction

The governing equations of nonlinear elasticity are formidably complicated and are amenable to analytic solutions only for very few problems. Semi-inverse methods have been particularly useful for obtaining exact solutions for nonlinear elasticity problems. One problem that has attracted several researchers in the last few decades is that of an infinite wedge made of a nonlinear elastic solid (either compressible or incompressible) under various boundary conditions and in the absence of body forces.

Tao and Rajagopal [57] studied the inhomogeneous deformation of a wedge made of a Blatz-Ko material. They assumed a specific form of deformations in which radial planes in the reference configuration remain radial planes after deformation. They found the only possible inhomogeneous solution, which turned out to be asymmetric with respect to the bisecting plane of the wedge. This specific class of deformations was further studied in the literature to find the inhomogeneous deformations in wedges and cones. Fu *et al.* [58] explored circumferentially-symmetric finite deformations of a wedge made of an incompressible Mooney-Rivlin material. To solve the problem, they specified the translation and rotation of the lateral faces of the wedge. They proved that the deformation is homogeneous when the pressure field associated with the incompressibility condition is uniform. For the inhomogeneous solutions, they were able to reduce the governing equations to a convenient form that allowed for a plane-phase analysis. They observed that for certain wedge angles, the deformation of the wedge is not radially-unidirectional, i.e., some parts of the wedge radially stretch, while others contract. Rajagopal and Carroll [59] assumed inhomogeneous circumferentially-symmetric finite deformations of a wedge made of an isotropic material. Using the displacement lateral boundary conditions and by applying the required tractions on the circular boundary, they obtained, when the material is compressible, a necessary condition that the energy function needs to satisfy for the assumed inhomogeneous deformation to be possible. For incompressible materials, they showed that such an inhomogeneous deformation is possible if the pressure field has a logarithmic singularity at the origin. Rajagopal and Tao [60] studied inhomogeneous circumferentially-symmetric finite deformations of a wedge made of an incompressible power law material. They showed that a "boundary layer solution", i.e., one that is homogeneous in the interior of the wedge but is inhomogeneous close to the boundary, is possible with a bounded pressure field. However, they showed that inhomogeneous solutions are possible only if the pressure field develops a logarithmic singularity at the apex of the wedge. Walton and Wilber [61] investigated the deformations of a neo-Hookean elastic wedge considering the aforementioned class of deformations. They observed that homogeneous non-unidirectional deformations are possible in every incompressible, isotropic, hyperelastic material. Assuming a more general class of deformations, where some restrictions on the form of the deformation were relaxed, they showed that there exist no additional solutions. Walton [62] studied the stability of this class of deformations under small amplitude vibrational perturbations of the lateral faces of a wedge. He found that even to the first order in an asymptotic expansion of the amplitude of the lateral sides of the wedge, the vibrations cannot remain planar; rather out-of-plane vibrational modes must be excited in the interior of the wedge.

To our best knowledge, finite eigenstrains in the framework of nonlinear elasticity have not been studied in any geometry other than spherical and cylindrical. In this chapter, we consider an infinite wedge made of an incompressible and isotropic solid and assume that it has a circumferentially-symmetric distribution of finite radial and circumferential eigenstrains. We derive the governing equilibrium equations of the wedge and using a semi-inverse method and assuming a specific class of deformations find the stresses that are induced by finite radial and circumferential eigenstrains. In particular, we solve for the stress field of both neo-Hookean and Mooney-Rivlin wedges with a symmetric inclusion or inhomogeneity with eigenstrains.

This chapter is organized as follows. In §3.2, we discuss the material manifold of a wedge with a circumferentially-symmetric distribution of finite eigenstrains and find the governing equations for an incompressible, isotropic wedge. In §3.2.1 and §3.2.2, we solve the problems of an inclusion and a Mooney-Rivlin inhomogeneity with uniform eigenstrains in a neo-Hookean wedge. In §3.2.3, we find the impotent (stress-free) circumferentially-symmetric finite eigenstrain distributions.

We assume that the body in the absence of eigenstrains is isotropic. Eigenstrains are modeled by a material metric $\mathbf{G}$ that explicitly depends on the distribution of eigenstrains [24, 34]. In other words, stress-free configuration of a body with a distribution of eigenstrains may not be globally realizable in the Euclidean ambient space. Note that the results of this section have been previously reported in our published work [25].

3.2 An infinite incompressible isotropic wedge with finite circumferentially symmetric eigenstrains

In this section we consider an infinitely long wedge of radius R_o and angle $2\Theta_o$ (see Figure 3.1). Let $(\bar{R}, \bar{\Theta}, \bar{Z})$ be the cylindrical coordinates for which $\bar{R} \geq 0\,,\ -\Theta_o \leq \bar{\Theta} \leq \Theta_o\,,$ and $\bar{Z} \in \mathbb{R}$ such that the axis of the wedge corresponds to $\bar{R} = 0$. In the cylindrical coordinates $(\bar{R}, \bar{\Theta}, \bar{Z})$, the material metric for the eigenstrain-free configuration reads

$$\boldsymbol{G}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \bar{R}^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{3.1}$$

We assume a circumferentially-symmetric eigenstrain (pre-strain) distribution in the wedge. With respect to the initial reference configuration and using the cylindrical coordinates (R, Θ, Z) for the material manifold $\mathbf{K}$ is assumed to have the following representation

$$\boldsymbol{K} = \begin{pmatrix} e^{\omega_R(\Theta)} & 0 & 0 \\ 0 & e^{\omega_\Theta(\Theta)} & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{3.2}$$

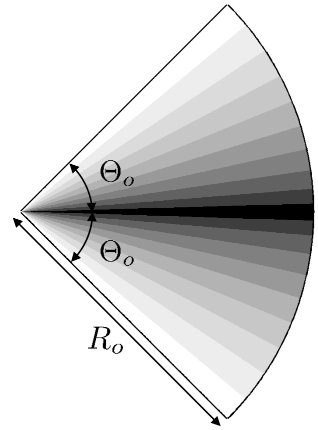


Figure 3.1: *A wedge with a finite circumferentially-symmetric eigenstrain distribution.*

where $\omega_R(\Theta)$ and $\omega_\Theta(\Theta)$ are arbitrary functions respectively describing the radial and circumferential eigenstrain distributions in the wedge. Now the material metric $\mathbf{G} = \mathbf{K}_*\mathbf{G}_0$ will have the following representation in the cylindrical coordinates (R, Θ, Z):

$$\boldsymbol{G} = \begin{pmatrix} e^{2\omega_R(\Theta)} & 0 & 0 \\ 0 & R^2 e^{2\omega_\Theta(\Theta)} & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{3.3}$$

This is the metric that was introduced by Yavari and Goriely [24].[1]

We endow the ambient space with the flat Euclidean metric, which in cylindrical coordinates (r, θ, z) reads

$$\boldsymbol{g} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{3.4}$$

Let us consider the class of deformations for which radial surfaces $\Theta = \mathrm{constant}$ in the reference configuration remain planar and are mapped to radial surfaces in the current configuration. That is,

[1]Similar constructions using non-trivial material geometries have been introduced in thermoelasticity, growth mechanics, and the mechanics of distributed defects [9, 20, 45, 63, 64, 65, 10, 66].

we assume an embedding of the material manifold into the ambient space with the following form

$$r = k(R, \Theta)\,, \qquad \theta = h(\Theta)\,, \qquad z = Z\,. \tag{3.5}$$

Therefore, the deformation gradient reads

$$\boldsymbol{F} = \begin{pmatrix} \frac{\partial k}{\partial R} & \frac{\partial k}{\partial \Theta} & 0 \\ \\ 0 & \frac{dh}{d\Theta} & 0 \\ \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.6}$$

Assuming incompressibility $J = \sqrt{\frac{\det \boldsymbol{g}}{\det \boldsymbol{G}}} \det \boldsymbol{F} = 1$, we find

$$h'(\Theta) \left[k^2(R, \Theta) - k^2(0, \Theta)\right] = R^2 e^{\omega_R(\Theta) + \omega_\Theta(\Theta)}. \tag{3.7}$$

Eliminating the rigid body translation by setting $r(0, \Theta) = 0$, we find that

$$r = k(R, \Theta) = R\zeta(\Theta), \tag{3.8}$$

where

$$\zeta^2(\Theta) = \frac{e^{\omega_R(\Theta) + \omega_\Theta(\Theta)}}{h'(\Theta)}\,. \tag{3.9}$$

This means that for an incompressible wedge within the class of deformations (3.5), and given the radial and circumferential eigenstrain distributions, the kinematics is fully determined after solving for the unknown function $\zeta = \zeta(\Theta)$. The right Cauchy-Green deformation tensor is written as

$$\boldsymbol{C} = \begin{pmatrix} e^{-2\omega_R(\Theta)}\zeta^2(\Theta) & Re^{-2\omega_R(\Theta)}\zeta(\Theta)\zeta'(\Theta) & 0 \\ \frac{e^{-2\omega_\Theta(\Theta)}}{R}\zeta(\Theta)\zeta'(\Theta) & \frac{e^{2\omega_R(\Theta)}}{\zeta^2(\Theta)} + \zeta'(\Theta)^2 e^{-2\omega_\Theta(\Theta)} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.10}$$

The invariants of $\boldsymbol{C}$ are

$$I_1 = \mathrm{tr}(\boldsymbol{C}) = 1 + \zeta(\Theta)^2 e^{-2\omega_R(\Theta)} + \zeta'(\Theta)^2 e^{-2\omega_\Theta(\Theta)} + \frac{e^{2\omega_R(\Theta)}}{\zeta(\Theta)^2}\,, \tag{3.11}$$

$$I_2 = = \frac{1}{2}(\mathrm{tr}(\boldsymbol{C}^2) - \mathrm{tr}(\boldsymbol{C})^2) = 1 + \zeta(\Theta)^2 e^{-2\omega_R(\Theta)} + \zeta'(\Theta)^2 e^{-2\omega_\Theta(\Theta)} + \frac{e^{2\omega_R(\Theta)}}{\zeta(\Theta)^2}\,, \tag{3.12}$$

$$I_3 = \det(\boldsymbol{C}) = 1\,. \tag{3.13}$$

Note that $I_1 = I_2$ depends only on Θ.

We assume that the wedge is made of an incompressible isotropic radially-homogenous material, i.e., the strain energy function has the form $W = W(\Theta, I_1, I_2)$. Following (2.10), for the class of deformations (3.5), the non-zero components of the Cauchy stress tensor read

$$\sigma^{rr} = -p + 2\left(W_{I_1} + W_{I_2}\right)\left(e^{-2\omega_\Theta(\Theta)}\zeta'(\Theta)^2 + \zeta(\Theta)^2 e^{-2\omega_R(\Theta)}\right) + 2W_{I_2}\,, \tag{3.14}$$

$$\sigma^{r\theta} = \frac{2\zeta'(\Theta)}{R\zeta(\Theta)^2}\left(W_{I_1} + W_{I_2}\right) e^{\omega_R(\Theta) - \omega_\Theta(\Theta)}\,, \tag{3.15}$$

$$\sigma^{\theta\theta} = \frac{1}{R^2\zeta(\Theta)^4}\left(2e^{2\omega_R(\Theta)}\left(W_{I_1} + W_{I_2}\right) - \zeta(\Theta)^2\left(p - 2W_{I_2}\right)\right)\,, \tag{3.16}$$

$$\sigma^{zz} = -p + 2W_{I_2}\left(e^{-2\omega_\Theta(\Theta)}\zeta'(\Theta)^2 + \zeta(\Theta)^2 e^{-2\omega_R(\Theta)} + \frac{e^{2\omega_R(\Theta)}}{\zeta(\Theta)^2}\right) + 2W_{I_1}\,. \tag{3.17}$$

The physical components of the Cauchy stress, i.e., $\hat{\sigma}^{ab} = \sigma^{ab}\sqrt{g_{aa}g_{bb}}$ (no summation) [67] read

$$\hat{\sigma}^{rr} = \sigma^{rr}\,, \quad \hat{\sigma}^{r\theta} = R\zeta(\Theta)\sigma^{r\theta}\,, \quad \hat{\sigma}^{\theta\theta} = R^2\zeta^2(\Theta)\sigma^{\theta\theta}\,, \quad \hat{\sigma}^{zz} = \sigma^{zz}\,. \tag{3.18}$$

The first Piola-Kirchhoff stress tensor $P^{aA} = J(F^{-1})^A{}_b \sigma^{ab}$ has the following non-zero components

$$
\begin{aligned}
P^{rR} =& \frac{e^{-2\omega_R(\Theta)}}{\zeta(\Theta)} \left(2\zeta(\Theta)^2 \left(W_{I_1} + W_{I_2}\right) - e^{2\omega_R(\Theta)} \left(p - 2W_{I_2}\right)\right) \,, & (3.19)\\
P^{r\Theta} =& \frac{2e^{-2\omega_\Theta(\Theta)}}{R} \zeta'(\Theta) \left(W_{I_1} + W_{I_2}\right) \,, & (3.20)\\
P^{\theta R} =& \frac{e^{-\omega_\Theta(\Theta)-\omega_R(\Theta)} \zeta'(\Theta)}{R\zeta(\Theta)} \left(p - 2W_{I_2}\right) \,, & (3.21)\\
P^{\theta\Theta} =& \frac{e^{-\omega_\Theta(\Theta)-\omega_R(\Theta)}}{R^2\, \zeta(\Theta)^2} \left(2e^{2\omega_R(\Theta)} \left(W_{I_1} + W_{I_2}\right) - \zeta(\Theta)^2 \left(p - 2W_{I_2}\right)\right) \,, & (3.22)\\
P^{zZ} =& -p + 2W_{I_2} \left(e^{-2\omega_\Theta(\Theta)} \zeta'(\Theta)^2 + \zeta(\Theta)^2 e^{-2\omega_R(\Theta)} + \frac{e^{2\omega_R(\Theta)}}{\zeta(\Theta)^2}\right) + 2W_{I_1} \,. & (3.23)
\end{aligned}
$$

In the absence of body forces, the non-trivial equilibrium equations are $\sigma^{rb}{}_{|b} = 0$ and $\sigma^{\theta b}{}_{|b} = 0$ (the axial equilibrium equation implies that $p = p(R, \Theta)$). Note that, following (3.5), (3.8), and (3.9), we have

$$
\begin{aligned}
\frac{\partial}{\partial r} &= \frac{1}{\zeta(\Theta)} \frac{\partial}{\partial R} \,, & (3.24)\\
\frac{\partial}{\partial \theta} &= \frac{\zeta^2(\Theta)}{e^{\omega_R(\Theta)+\omega_\Theta(\Theta)}} \left(\frac{\partial}{\partial \Theta} - \frac{R\zeta'(\Theta)}{\zeta(\Theta)} \frac{\partial}{\partial R}\right) \,. & (3.25)
\end{aligned}
$$

Therefore, the non-trivial equilibrium equations read

$$
\begin{aligned}
& 2\zeta e^{-2\omega_\Theta} \Big[\zeta'' \left(W_{I_1} + W_{I_2}\right) + \zeta' \left(W_{I_1} + W_{I_2}\right) \left(\omega'_R - \omega'_\Theta\right) + \zeta' \left(W'_{I_1} + W'_{I_2}\right)\Big] \\
& \qquad + 2\left(W_{I_1} + W_{I_2}\right) \left[\zeta^2 e^{-2\omega_R} - \frac{e^{2\omega_R}}{\zeta^2}\right] - R\frac{\partial p}{\partial R} = 0 \,, & (3.26a)\\
& R\zeta\zeta' \frac{\partial p}{\partial R} - \zeta^2 \frac{\partial p}{\partial \Theta} + 2e^{2\omega_R} \left(W'_{I_1} + W'_{I_2}\right) + 4e^{2\omega_R} \omega'_R \left(W_{I_1} + W_{I_2}\right) + 2\zeta^2 W'_{I_2} = 0 \,, & (3.26b)
\end{aligned}
$$

where by using the chain rule, one can write

$$
\begin{aligned}
W'_{I_1}(\Theta) &= \frac{\partial W_{I_1}}{\partial I_1} \frac{\partial I_1}{\partial \Theta} + \frac{\partial W_{I_1}}{\partial I_2} \frac{\partial I_2}{\partial \Theta} + \frac{\partial W_{I_1}}{\partial \Theta} \,, \\
W'_{I_2}(\Theta) &= \frac{\partial W_{I_2}}{\partial I_1} \frac{\partial I_1}{\partial \Theta} + \frac{\partial W_{I_2}}{\partial I_2} \frac{\partial I_2}{\partial \Theta} + \frac{\partial W_{I_2}}{\partial \Theta} \,.
\end{aligned} \qquad (3.27)
$$

It follows from (4.35) that

$$p(R,\Theta) = f(\Theta)\ln R + \Phi(\Theta)\,, \tag{3.28}$$

where

$$\begin{aligned} f(\Theta) = 2\zeta e^{-2\omega_\Theta}\Big[\zeta''\left(W_{I_1}+W_{I_2}\right) + \zeta'\left(W_{I_1}+W_{I_2}\right)\left(\omega_R'-\omega_\Theta'\right) \\ + \zeta'\left(W_{I_1}'+W_{I_2}'\right)\Big] + 2\left(W_{I_1}+W_{I_2}\right)\left[\zeta^2 e^{-2\omega_R} - \frac{e^{2\omega_R}}{\zeta^2}\right], \end{aligned} \tag{3.29}$$

and $\Phi(\Theta)$ is an arbitrary function of Θ to be determined. Substituting the pressure field into (6.52) yields

$$\zeta\zeta' f - \zeta^2\Phi' + 2e^{2\omega_R}\left(W_{I_1}'+W_{I_2}'\right) + 4e^{2\omega_R}\omega_R'\left(W_{I_1}+W_{I_2}\right) + 2\zeta^2 W_{I_2}' - \zeta^2 f'\ln R = 0\,. \tag{3.30}$$

Note that (3.30) must hold for any R and $\zeta(\Theta)\neq 0$. Therefore, f is constant, i.e., $f(\Theta) = f_o$ and hence

$$p(R,\Theta) = f_o \ln R + \Phi(\Theta). \tag{3.31}$$

Therefore, the equilibrium equation (3.30) is reduced to the following ODE

$$\zeta\zeta' f_o - \zeta^2\Phi' + 4e^{2\omega_R}\omega_R'\left(W_{I_1}+W_{I_2}\right) + 2e^{2\omega_R}W_{I_1}' + 2\left[e^{2\omega_R}+\zeta^2\right]W_{I_2}' = 0\,. \tag{3.32}$$

One then obtains Φ' as

$$\Phi'(\Theta) = f_o\frac{\zeta'}{\zeta} + \frac{2e^{2\omega_R}}{\zeta^2}\left(W_{I_1}'+W_{I_2}'\right) + \frac{4e^{2\omega_R}\omega_R'}{\zeta^2}\left(W_{I_1}+W_{I_2}\right) + 2W_{I_2}'\,. \tag{3.33}$$

Equation (3.29) gives us the following nonlinear second-order ODE for $\zeta(\Theta)$.

$$\begin{aligned} 2\zeta e^{-2\omega_\Theta}\Big[\zeta''\left(W_{I_1}+W_{I_2}\right) + \zeta'\left(W_{I_1}+W_{I_2}\right)\left(\omega_R'-\omega_\Theta'\right) \\ + \zeta'\left(W_{I_1}'+W_{I_2}'\right)\Big] + 2\left(W_{I_1}+W_{I_2}\right)\left[\zeta^2 e^{-2\omega_R} - \frac{e^{2\omega_R}}{\zeta^2}\right] = f_o\,. \end{aligned} \tag{3.34}$$

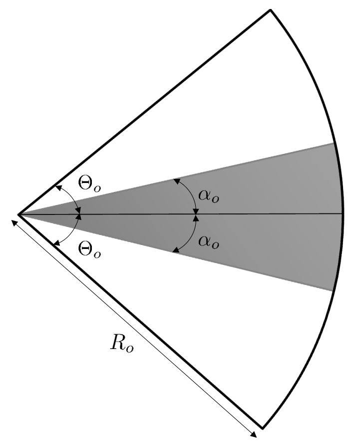


Figure 3.2: *A wedge with uniform eigenstrains in the shaded region.*

In the next section, we will solve for the residual stress field of a neo-Hookean wedge with a symmetric inclusion with uniform eigenstrains.

3.2.1 An inclusion with uniform eigenstrains in a neo-Hookean wedge with traction-free lateral boundaries

Let us consider the following distribution of eigenstrains in the wedge (see Figure 3.2)

$$\omega_R(\Theta) = \begin{cases} \omega_1\,, & |\Theta| \leq \alpha_o \\ 0\,, & |\Theta| > \alpha_o \end{cases}\,, \quad \omega_\Theta(\Theta) = \begin{cases} \omega_2\,, & |\Theta| \leq \alpha_o \\ 0\,, & |\Theta| > \alpha_o \end{cases}\,, \tag{3.35}$$

where ω_1 and ω_2 are constants. Let us assume that the wedge is made of an incompressible homogeneous neo-Hookean solid, i.e., $W = W(I_1) = \frac{\mu}{2}(I_1 - 3)$. Thus, $W_{I_1} = \frac{\mu}{2}\,,\ W_{I_2} = 0$. Simplifying (3.34), we find the following non-linear second-order ODEs inside and outside the inclusion

$$\begin{aligned} \zeta\zeta'' e^{-2\omega_2} + \zeta^2 e^{-2\omega_1} - \frac{e^{2\omega_1}}{\zeta^2} &= \frac{f_o}{\mu}\,, \quad |\Theta| \leq \alpha_o\,, \\ \zeta\zeta'' + \zeta^2 - \frac{1}{\zeta^2} &= \frac{f_o}{\mu}\,, \quad |\Theta| > \alpha_o\,. \end{aligned} \tag{3.36}$$

Note that in the absence of eigenstrains ($\omega_1 = \omega_2 = 0$), the above equations reduce to the equation for the deformation of a wedge derived by Fu *et al.* [58], Rajagopal and Tao [60], and Rajagopal and

Carroll [59]. We integrate (3.33) for the assumed eigenstrain distribution and find that the pressure field has the following distribution

$$p(R,\Theta) = f_o \ln R + \Phi(\Theta) = \begin{cases} f_o \ln(R\zeta(\Theta)) + p_i\,, & |\Theta| \leq \alpha_o\,, \\ f_o \ln(R\zeta(\Theta)) + p_o\,, & |\Theta| > \alpha_o\,, \end{cases} \tag{3.37}$$

where p_i and p_o are constants of integration. We integrate (3.36) once and obtain

$$\zeta'(\Theta)^2 = \begin{cases} c_{i_1} + \frac{2f_o e^{2\omega_2}}{\mu} \ln\zeta - \zeta^2 e^{2(\omega_2-\omega_1)} - \frac{e^{2(\omega_1+\omega_2)}}{\zeta^2}\,, & |\Theta| \leq \alpha_o\,, \\ c_{o_1} + \frac{2f_o}{\mu} \ln\zeta - \zeta^2 - \frac{1}{\zeta^2}\,, & |\Theta| > \alpha_o\,, \end{cases} \tag{3.38}$$

where c_{i_1} and c_{o_1} are constants of integration. In order to solve (3.38) for ζ, we next examine the boundary and continuity conditions.

Boundary conditions. The traction vector is defined as

$$\boldsymbol{t} = \langle \boldsymbol{\sigma}, \boldsymbol{n} \rangle_{\boldsymbol{g}}\,. \tag{3.39}$$

In components, $t^a(x,\boldsymbol{n}) = \sigma^{ac} g_{bc} n^b$. From (3.39), the continuity of the traction vector on the boundary of the inclusion (or inhomogeneity) implies that both $\sigma^{r\theta}$ and $\sigma^{\theta\theta}$ must be continuous at $\Theta=\pm\alpha_o$. Thus, after some simplifications, (3.15) and (6.32) give us

$$e^{\omega_1-\omega_2}\zeta'(\Theta)|_{\Theta=\alpha_o^-} = \zeta'(\Theta)|_{\Theta=\alpha_o^+}, \tag{3.40}$$

and

$$\mu\left(e^{2\omega_1}-1\right) = (p_i - p_o)\,\zeta(\pm\alpha_o)^2. \tag{3.41}$$

Remark 3.2.1. From (3.41), it is clear that when the eigenstrain distribution is purely circumferential, i.e., $\omega_1 = 0$, one finds that $p_i = p_o = c$. Hence, the pressure field is continuous at the inclusion boundary and reads $p(R,\Theta) = f_o \ln\left(R\zeta(\Theta)\right) + c$.

Remark 3.2.2. Note that although the Cauchy traction vector $\boldsymbol{t}\,(x,\boldsymbol{n}) = \langle \boldsymbol{\sigma},\boldsymbol{n}\rangle_{\boldsymbol{g}}$ is continuous at the inclusion boundary, the first Piola-Kirchhoff traction vector $\boldsymbol{t}_o\,(X,\boldsymbol{N}) = \langle \boldsymbol{P},\boldsymbol{N}\rangle_{\boldsymbol{G}}$ is not. This is due to the fact that $\boldsymbol{t}_o$ is defined with respect to the undeformed surface element dA in the reference configuration. Since the material metric is discontinuous at the inclusion boundary, dA is discontinuous as well. However, $\boldsymbol{t}_o\,(X,\boldsymbol{N})\,dA = \boldsymbol{t}\,(x,\boldsymbol{n})\,da$ is continuous. Hence the first Piola-Kirchhoff traction vector must be discontinuous at the inclusion boundary to account for the discontinuity of dA and make $\boldsymbol{t}_o\,(X,\boldsymbol{N})\,dA$ continuous. On the other hand, $\boldsymbol{t}$ is continuous because it is defined per unit of deformed area in the current configuration da, which is continuous at the inclusion boundary.

The continuity of the displacement field implies that $\zeta(\Theta)$ and $h(\Theta)$ are both continuous at $\Theta = \pm\alpha_o$. For boundary conditions, we can either prescribe the tractions or the resultant forces acting on the boundary of the wedge. Alternatively, we may specify the boundary displacements and then find the required surface tractions. We assume the special case of symmetric boundary conditions with respect to the bisecting plane of the wedge, and then find the boundary tractions required to maintain such a deformation. Note, however, that Tao and Rajagopal [57] showed that for Blatz-Ko (compressible) materials, only asymmetric inhomogeneous solutions are admitted by the equilibrium equations.

Let us assume that the lateral boundaries are traction-free, i.e.

$$P^{r\Theta} = P^{\theta\Theta} = 0\,, \quad 0 \leq R \leq R_o\,, \quad \Theta = \pm\Theta_o\,. \tag{3.42}$$

Imposing (3.42), we find that $p(R,\Theta)$ must be bounded (f_o=0) and

$$\zeta'(\pm\Theta_o) = 0\,, \quad p_o = \frac{\mu}{\zeta(\Theta_o)^2}\,. \tag{3.43}$$

Furthermore, (3.37) implies that the pressure is equal to p_i inside the inclusion and is equal to p_o outside the inclusion. Note that due to the symmetry of the problem, $\zeta(\Theta)$ and $h(\Theta)$ must be even and odd, respectively. Thus, since (3.36) implies that $\zeta(\Theta)$ must be at least C^2 inside the inclusion, we have $\zeta'(0) = 0$. Hence, we can solve the problem by imposing the above boundary conditions, which in turn specify the required traction distribution on the circular boundary of the wedge. Then, we find the resultant force acting on the circular boundary of the wedge, which is equal to the force that needs be applied at the apex of the wedge to maintain the equilibrium. The radial material traction per unit undeformed area acting on the circular boundary is calculated using the relation, $t_o^a = P^{aA}N^B G_{BA}$. Thus[2]

$$t_o^\theta = \hat{P}^{\theta R}\,,\ \ t_o^r = \hat{P}^{rR}, \qquad R = R_o\,,\ -\Theta_o < \Theta < \Theta_o\,. \tag{3.44}$$

Therefore, the radial force per unit undeformed area reads[3]

$$F_r = \int t_o^r dA_{\boldsymbol{G}}\,, \tag{3.45}$$

where $dA_{\boldsymbol{G}} = R_o e^{\omega_\Theta(\Theta)} d\Theta \wedge dZ$ is the Riemannian area element.[4] Hence, for the infinite cylinder (in the Z-direction), the radial force per unit length of the cylinder in the Z-direction is written as

$$F_r = \int_{-\Theta_o}^{\Theta_o} \left(R_o e^{\omega_\Theta(\Theta)} e^{\omega_R(\Theta)} P^{rR}|_{(R_o,\Theta)} \right) d\Theta\,, \tag{3.46}$$

which is simplified to read

$$F_r = 2\mu R_o \left[e^{\omega_2-\omega_1} \int_0^{\alpha_o} \left(\zeta(\Theta) - \frac{p_i e^{2\omega_1}}{\mu\zeta(\Theta)} \right) d\Theta + \int_{\alpha_o}^{\Theta_o} \left(\zeta(\Theta) - \frac{p_o}{\mu\zeta(\Theta)} \right) d\Theta \right]. \tag{3.47}$$

Remark 3.2.3. It is worth mentioning that only if $\zeta(\Theta) = \text{constant}$ one can enforce pointwise zero traction boundary conditions on the whole boundary of the wedge for any values of ω_1. In this case, we can only have Θ_o=α_o and $\zeta = e^{\omega_1}$, which in turn gives $h(\Theta) = e^{\omega_2-\omega_1}\Theta$. Hence, all the stress components vanish point-wise.

[2]Note that the physical components of the first Piola-Kirchhoff stress tensor are defined as $\hat{P}^{aA} = P^{aA}\sqrt{G_{AA}g_{aa}}$ (no summation).

[3]The resultant force acting in the θ-direction on the circular boundary is trivially zero as $\zeta'(\Theta)$ is an odd function. In addition, $t^z = 0$ as $P^{zR} = 0$.

[4]The volume form of a Riemannian manifold is defined as $\Omega = \sqrt{\det(g_{ij})}dx^1 \wedge dx^2 \wedge ... \wedge dx^n$.

Solving (3.38), one obtains $\zeta(\Theta)$ in the upper half region of the wedge as[5]

$$\zeta(\Theta)=\begin{cases} e^{\frac{1}{2}(\omega_1-\omega_2)}\left[\left(\sqrt{\left(\frac{1}{2}c_{i_1}e^{\omega_1-\omega_2}\right)^2-e^{2(\omega_1+\omega_2)}}\right)\cos\left(2e^{\omega_2-\omega_1}\Theta+c_{i_2}\right)\right. \\ \qquad\qquad \left.+\frac{1}{2}c_{i_1}e^{\omega_1-\omega_2}\right]^{\frac{1}{2}}, & 0\le\Theta\le\alpha_o, \\ \\ \left[\frac{1}{2}c_{o_1}+\left(\sqrt{\frac{1}{4}c_{o_1}^2-1}\right)\cos\left(2\Theta+c_{o_2}\right)\right]^{\frac{1}{2}}, & \alpha_o\le\Theta\le\Theta_o, \end{cases} \tag{3.48}$$

where c_{i_2} and c_{o_2} are constants.

Equation (3.43), i.e., $\zeta'(\Theta_o)=0$, and $\zeta'(0)=0$ give us $c_{o_2}=k_1\pi-2\Theta_o$ and $c_{i_2}=k_2\pi$, respectively, where $k_1,k_2\in\mathbb{Z}$. Upon using the continuity of $\zeta(\Theta)$ at $\Theta=\alpha_o$ as well as (3.40), we find c_{i_1} and c_{o_1}. They read

$$\begin{aligned} c_{i_1}=2e^{2\omega_2}\Bigg[1+\left(e^{4\omega_1}-1\right)^2\Bigg\{&-e^{4\omega_1}\left(e^{4\omega_1}+1\right)\cot\left(2\left(\alpha_o-\Theta_o\right)\right)\sin\left(4e^{\omega_2-\omega_1}\alpha_o\right) \\ &+e^{4\omega_1}\csc^2\left(2\left(\alpha_o-\Theta_o\right)\right)\sin^2\left(2e^{\omega_2-\omega_1}\alpha_o\right)\Big(\left(e^{4\omega_1}+1\right)\cos^2\left(2\left(\alpha_o-\Theta_o\right)\right)+e^{4\omega_1}-1\Big) \\ &\pm\sqrt{2}\Big[-e^{8\omega_1}\csc^4\left(2\left(\alpha_o-\Theta_o\right)\right)\sin^2\left(2\left(1-e^{\omega_2-\omega_1}\right)\alpha_o-2\Theta_o\right)\Big\{\left(1-e^{4\omega_1}\right)\cos\left(4\left(\alpha_o-\Theta_o\right)\right) \\ &+e^{4\omega_1}\cos\left(4\left(1-e^{\omega_2-\omega_1}\right)\alpha_o-4\Theta_o\right)+\left(e^{4\omega_1}-1\right)\left(e^{4\omega_1}\cos\left(4e^{\omega_2-\omega_1}\alpha_o\right)+1\right)-e^{8\omega_1}\Big\}\Big]^{\frac{1}{2}} \\ &-e^{8\omega_1}\sin^2\left(2e^{\omega_2-\omega_1}\alpha_o\right)+e^{4\omega_1}\left(\cos^2\left(2e^{\omega_2-\omega_1}\alpha_o\right)+1\right)\Bigg\}^{-1}\Bigg]^{\frac{1}{2}}, \end{aligned} \tag{3.49}$$

$$c_{o_1}=2\left[1+\left(\frac{e^{2\omega_1}\sin\left(2e^{\omega_2-\omega_1}\alpha_o\right)}{\sin\left(2\left(\Theta_o-\alpha_o\right)\right)}\right)^2\left(\left(\frac{c_{i_1}}{2e^{2\omega_2}}\right)^2-1\right)\right]^{\frac{1}{2}}. \tag{3.50}$$

---

[5]Here, it suffices to specify $\zeta(\Theta)$ and $h(\Theta)$ only in the upper half region of the wedge as these functions are even and odd, respectively.

Using (3.8), one finds

$$h(\Theta) = \begin{cases} \tan^{-1}\left[e^{-(\omega_1+\omega_2)}\left(-\sqrt{\left(\frac{1}{2}c_{i_1}e^{\omega_1-\omega_2}\right)^2 - e^{2(\omega_1+\omega_2)}}\right.\right. \\ \qquad\qquad \left.\left. + \frac{1}{2}c_{i_1}e^{\omega_1-\omega_2}\right)\tan\left(e^{\omega_2-\omega_1}\Theta + \frac{1}{2}c_{i_2}\right)\right] + c_{i_3}, & 0 \leq \Theta \leq \alpha_o, \\ \\ \tan^{-1}\left[\left(\frac{1}{2}c_{o_1} - \sqrt{\frac{1}{4}c_{o_1}^2 - 1}\right)\tan(\Theta + \frac{1}{2}c_{o_2})\right] + c_{o_3}, & \alpha_o \leq \Theta \leq \Theta_o, \end{cases} \tag{3.51}$$

where c_{i_3} and c_{o_3} are constants of integration. Imposing the condition $h(0) = 0$, implies that $c_{i_3} = -k_2\frac{\pi}{2} - k_3\pi$, where $k_3 \in \mathbb{Z}$. Using the continuity of $h(\Theta)$ at $\Theta = \alpha_o$, we have

$$\begin{aligned} c_{o_3} = \tan^{-1}&\left[e^{-(\omega_1+\omega_2)}\left(-\sqrt{\left(\frac{1}{2}c_{i_1}e^{\omega_1-\omega_2}\right)^2 - e^{2(\omega_1+\omega_2)}} + \frac{1}{2}c_{i_1}e^{\omega_1-\omega_2}\right)\tan\left(e^{\omega_2-\omega_1}\alpha_o + \frac{k_2\pi}{2}\right)\right] \\ &- \tan^{-1}\left[\left(\frac{1}{2}c_{o_1} - \sqrt{\frac{1}{4}c_{o_1}^2 - 1}\right)\tan(\alpha_o - \Theta_o + \frac{k_1\pi}{2})\right] - k_2\frac{\pi}{2} - k_3\pi. \end{aligned} \tag{3.52}$$

Remark 3.2.4. From (3.49) and (3.50), it can be seen that $\omega_1 = 0$ implies that $c_{i_1} = 2e^{2\omega_2}$ and $c_{o_1} = 2$. In this case, the radius of the wedge does not change, and the inclusion deforms independently of the matrix in the circumferential direction, such that $h(\Theta) = e^{\omega_2}\Theta$ in the inclusion, and $h(\Theta) = (e^{\omega_2} - 1)\,\alpha_o + \Theta$ outside the inclusion. Furthermore, all the components of the stress tensor are zero point-wise.

Using (3.18), (3.37), (3.41), and (3.43), one finds the physical components of the Cauchy stress, along with the pressure field as follows

$$\hat{\sigma}^{rr} = \begin{cases} -p_i + \mu\left(e^{-2\omega_2}\zeta'(\Theta)^2 + e^{-2\omega_1}\zeta(\Theta)^2\right)\,, & |\Theta| \leq \alpha_o\,, \\ -p_o + \mu\left(\zeta'(\Theta)^2 + \zeta(\Theta)^2\right)\,, & |\Theta| > \alpha_o\,, \end{cases} \tag{3.53}$$

$$\hat{\sigma}^{r\theta} = \begin{cases} \frac{\mu\zeta'(\Theta)}{\zeta(\Theta)} e^{\omega_1 - \omega_2}\,, & |\Theta| \leq \alpha_o\,, \\ \frac{\mu\zeta'(\Theta)}{\zeta(\Theta)}\,, & |\Theta| > \alpha_o\,, \end{cases} \tag{3.54}$$

$$\hat{\sigma}^{\theta\theta} = \begin{cases} \frac{1}{\zeta(\Theta)^2}\left(\mu e^{2\omega_1} - p_i\zeta(\Theta)^2\right)\,, & |\Theta| \leq \alpha_o\,, \\ \frac{1}{\zeta(\Theta)^2}\left(\mu - p_o\zeta(\Theta)^2\right)\,, & |\Theta| > \alpha_o\,, \end{cases} \tag{3.55}$$

$$\hat{\sigma}^{zz} = \begin{cases} -p_i + \mu\,, & |\Theta| \leq \alpha_o\,, \\ -p_o + \mu\,, & |\Theta| > \alpha_o\,, \end{cases} \tag{3.56}$$

where

$$p(\Theta) = \begin{cases} p_i = \frac{\mu}{\zeta(\Theta_o)^2} + \frac{\mu\left(e^{2\omega_1} - 1\right)}{\zeta(\alpha_o)^2}\,, & |\Theta| \leq \alpha_o\,, \\ p_o = \frac{\mu}{\zeta(\Theta_o)^2}\,, & |\Theta| > \alpha_o\,. \end{cases} \tag{3.57}$$

Remark 3.2.5. Note that the physical components of the Cauchy stress are independent of the radial coordinate R. Therefore, the stress components at the apex of the wedge do not have a unique value. In fact, this should not be surprising given that the eigenstrain distribution (3.35) is multi-valued at the apex.

Numerical results. We now consider some specific examples and find the deformed shape of the wedge and the corresponding residual stress field. A comparison of the deformations and the dis-

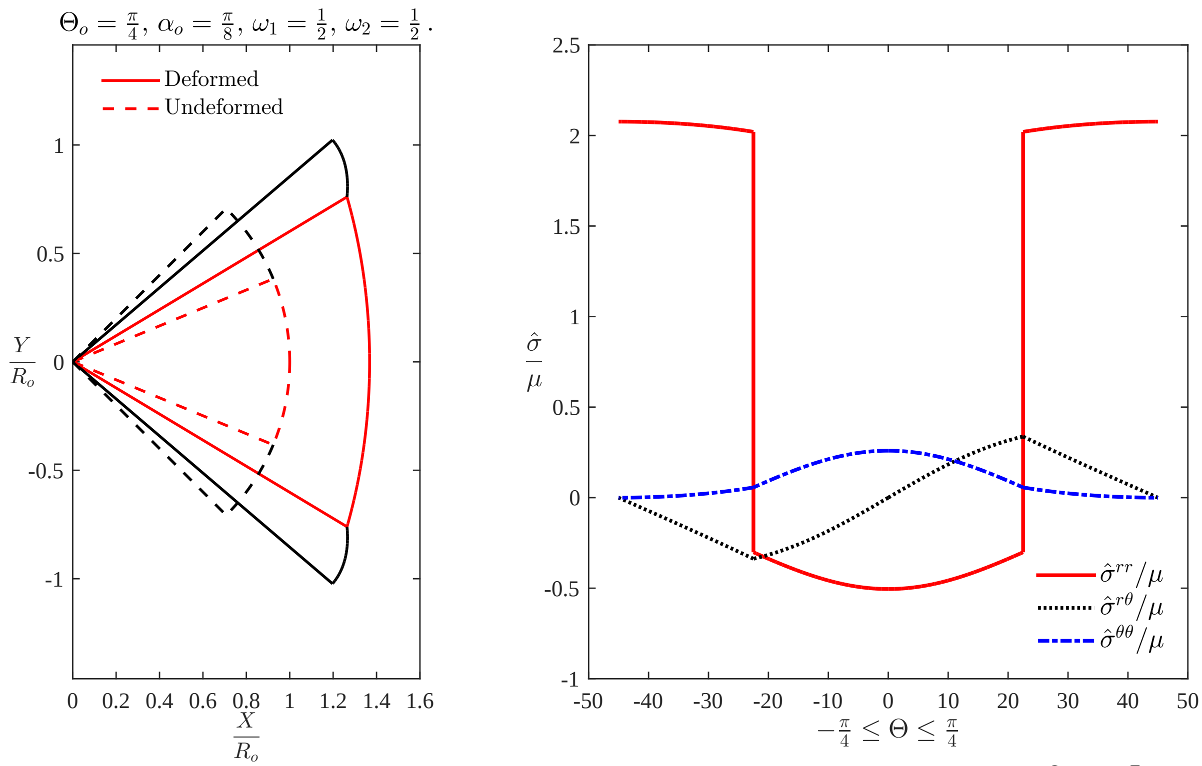


Figure 3.3: *Left: The initial and deformed configurations of a wedge with the initial half-angle $\Theta_o = \frac{\pi}{4}$ having an inclusion with $\alpha_o = \frac{\pi}{8}$ and the pure dilatational eigenstrain distribution $\omega_1 = \omega_2 = \frac{1}{2}$. Right: Variation of the physical components of the Cauchy stress tensor versus Θ.*

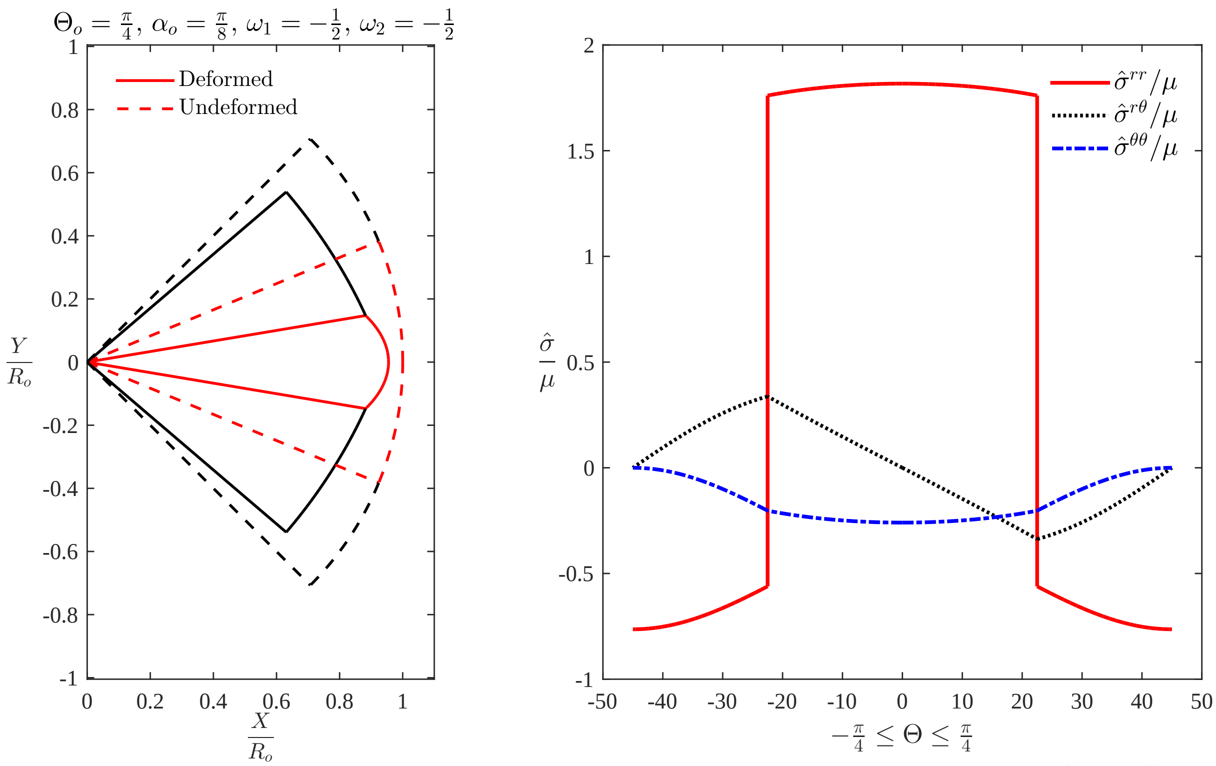


Figure 3.4: *Left: The initial and deformed configurations of a wedge with the initial half-angle $\Theta_o = \frac{\pi}{4}$ having an inclusion with $\alpha_o = \frac{\pi}{8}$ and the pure dilatational eigenstrain distribution $\omega_1 = \omega_2 = -\frac{1}{2}$. Right: Variation of the physical components of the Cauchy stress tensor versus Θ.*

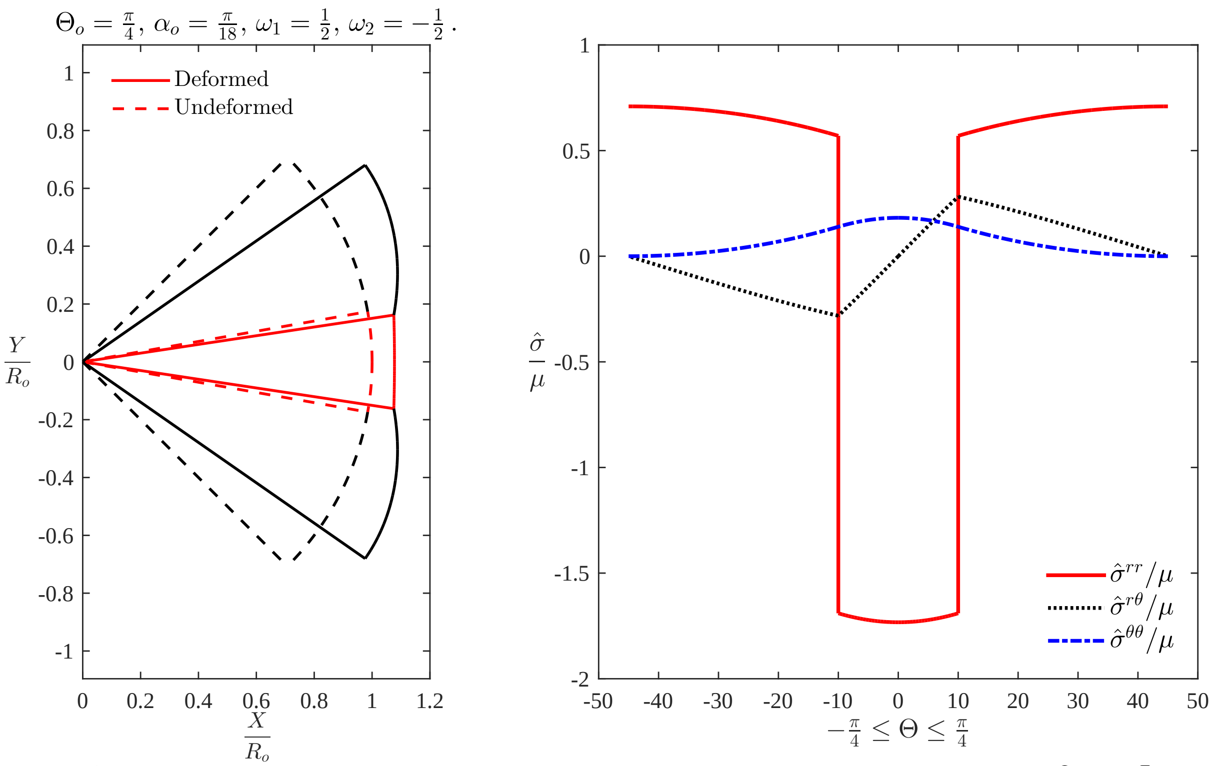


Figure 3.5: *Left: The initial and deformed configurations of a wedge with the initial half-angle* $\Theta_o = \frac{\pi}{4}$ *having an inclusion with* $\alpha_o = \frac{\pi}{18}$ *and the constant eigenstrain distribution* $\omega_1 = \frac{1}{2}$ *and* $\omega_2 = -\frac{1}{2}$*. Right: Variation of the physical components of the Cauchy stress tensor versus* Θ*.*

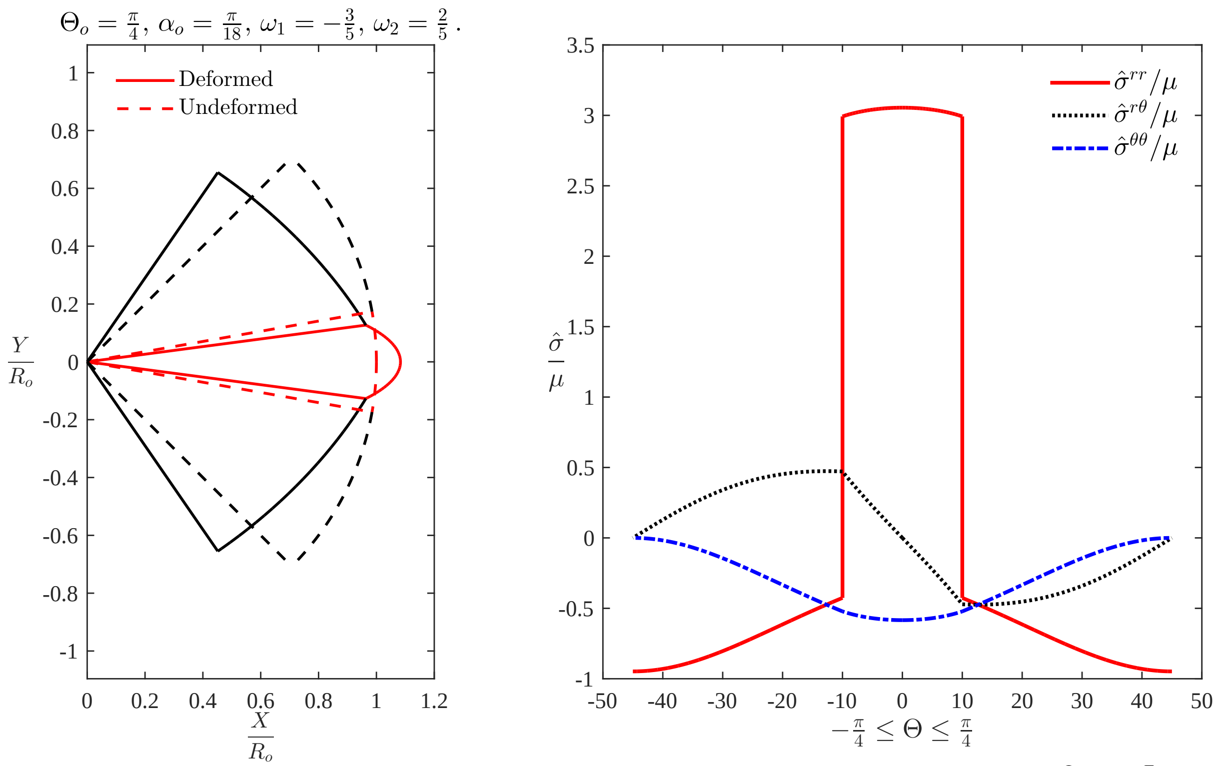


Figure 3.6: *Left: The initial and deformed configurations of a wedge with the initial half-angle* $\Theta_o = \frac{\pi}{4}$ *having an inclusion with* $\alpha_o = \frac{\pi}{18}$ *and the constant eigenstrain distribution* $\omega_1 = -\frac{3}{5}$ *and* $\omega_2 = \frac{2}{5}$. *Right: Variation of the physical components of the Cauchy stress tensor versus* Θ. *In this example, the deformation is non-unidirectional.*

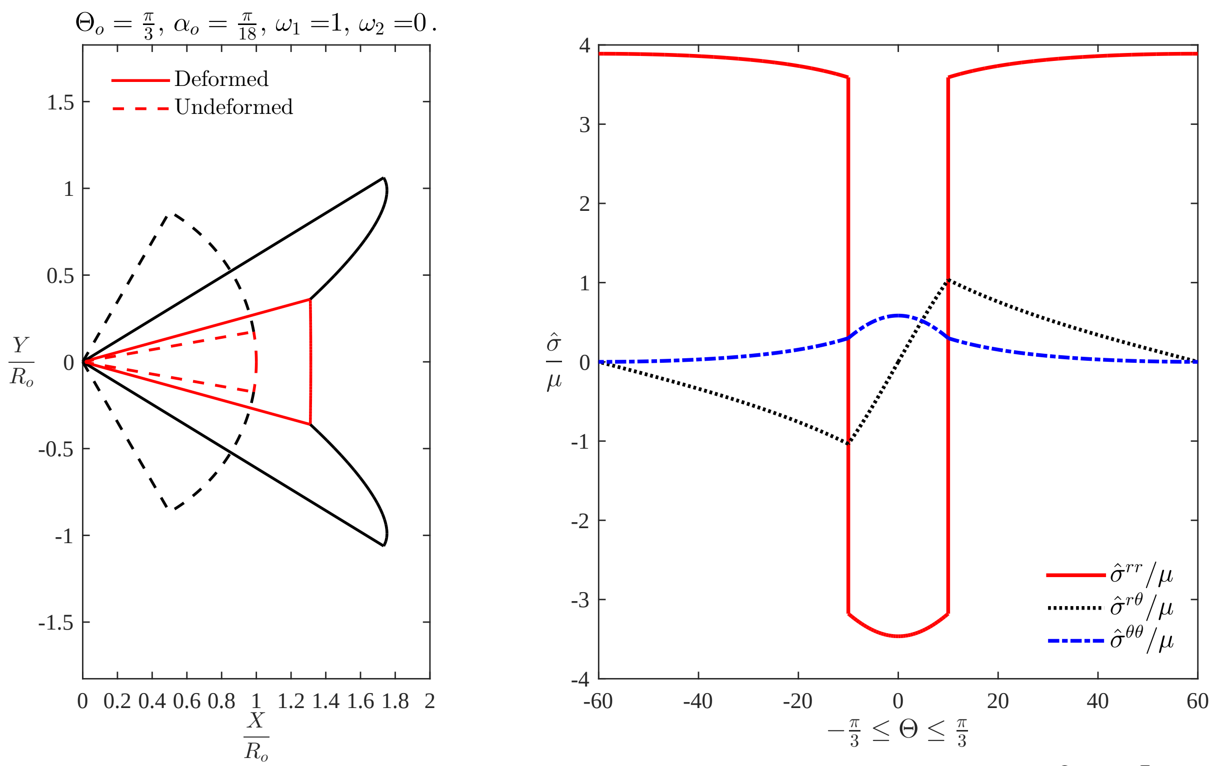


Figure 3.7: *Left: The initial and deformed configurations of a wedge with the initial half-angle $\Theta_o = \frac{\pi}{3}$ having an inclusion with $\alpha_o = \frac{\pi}{18}$ and the constant eigenstrain distribution $\omega_1 = 1$ and $\omega_2 = 0$. Right: Variation of the physical components of the Cauchy stress tensor versus Θ.*

tribution of the stress components for different values of eigenstrains ω_1 and ω_2, and various wedge geometries are presented in Figures 3.3, 3.4, 3.5, 3.6, and 3.7. A wedge having an inclusion with positive pure dilatational eigenstrains is depicted in Figure 3.3. As expected both the inclusion and the matrix regions are pushed outward in the radial direction, with the matrix filaments stretched more than those of the inclusion. Although the circumferential eigenstrain is positive in this case, the total wedge angle is decreased. As a matter of fact, for any positive value of pure dilatational eigenstrains the angle of the wedge is reduced after deformation. Moreover, $\hat{\sigma}^{rr}$ is compressive in the inclusion and tensile in the matrix, and undergoes a jump at the inclusion-matrix interface, which is also the case as illustrated in other figures.[6]

For an inclusion with negative purely dilatational eigenstrains, all the radial planes of the wedge displace inward, with the matrix region being shortened more than the inclusion (Figure 3.4). Undeformed and deformed configurations of a wedge with positive radial and negative circumferential eigenstrains is shown in Figure 3.5. Note that $\hat{\sigma}^{\theta\theta}$ is tensile throughout the wedge, and $\hat{\sigma}^{rr}$ is compressive and tensile in the inclusion and the matrix, respectively. A wedge containing an inclusion with a negative radial and positive circumferential eigenstrains is shown in Figure 3.6. Notice that unlike other cases for which the deformation was purely inward or purely outward, in this example, the deformation is no longer unidirectional. In this example, the central region of the inclusion moves outward, while the region close to the inclusion-matrix interface moves inward. Moreover, this trend continues even for the large negative values of the radial eigenstrain. Although the circumferential eigenstrain is positive, the inclusion shrinks in the circumferential direction, while the matrix expands in this direction such that the total angle of the wedge is increased. Figure 3.7 shows an inclusion with a purely radial eigenstrain. Note that although the eigenstrian is purely radial, the wedge is deformed considerably in the circumferential direction, with the inclusion expanding and the matrix shrinking circumferentially such that the total angle of the wedge is reduced.

[6]Note that the undeformed (reference) configuration shown in the following figures has a metric different from that of the deformed configuration, and hence, the area of the body seen in the figures is not representative of the actual volume of the body in the (non-flat) reference configuration. In particular, the material manifold is equipped with the non-trivial Riemannian metric (6.26), giving a volume for the body in the reference configuration different from that given by the flat Euclidean metric.

3.2.2 A Mooney-Rivlin inhomogeneity with uniform eigenstrains in a neo-Hookean wedge with clamped lateral boundaries

In this example, we consider an inhomogeneity made of a Mooney-Rivlin material in a neo-Hookean wedge with fixed (clamped) lateral boundaries such that they cannot move in the radial and circumferential directions. The energy function has the following Θ-dependence in the wedge

$$W(I_1, I_2, \Theta) = \begin{cases} \frac{\mu_1}{2}(I_1 - 3) + \frac{\mu_2}{2}(I_2 - 3)\,, & |\Theta| \leq \alpha_o\,, \\ \frac{\mu_o}{2}(I_1 - 3)\,, & |\Theta| > \alpha_o\,. \end{cases} \tag{3.58}$$

Moreover, we consider the eigenstrain distribution in the wedge given by (3.35). Looking at (3.33) and (3.34) one observes that equations (3.37) and (3.38) of Section 3.2.1 hold for this example as well if μ in (3.38) is replaced by $\mu_1 + \mu_2$ and μ_o in the inhomogeneity and the matrix, respectively. Therefore

$$p(R, \Theta) = \begin{cases} f_o \ln(R\zeta(\Theta)) + p_i\,, & |\Theta| \leq \alpha_o\,, \\ f_o \ln(R\zeta(\Theta)) + p_o\,, & |\Theta| > \alpha_o\,, \end{cases} \tag{3.59}$$

and

$$\zeta'(\Theta)^2 = \begin{cases} c_{i_1} + \frac{2f_o e^{2\omega_2}}{\mu_1+\mu_2} \ln\zeta(\Theta) - \zeta^2(\Theta)e^{2(\omega_2-\omega_1)} - \frac{e^{2(\omega_1+\omega_2)}}{\zeta^2(\Theta)}\,, & |\Theta| \leq \alpha_o\,, \\ c_{o_1} + \frac{2f_o}{\mu_o} \ln\zeta(\Theta) - \zeta^2(\Theta) - \frac{1}{\zeta^2(\Theta)}\,, & |\Theta| > \alpha_o\,. \end{cases} \tag{3.60}$$

Boundary conditions. The continuity of the traction vector at the inhomogeneity-matrix interface implies that

$$\frac{\mu_1 + \mu_2}{\mu_o} e^{\omega_1-\omega_2} \zeta'(\Theta)|_{\Theta=\alpha_o^-} = \zeta'(\Theta)|_{\Theta=\alpha_o^+}, \tag{3.61}$$

and

$$\frac{\mu_1 + \mu_2}{\mu_o} e^{2\omega_1} - 1 = \frac{p_i - p_o - \mu_2}{\mu_o} \zeta(\pm\alpha_o)^2. \tag{3.62}$$

We assume that the lateral boundaries of the wedge are clamped, i.e.[7]

$$\zeta(\Theta_o) = 1, \quad h(\Theta_o) = \Theta_o\,. \tag{3.63}$$

In order to determine the pressure constants p_i and p_o, we assume that the resultant force acting on the circular boundary of the wedge vanishes. Using (3.46), the radial force per unit length of the cylinder in the Z-direction is simplified to read

$$\begin{aligned} F_r = 2\mu_o R_o \Bigg[& e^{\omega_1+\omega_2} \int_0^{\alpha_o} \left\{ e^{-2\omega_1} \left(\frac{\mu_1}{\mu_o} + \frac{\mu_2}{\mu_o} \right) \zeta(\Theta) - \frac{f_o}{\mu_o} \frac{\ln\left(R_o \zeta(\Theta)\right)}{\zeta(\Theta)} - \left(\frac{p_i}{\mu_o} - \frac{\mu_2}{\mu_o} \right) \frac{1}{\zeta(\Theta)} \right\} d\Theta \\ & + \int_{\alpha_o}^{\Theta_o} \left\{ \zeta(\Theta) - \frac{f_o}{\mu_o} \frac{\ln\left(R_o \zeta(\Theta)\right)}{\zeta(\Theta)} - \frac{p_o}{\mu_o \zeta(\Theta)} \right\} d\Theta \Bigg]\,. \end{aligned} \tag{3.64}$$

We proceed to numerically solve the boundary-value problem (3.60) along with the above boundary conditions and the constraint of a zero-boundary resultant. The physical components of the Cauchy stress read

$$\hat{\sigma}^{rr} = \begin{cases} -f_o \ln(R\zeta(\Theta)) - p_i + (\mu_1 + \mu_2) \left(e^{-2\omega_2} \zeta'(\Theta)^2 + e^{-2\omega_1} \zeta(\Theta)^2 \right) + \mu_2\,, & |\Theta| \leq \alpha_o\,, \\ -f_o \ln(R\zeta(\Theta)) - p_o + \mu_o \left(\zeta'(\Theta)^2 + \zeta(\Theta)^2 \right)\,, & |\Theta| > \alpha_o\,, \end{cases} \tag{3.65}$$

$$\hat{\sigma}^{r\theta} = \begin{cases} \frac{(\mu_1+\mu_2)\zeta'(\Theta)}{\zeta(\Theta)} e^{\omega_1-\omega_2}\,, & |\Theta| \leq \alpha_o\,, \\ \frac{\mu_o \zeta'(\Theta)}{\zeta(\Theta)}\,, & |\Theta| > \alpha_o\,, \end{cases} \tag{3.66}$$

$$\hat{\sigma}^{\theta\theta} = \begin{cases} \frac{1}{\zeta(\Theta)^2} \left((\mu_1 + \mu_2)\, e^{2\omega_1} - \zeta(\Theta)^2 \left(f_o \ln(R\zeta(\Theta)) + p_i - \mu_2 \right) \right)\,, & |\Theta| \leq \alpha_o\,, \\ \frac{1}{\zeta(\Theta)^2} \left(\mu_o - \zeta(\Theta)^2 \left(f_o \ln(R\zeta(\Theta)) + p_o \right) \right)\,, & |\Theta| > \alpha_o\,, \end{cases} \tag{3.67}$$

[7] As in Section 3.2.1, the functions $\zeta(\Theta)$ and $h(\Theta)$ are even and odd, respectively, and hence, $\zeta'(0) = 0$ and $h(0) = 0$.

$$\hat{\sigma}^{zz} = \begin{cases} -f_o \ln(R\zeta(\Theta)) - p_i + \mu_2 \left(e^{-2\omega_2} \zeta'(\Theta)^2 + e^{-2\omega_1} \zeta(\Theta)^2 + \frac{e^{2\omega_1}}{\zeta(\Theta)^2} \right) + \mu_1 \,, & |\Theta| \leq \alpha_o \,, \\ -f_o \ln(R\zeta(\Theta)) - p_o + \mu_o \,, & |\Theta| > \alpha_o \,, \end{cases} \tag{3.68}$$

Remark 3.2.6. Note that $\hat{\sigma}^{r\theta}$ depends only on Θ. Moreover, the radial dependence of $\hat{\sigma}^{rr}$, $\hat{\sigma}^{\theta\theta}$, and $\hat{\sigma}^{zz}$ is linear with respect to $\ln R$. We use this property and plot the stress components at $R = R_o$.

Numerical results. The deformation of the wedge and the variation of the stress components for various eigenstrain distributions in the inhomogeneity with different elastic constants are examined and are presented in Figures 3.8, 3.10, 3.11, 3.12, and 3.9. A wedge containing an inhomogeneity stiffer than the matrix with positive eigenstrains such that the circumferential eigenstrain is twice the radial one is shown in Figure 3.8. As expected all the radial planes of the wedge displace outward, with the inhomogeneity expanding more than the matrix. Furthermore, on the circular boundary $\hat{\sigma}^{rr}$ is negative in the inhomogeneity, positive in the matrix, and discontinuous at the inhomogeneity-matrix interface. Note that $\hat{\sigma}^{\theta\theta}$ is compressive almost everywhere on the circular boundary except for some small regions close to the lateral boundaries.

Figure 3.9 depicts an inhomogeneity placed in a stiffer matrix with anisotropic eigenstrains such that the radial eigenstrain is twice the circumferential one. It is observed that $\hat{\sigma}^{\theta\theta}$ is tensile on the circular boundary. Moreover, $\hat{\sigma}^{\theta\theta}$ and $\hat{\sigma}^{rr}$ are almost uniform in the inhomogeneity. For a wedge having an inhomogeneity stiffer than the matrix with negative circumferentially dominated eigenstrains, all the radial planes are contracted. In addition, $\hat{\sigma}^{rr}$ and $\hat{\sigma}^{\theta\theta}$ are almost uniform, and $\hat{\sigma}^{r\theta}$ is almost zero in the inhomogeneity (Figure 3.10).

Inhomogeneities with purely radial and purely circumferential eigenstrains are shown in Figures 3.11 and 3.12, respectively. For both cases, all the radial planes are elongated, with the inhomogeneity expanded and the matrix shrunk in the circumferential direction. Interestingly, the circumferential deformation is more pronounced in the purely radial eigenstrain case while radial deformation

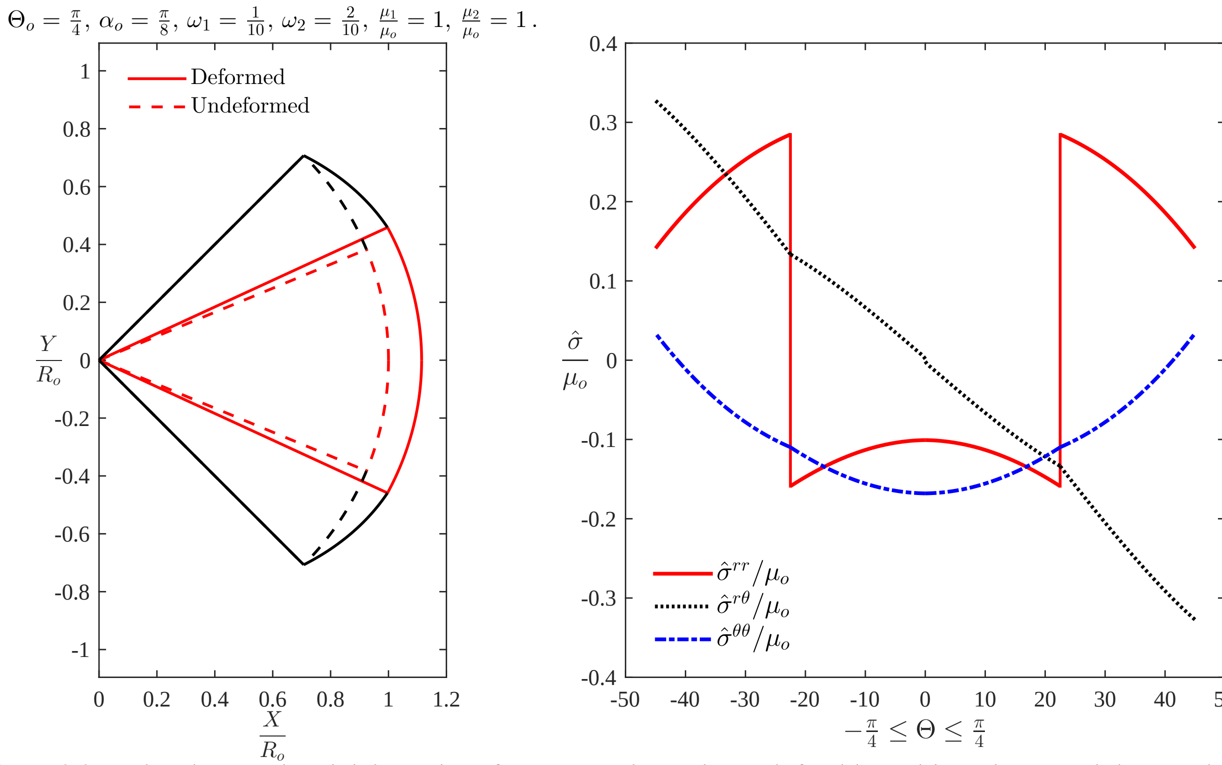


Figure 3.8: *Left: The initial and deformed configurations of a wedge with fixed lateral boundaries and the initial half-angle* $\Theta_o = \frac{\pi}{4}$ *having an inhomogenity with* $\alpha_o = \frac{\pi}{8}$, $\frac{\mu_1}{\mu_o} = \frac{\mu_2}{\mu_o} = 1$, *and the constant eigenstrain distribution* $\omega_1 = \frac{1}{10}$ *and* $\omega_2 = \frac{2}{10}$. ($\frac{f_o}{\mu_o} = -0.2761$, $\frac{p_i}{\mu_o} = 3.1646$). *Right: Variation of the physical components of the Cauchy stress tensor versus* Θ *at* $R = R_o$.

is more pronounced in the purely circumferential eigenstrain case.[8] Unlike wedges with traction-free lateral boundaries for which a purely circumferential eigenstrain does not induce any residual stresses in the wedge, here residual stress is developed due to a purely circumferential eigenstrain because the wedge can no longer move freely in the circumferential direction. Note that $\hat{\sigma}^{rr}$ and $\hat{\sigma}^{\theta\theta}$ are almost uniform in the inhomogeneity for the purely radial eigenstrain case, with $\hat{\sigma}^{rr}$ undergoing a jump at the inhomogeneity-matrix interface. For the purely circumferential eigenstrain case, however, the stress components exhibit a quite different behavior in the inhomogeneity. For instance, $\hat{\sigma}^{rr}$ remains continuous at the inhomogeneity-matrix interface and does not tend to be uniform in the inhomogeneity.

[8]A similar observation was made for the wedge with traction-free lateral boundaries having an inclusion with a purely radial eigenstrain (cf. Figure 3.7).

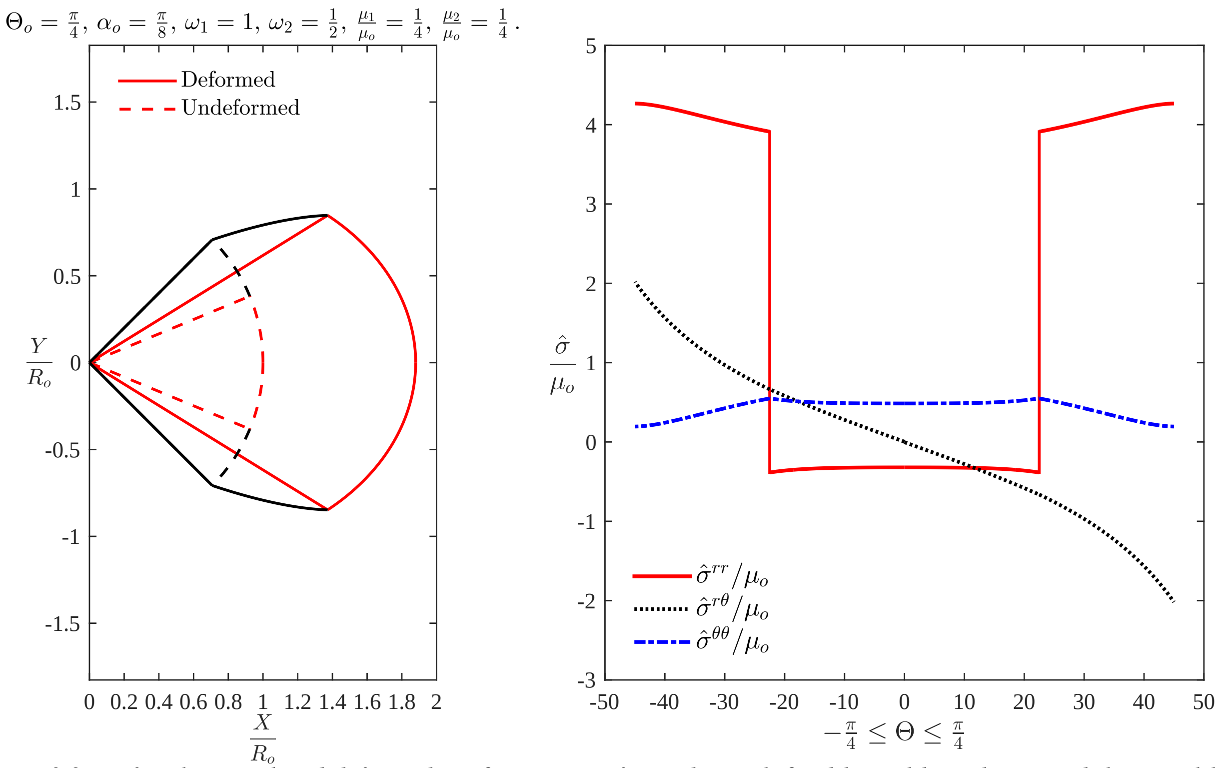


Figure 3.9: *Left: The initial and deformed configurations of a wedge with fixed lateral boundaries and the initial half-angle* $\Theta_o = \frac{\pi}{4}$ *having an inhomogenity with* $\alpha_o = \frac{\pi}{8}$, $\frac{\mu_1}{\mu_o} = \frac{\mu_2}{\mu_o} = \frac{1}{4}$, *and the constant eigenstrain distribution* $\omega_1 = 1$ *and* $\omega_2 = \frac{1}{2}$. ($\frac{f_o}{\mu_o} = -2.0293$, $\frac{p_i}{\mu_o} = 2.0909$). *Right: Variation of the physical components of the Cauchy stress tensor versus* Θ *at* $R = R_o$.

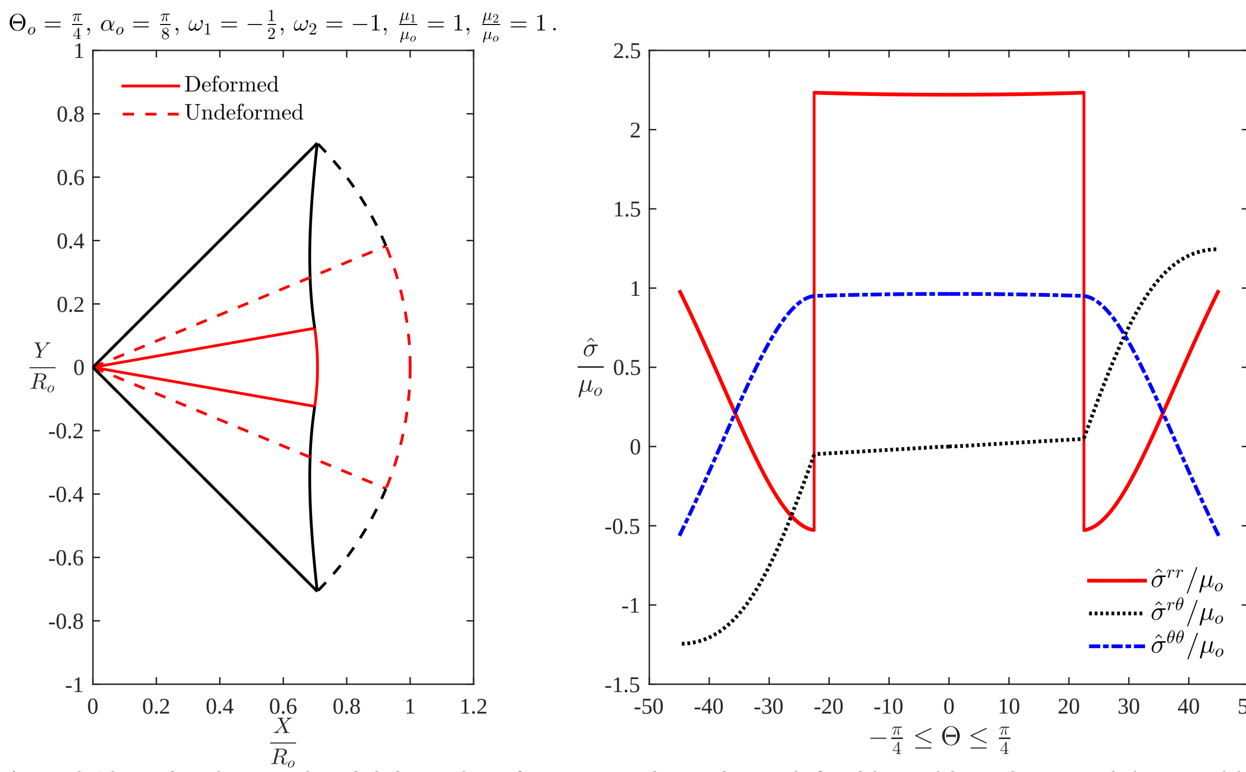


Figure 3.10: *Left: The initial and deformed configurations of a wedge with fixed lateral boundaries and the initial half-angle* $\Theta_o = \frac{\pi}{4}$ *having an inhomogenity with* $\alpha_o = \frac{\pi}{8}$, $\frac{\mu_1}{\mu_o} = \frac{\mu_2}{\mu_o} = 1$, *and the constant eigenstrain distribution* $\omega_1 = -\frac{1}{2}$ *and* $\omega_2 = -1$. ($\frac{f_o}{\mu_o} = 1.5500$, $\frac{p_i}{\mu_o} = 2.0399$). *Right: Variation of the physical components of the Cauchy stress tensor versus* Θ *at* $R = R_o$.

$\Theta_o = \frac{\pi}{4}$, $\alpha_o = \frac{\pi}{8}$, $\omega_1 = 1$, $\omega_2 = 0$, $\frac{\mu_1}{\mu_o} = \frac{1}{2}$, $\frac{\mu_2}{\mu_o} = \frac{1}{2}$.

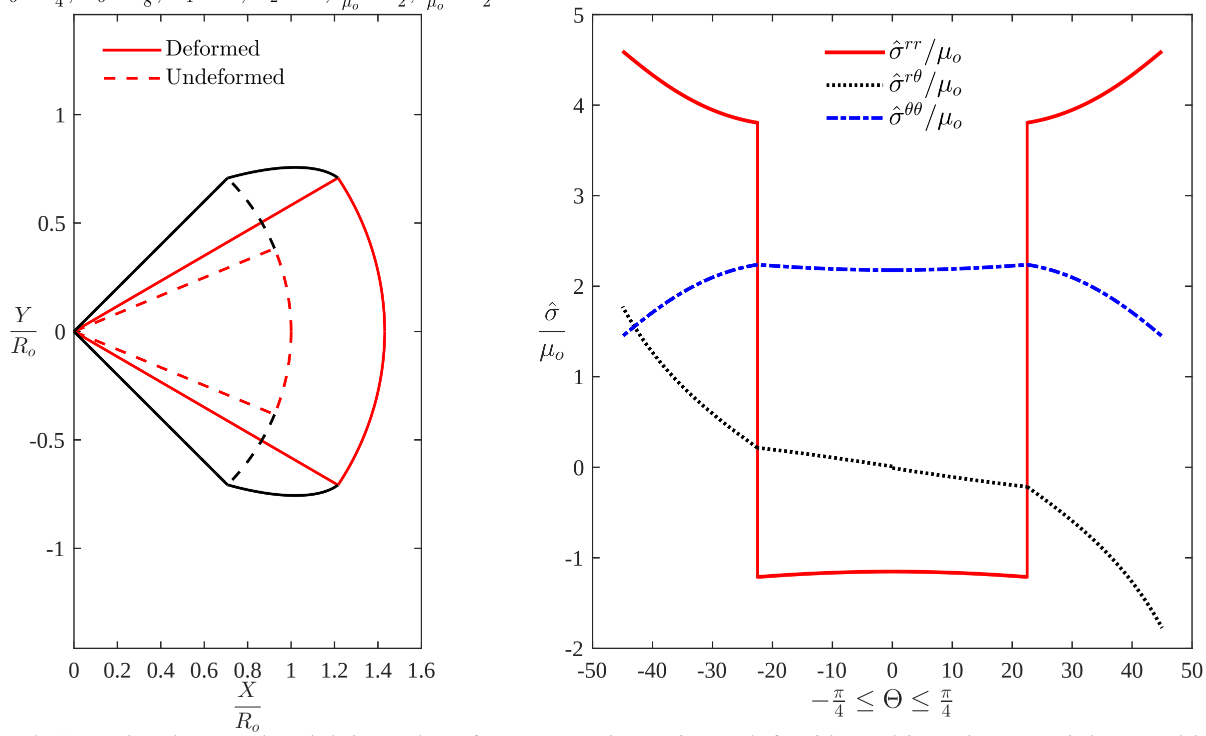


Figure 3.11: *Left: The initial and deformed configurations of a wedge with fixed lateral boundaries and the initial half-angle* $\Theta_o = \frac{\pi}{4}$ *having an inhomogenity with* $\alpha_o = \frac{\pi}{8}$, $\frac{\mu_1}{\mu_o} = \frac{\mu_2}{\mu_o} = \frac{1}{2}$, *and the constant eigenstrain distribution* $\omega_1 = 1$ *and* $\omega_2 = 0$. ($\frac{f_o}{\mu_o} = -3.7629$, $\frac{p_i}{\mu_o} = 3.2785$). *Right: Variation of the physical components of the Cauchy stress tensor versus* Θ *at* $R = R_o$.

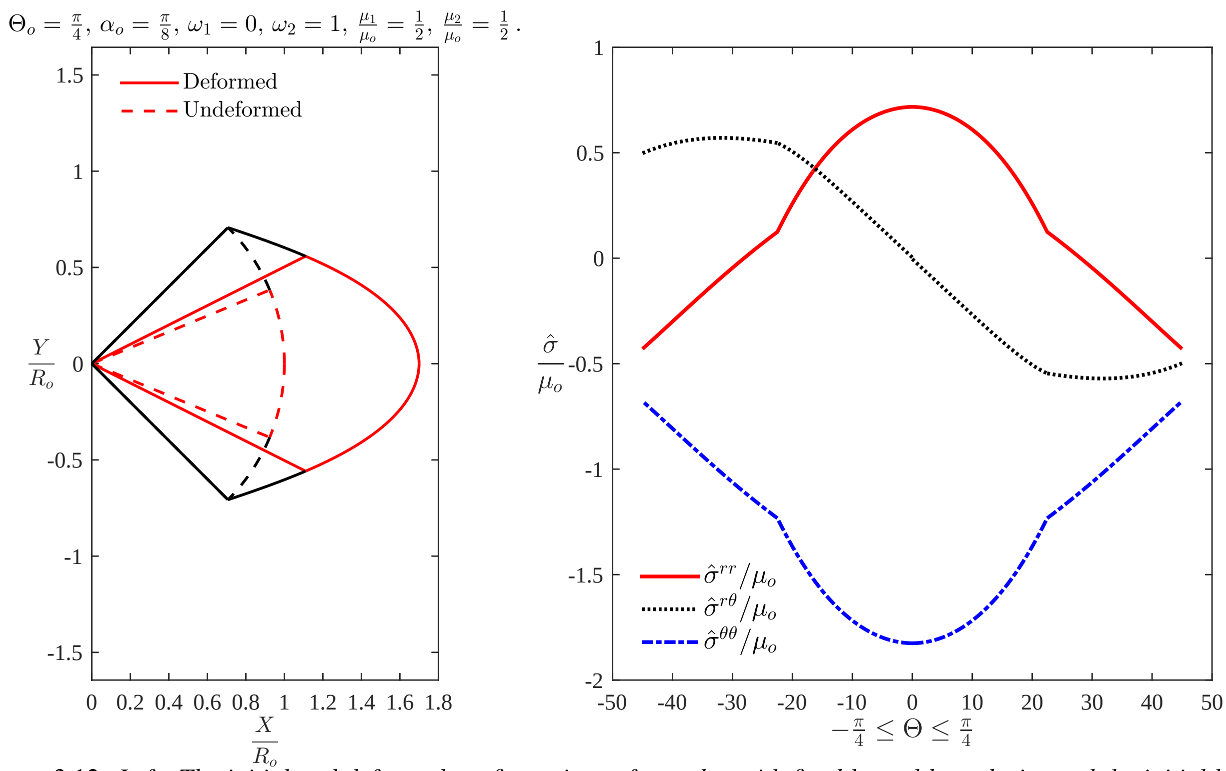


Figure 3.12: *Left: The initial and deformed configurations of a wedge with fixed lateral boundaries and the initial half-angle* $\Theta_o = \frac{\pi}{4}$ *having an inhomogenity with* $\alpha_o = \frac{\pi}{8}$, $\frac{\mu_1}{\mu_o} = \frac{\mu_2}{\mu_o} = \frac{1}{2}$, *and the constant eigenstrain distribution* $\omega_1 = 0$ *and* $\omega_2 = 1$. ($\frac{f_o}{\mu_o} = 0.9305$, $\frac{p_i}{\mu_o} = 2.1780$). *Right: Variation of the physical components of the Cauchy stress tensor versus* Θ *at* $R = R_o$.

3.2.3 Stress-free eigenstrain distributions in a wedge

In this section, we find those eigenstrain distributions that induce no residual stresses. For such eigenstrain distributions, the material manifold can be isometrically embedded into the ambient space, i.e, $\boldsymbol{G} = \varphi^*\boldsymbol{g}$.[9] Hence, for a simply-connected body, a stress-free eigenstrain distribution corresponds to a material metric with vanishing Riemannian curvature. Note that the wedge is a simply-connected body. Curvature tensor has the following components

$$R^A{}_{BCD} = \frac{\partial \Gamma^A{}_{DB}}{\partial X^C} - \frac{\partial \Gamma^A{}_{CB}}{\partial X^D} + \Gamma^A{}_{CE}\Gamma^E{}_{DB} - \Gamma^A{}_{DE}\Gamma^E{}_{CB}\,, \tag{3.69}$$

where the Christoffel symbols are defined as

$$\Gamma^A{}_{BC} = \frac{1}{2} G^{AK} \left(\frac{\partial G_{KB}}{\partial X^C} + \frac{\partial G_{KC}}{\partial X^B} - \frac{\partial G_{BC}}{\partial X^K} \right) . \tag{3.70}$$

[9]Equivalently, the Lagrangian strain tensor must vanish.

For the material metric of the wedge, the Christoffel symbol matrices read

$$\boldsymbol{\Gamma}^R = \left[\Gamma^R{}_{AB}\right] = \begin{pmatrix} 0 & \omega_R'(\Theta) & 0 \\ \omega_R'(\Theta) & -Re^{2(\omega_\Theta(\Theta)-\omega_R(\Theta))} & 0 \\ 0 & 0 & -Re^{2(\omega_\Theta(\Theta)-\omega_R(\Theta))}\sin^2\Theta \end{pmatrix},$$

$$\boldsymbol{\Gamma}^\Theta = \left[\Gamma^\Theta{}_{AB}\right] = \begin{pmatrix} -\frac{1}{R^2}e^{2(\omega_R(\Theta)-\omega_\Theta(\Theta))}\omega_R'(\Theta) & \frac{1}{R} & 0 \\ \frac{1}{R} & \omega_\Theta'(\Theta) & 0 \\ 0 & 0 & -\sin\Theta\left[\cos\Theta+\sin\Theta\omega_\Theta'(\Theta)\right] \end{pmatrix},$$

$$\boldsymbol{\Gamma}^\Phi = \left[\Gamma^\Phi{}_{AB}\right] = \begin{pmatrix} 0 & 0 & \frac{1}{R} \\ 0 & 0 & \cot\Theta+\omega_\Theta'(\Theta) \\ \frac{1}{R} & \cot\Theta+\omega_\Theta'(\Theta) & 0 \end{pmatrix}. \tag{3.71}$$

The non-trivially non-zero components of the curvature tensor for the cylinder with the metric $\boldsymbol{G}$ are

$$R^\Theta_{\Theta RR} = -\,R^\Theta_{R\Theta R} = \frac{e^{2(\omega_R(\Theta)-\omega_\Theta(\Theta))}}{R^2}\left[-\omega_\Theta'(\Theta)\omega_R'(\Theta)+\omega_R'(\Theta)^2+\omega_R''(\Theta)\right], \tag{3.72}$$

$$R^R_{R\Theta\Theta} = -\,R^R_{\Theta R\Theta} = -\omega_\Theta'(\Theta)\omega_R'(\Theta)+\omega_R'(\Theta)^2+\omega_R''(\Theta). \tag{3.73}$$

Therefore, in order for an eigenstrain distribution to be stress-free in a wedge, it needs to satisfy the following non-linear ordinary differential equation:

$$-\omega_\Theta'(\Theta)\omega_R'(\Theta)+\omega_R'(\Theta)^2+\omega_R''(\Theta)=0. \tag{3.74}$$

Using this ODE, given $\omega_\Theta(\Theta)$, $\omega_R(\Theta)$ is expressed as

$$\omega_R(\Theta) = c_2 + \ln\left(c_1+\int_0^\Theta e^{\omega_\Theta(\phi)}d\phi\right). \tag{3.75}$$

Remark 3.2.7. In the special case of $\omega = \omega_R(\Theta) = \omega_\Theta(\Theta)$, we have a linear solution ($\omega(\Theta) = c_1\Theta + c_2$) for the stress-free eigenstrain distribution, where c_1 and c_2 are constants.

CHAPTER 4
A NONLINEAR ELASTIC SOLID TORUS WITH A TOROIDAL INCLUSION

4.1 Introduction

In a seminal paper, Eshelby [4] showed that for an ellipsoidal inclusion in an infinite linear elastic solid, for uniform eigenstrains the stress inside the inclusion is uniform as well. There have been many investigations in recent years on the validity of this uniformity property for nonlinear elastic solids and inclusions with finite eigenstrains. There are several results in 2D in the case of harmonic solids [28, 29, 30, 31, 32]. In 3D, recently Yavari and Goriely [24] showed that in the case of cylindrical bars (finite or infinitely-long) and spherical balls with cylindrical and spherical inclusions, respectively, with pure dilatational finite eigenstrains, the stress uniformity property holds for both incompressible isotropic solids and some special classes of compressible isotropic solids. These geometries are simply-connected. Perhaps the simplest example of a non-simply connected body is a hollow cylinder. However, in that case one can only have an annular inclusion. Another simple example of a non-simply connected body is a solid torus. To our best knowledge, finite (or infinitesimal) eigenstrains in a solid torus and their induced residual stresses have not been studied in the literature. In this chapter we investigate this problem in the case of incompressible solids (see Fig. 4.1).

Kydoniefs and Spencer [68] and Kydoniefs [69] studied the finite deformation of a torus made of a homogeneous, isotropic, incompressible elastic solid under inflation by uniform internal pressure and under inflation and rotation with a constant angular velocity, respectively. They assumed that the torus in its deformed state is generated by rotating two concentric circles about a line in their plane. Assuming that the radii of the generating circles are small compared to the overall radius of the torus, they obtained approximate solutions for the stress and deformation fields in the torus. Their work was further extended to a solid torus inflated from a torus in its undeformed state by Hill [70]. He too assumed that the ratio of the radius of the generating circles and the overall radius of the torus (thinness ratio) is small and obtained the solutions to the first order of this small ratio. Under the same assumption, Kydoniefs and Spencer [71] explored the finite inflation of an elastic toroidal membrane due to uniform internal pressure such that it has a circular cross section in its reference configuration. They obtained the solutions to the second order in the thinness ratio and presented some numerical results for a toroidal membrane made of a Mooney-Rivlin material, describing the dependence of the deformation and the generated stresses on the internal pressure. Krokhmal [72] studied the displacement boundary-value problem of a linear elastic torus. He reduced the boundary-value problem to an infinite system of linear algebraic equations and developed an analytical technique to solve it.

Toroidal inclusions and inhomogeneities have been observed in the microstructures of both natural and engineered materials [73]. Onaka *et al.* [74] investigated the problem of elastic toroidal inclusions in an infinite linear elastic medium using averaged Eshelby tensor. They found that the averaged Eshelby tensor of toroidal inclusions on an arbitrary plane is nearly the same as the average of the Eshelby tensors of randomly oriented rod-like inclusions on that plane. Onaka [75] considered an infinitely extended material having a doughnut-like inclusion with purely dilatational eigenstrains. They observed that near the inclusion there are two points at which all the components of the strain tensor vanish. Note that this is not the case for spherical inclusions with purely dilatational eigenstrains placed in an infinite linear elastic medium, for which strains become null only at infinitely far distances from the inclusion. In another paper by Onaka [76], the strain field generated by elongated toroidal inclusions were studied and compared with that of doughnut-like and spherical inclusions. It was observed that for an infinitely elongated tubular inclusion, all the strain tensor components in the matrix region surrounded by the inclusion vanish. The reinforcing effects of rigid toroidal inhomogeneities in a linear elastic medium was studied by Argatov and Sevostianov [77]. They observed that there is no noticeable difference in the reinforcing properties of toroidal and spheroidal inhomogeneities with the same volume and diameter. Kirilyuk [78] investigated the effects of a toroidal inhomogeneity on the stress concentration in an infinite isotropic medium. They considered two cases: perfect bonding and slipping at the inhomogeneity-matrix interface. As an example, they showed that

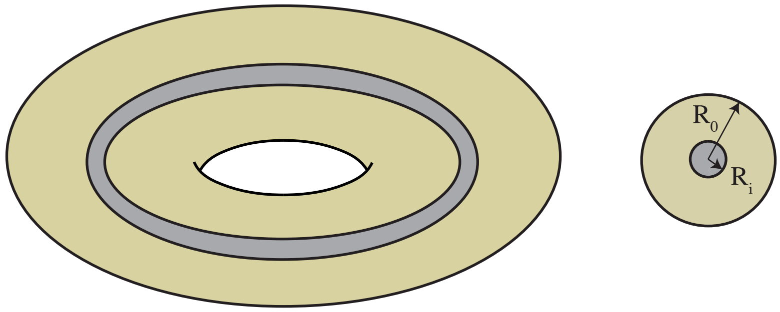


Figure 4.1: *A solid torus with a toroidal inclusion that is concentric with it.*

the difference in the maximum stress could differ up to 40% for the two cases.

In the setting of linearized elasticity, it is known that for a single inclusion with uniform eigenstrain in an infinite domain to have a uniform stress field the inclusion must be an ellipsoid [79, 80]. In particular, a toroidal inclusion with uniform eigenstrain in an infinite solid would have a non-uniform stress field. One may now consider a solid torus with an inclusion whose generating circle is concentric with the boundary circle of the solid torus (see Fig. 4.1). Is the stress field inside such an inclusion with uniform and pure dilatational eigenstrain uniform? We solve this problem for finite dilatational eigenstrains in the case of a "thin" solid torus made of an incompressible isotropic nonlinear elastic solid. We will show that to the first order in the thinness ratio, stress inside the inclusion is not uniform. We then study the same problem for a solid torus made of an incompressible linear elastic solid with a toroidal inclusion with a uniform infinitesimally small pure dilatational eigenstrain. We show that for any size of the solid torus (not necessarily thin) the stress inside the inclusion cannot be uniform.

This chapter is organized as follows. In §4.2 we formulate the governing equilibrium equations of a solid torus with an axially-symmetric distribution of finite eigenstrains. In §4.2.1 we consider a toroidal inclusion that is concentric with the solid torus and calculate the residual stress field using a perturbation analysis. We then present some numerical examples for neo-Hookean solids. Finally, we solve the corresponding problem in linear elasticity in §4.2.2.

In this chapter we model finite eigenstrains in a nonlinear elastic solid by defining a Riemannian material manifold, which has a metric that explicitly depends on the distribution of eigenstrains. This

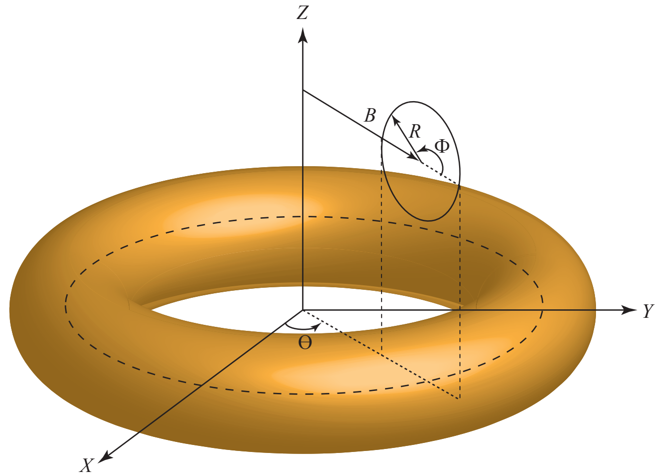


Figure 4.2: *A solid torus and its toroidal coordinates in the undeformed configuration.*

idea has been discussed in detail in our previous works [24, 65, 34, 25]. Note that the results of this section have been previously reported in our published work [26].

4.2 An incompressible isotropic solid torus with axially-symmetric finite eigenstrains

In this section we consider a solid torus generated by rotating a circle with radius R_o about a line in its plane such that the distance from the origin to the center of the circle is B. Let (R, Θ, Φ) and (r, θ, ϕ) be the material and spatial toroidal coordinates as illustrated in Figure 4.2. In the toroidal coordinates (R, Θ, Φ), the metric of the eigenstrain-free torus is written as

$$\mathbf{G}_o = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (B + R\cos\Phi)^2 & 0 \\ 0 & 0 & R^2 \end{pmatrix} . \tag{4.1}$$

We assume an axially-symmetric (Θ-independent) eigenstrain (pre-strain) distribution in the torus. Following the construction suggested by Yavari and Goriely [24] to model eigenstrains, we consider the following material metric[1]

$$\mathbf{G} = e^{\Omega(R,\Phi)}\mathbf{G}_o \,, \tag{4.2}$$

[1]Similar constructions have been discussed in [9, 20, 44, 63, 64, 65, 10] to address problems in growth mechanics, thermoelasticity, and the nonlinear mechanics of distributed defects.

where $\Omega(R,\Phi)$ is an arbitrary function describing the inhomogeneous dilatational eigenstrain distribution in the torus. The ambient space is endowed with the Euclidean metric, which in the toroidal coordinates (r,θ,ϕ) has the following representation

$$\mathbf{g} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (b+r\cos\phi)^2 & 0 \\ 0 & 0 & r^2 \end{pmatrix} . \tag{4.3}$$

Let us consider an axially-symmetric class of deformations of the form

$$r = r(R,\Phi)\,, \quad \theta = \Theta\,, \quad \phi = \phi(R,\Phi)\,. \tag{4.4}$$

The deformation gradient for this class of deformations reads

$$\mathbf{F} = \begin{pmatrix} \frac{\partial r}{\partial R} & 0 & \frac{\partial r}{\partial \Phi} \\ 0 & 1 & 0 \\ \frac{\partial \phi}{\partial R} & 0 & \frac{\partial \phi}{\partial \Phi} \end{pmatrix} . \tag{4.5}$$

We assume an incompressible solid, i.e., $J = \sqrt{\frac{\det\mathbf{g}}{\det\mathbf{G}}}\det\mathbf{F} = 1$, which gives us

$$r_{,R}\phi_{,\Phi} - r_{,\Phi}\phi_{,R} = \frac{Re^{\frac{3}{2}\Omega(R,\Phi)}(B + R\cos\Phi)}{r(b+r\cos\phi)}. \tag{4.6}$$

The Finger deformation tensor reads

$$\mathbf{b}^\sharp = e^{-\Omega(R,\Phi)} \begin{pmatrix} {r_{,R}}^2 + \frac{{r_{,\Phi}}^2}{R^2} & 0 & r_{,R}\phi_{,R} + \frac{r_{,\Phi}\phi_{,\Phi}}{R^2} \\ 0 & \frac{1}{(B+R\cos\Phi)^2} & 0 \\ r_{,R}\phi_{,R} + \frac{r_{,\Phi}\phi_{,\Phi}}{R^2} & 0 & {\phi_{,R}}^2 + \frac{{\phi_{,\Phi}}^2}{R^2} \end{pmatrix} . \tag{4.7}$$

The first two principal invariants of $\mathbf{b}$ are ($I_3 = 1$)

$$I_1 = I + \beta^2 , \qquad I_2 = \beta^2 I + \frac{1}{\beta^2} , \tag{4.8}$$

where

$$I = e^{-\Omega(R,\Phi)} \left({r_{,R}}^2 + r^2 {\phi_{,R}}^2 + \frac{{r_{,\Phi}}^2 + r^2 {\phi_{,\Phi}}^2}{R^2} \right) , \qquad \beta = \frac{R e^{\Omega(R,\Phi)}}{r\left(r_{,R}\phi_{,\Phi} - r_{,\Phi}\phi_{,R}\right)} . \tag{4.9}$$

The inverse of Finger tensor $\mathbf{b}^{-1} = \mathbf{c}$ is written as

$$\mathbf{b}^{-1} = e^{\Omega(R,\Phi)} \begin{pmatrix} \frac{{\phi_{,\Phi}}^2 + R^2 {\phi_{,R}}^2}{\left(r_{,R}\phi_{,\Phi} - r_{,\Phi}\phi_{,R}\right)^2} & 0 & -\frac{R^2 r_{,R}\phi_{,R} + r_{,\Phi}\phi_{,\Phi}}{r^2\left(r_{,R}\phi_{,\Phi} - r_{,\Phi}\phi_{,R}\right)^2} \\ 0 & \frac{(B+R\cos\Phi)^2}{(b+r\cos\phi)^4} & 0 \\ -\frac{R^2 r_{,R}\phi_{,R} + r_{,\Phi}\phi_{,\Phi}}{r^2\left(r_{,R}\phi_{,\Phi} - r_{,\Phi}\phi_{,R}\right)^2} & 0 & \frac{R^2 {r_{,R}}^2 + {r_{,\Phi}}^2}{r^4\left(r_{,R}\phi_{,\Phi} - r_{,\Phi}\phi_{,R}\right)^2} \end{pmatrix} . \tag{4.10}$$

Following (2.9), the non-zero components of the Cauchy stress read

$$\sigma^{rr} = -p(R,\Phi) + 2\left(W_{I_1} + \beta^2 W_{I_2}\right) \left({r_{,R}}^2 + \frac{{r_{,\Phi}}^2}{R^2} \right) e^{-\Omega(R,\Phi)} + \frac{2W_{I_2}}{\beta^2} , \tag{4.11a}$$

$$\sigma^{r\phi} = 2e^{-\Omega(R,\Phi)}\left(W_{I_1} + \beta^2 W_{I_2}\right)\left(r_{,R}\phi_{,R} + \frac{r_{,\Phi}\phi_{,\Phi}}{R^2}\right), \tag{4.11b}$$

$$\sigma^{\theta\theta} = -\frac{p(R,\Phi)}{\left(b + r\cos\phi\right)^2} + \frac{2e^{-\Omega(R,\Phi)}\left(W_{I_1} + IW_{I_2}\right)}{\left(B + R\cos\Phi\right)^2}, \tag{4.11c}$$

$$\sigma^{\phi\phi} = -\frac{p(R,\Phi)}{r^2} + 2\left(W_{I_1} + \beta^2 W_{I_2}\right)\left({\phi_{,R}}^2 + \frac{{\phi_{,\Phi}}^2}{R^2}\right)e^{-\Omega(R,\Phi)} + \frac{2W_{I_2}}{\beta^2 r^2}. \tag{4.11d}$$

The physical components of the Cauchy stress are calculated using the relation $\hat{\sigma}^{ab} = \sigma^{ab}\sqrt{g_{aa}g_{bb}}$ [67]. Thus

$$\hat{\sigma}^{rr} = \sigma^{rr}, \quad \hat{\sigma}^{r\phi} = r\sigma^{r\phi}, \quad \hat{\sigma}^{\theta\theta} = \left(b + r\cos\phi\right)^2\sigma^{\theta\theta}, \quad \hat{\sigma}^{\phi\phi} = r^2\sigma^{\phi\phi}. \tag{4.12}$$

The non-zero components of the first Piola-Kirchhoff stress tensor, i.e., $P^{aA} = J(F^{-1})^A{}_b\sigma^{ab}$ are written as

$$P^{rR} = e^{-\Omega(R,\Phi)}\left[2r_{,R}\left(W_{I_1} + \beta^2 W_{I_2}\right) + \frac{r\beta\phi_{,\Phi}}{R}\left(\frac{2W_{I_2}}{\beta^2} - p\left(R,\Phi\right)\right)\right], \tag{4.13a}$$

$$P^{r\Phi} = \frac{e^{-\Omega(R,\Phi)}}{R^2}\left[2r_{,\Phi}\left(W_{I_1} + \beta^2 W_{I_2}\right) + rR\beta\phi_{,R}\left(p\left(R,\Phi\right) - \frac{2W_{I_2}}{\beta^2}\right)\right], \tag{4.13b}$$

$$P^{\phi R} = e^{-\Omega(R,\Phi)}\left[2\phi_{,R}\left(W_{I_1} + \beta^2 W_{I_2}\right) + \frac{\beta r_{,\Phi}}{rR}\left(p\left(R,\Phi\right) - \frac{2W_{I_2}}{\beta^2}\right)\right], \tag{4.13c}$$

$$P^{\theta\Theta} = \sigma^{\theta\theta}\,, \tag{4.13d}$$

$$P^{\phi\Phi} = \frac{e^{-\Omega(R,\Phi)}}{rR^2}\left[2r\phi_{,\Phi}\left(W_{I_1}+\beta^2 W_{I_2}\right) + R\beta r_{,R}\left(\frac{2W_{I_2}}{\beta^2} - p\left(R,\Phi\right)\right)\right]. \tag{4.13e}$$

The Christoffel symbol matrices of g read (cf. 2.8)

$$\boldsymbol{\gamma}^r = \left[\gamma^r{}_{ab}\right] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\left(b + r\cos\phi\right)\cos\phi & 0 \\ 0 & 0 & -r \end{pmatrix}, \ \boldsymbol{\gamma}^\theta = \left[\gamma^\theta{}_{ab}\right] = \begin{pmatrix} 0 & \frac{\cos\phi}{b+r\cos\phi} & 0 \\ \frac{\cos\phi}{b+r\cos\phi} & 0 & -\frac{r\sin\phi}{b+r\cos\phi} \\ 0 & -\frac{r\sin\phi}{b+r\cos\phi} & 0 \end{pmatrix},$$

$$\boldsymbol{\gamma}^\phi = \left[\gamma^\phi{}_{ab}\right] = \begin{pmatrix} 0 & 0 & \frac{1}{r} \\ 0 & \left(\frac{b}{r} + \cos\phi\right)\sin\phi & 0 \\ \frac{1}{r} & 0 & 0 \end{pmatrix}. \tag{4.14}$$

In the absence of body forces, the non-trivial equilibrium equations are $\sigma^{rb}{}_{|b} = 0$ and $\sigma^{\phi b}{}_{|b} = 0$, which after simplification read (the equilibrium equation in the θ-direction gives $p = p\left(R,\Phi\right)$)

$$\frac{\partial\sigma^{rr}}{\partial r} + \frac{\partial\sigma^{r\phi}}{\partial\phi} + \left(\frac{1}{r} + \frac{\cos\phi}{b + r\cos\phi}\right)\sigma^{rr} - \cos\phi\left(b + r\cos\phi\right)\sigma^{\theta\theta} - \frac{r\sin\phi}{b + r\cos\phi}\sigma^{r\phi} - r\sigma^{\phi\phi} = 0\,, \tag{4.15}$$

$$\frac{\partial\sigma^{r\phi}}{\partial r} + \frac{\partial\sigma^{\phi\phi}}{\partial\phi} + \left(\frac{3}{r} + \frac{\cos\phi}{b + r\cos\phi}\right)\sigma^{r\phi} + \frac{\sin\phi}{r}\left(b + r\cos\phi\right)\sigma^{\theta\theta} - \frac{r\sin\phi}{b + r\cos\phi}\sigma^{\phi\phi} = 0\,. \tag{4.16}$$

Note that

$$\frac{\partial}{\partial r}=\frac{\phi_{,\Phi}}{\phi_{,\Phi}r_{,R}-\phi_{,R}r_{,\Phi}}\frac{\partial}{\partial R}+\frac{\phi_{,R}}{\phi_{,R}r_{,\Phi}-\phi_{,\Phi}r_{,R}}\frac{\partial}{\partial \Phi}\,,\quad \frac{\partial}{\partial \phi}=\frac{r_{,\Phi}}{r_{,\Phi}\phi_{,R}-r_{,R}\phi_{,\Phi}}\frac{\partial}{\partial R}+\frac{r_{,R}}{r_{,R}\phi_{,\Phi}-r_{,\Phi}\phi_{,R}}\frac{\partial}{\partial \Phi}\,. \tag{4.17}$$

Boundary conditions. As we are interested in finding the residual stress field, we assume that the boundary of the torus is traction-free, i.e.

$$P^{rR}=0\,,\quad P^{\phi R}=0\,,\qquad R=R_o\,,\quad -\pi\leq\Phi\leq\pi\,. \tag{4.18}$$

Finding an exact solution of the PDEs (4.15) and (4.16) does not seem feasible. Therefore, we seek approximate solutions assuming that the radius of the cross section generating the torus is small compared to the radius of revolution [68, 69, 71, 81, 70]. Hence, we find the solution assuming that $\frac{R}{B}$ and $\frac{r}{b}$, which are of the same order, are sufficiently small so that the second and higher powers of $\frac{R}{B}$ (and $\frac{r}{b}$) can be neglected. In doing so, the problem is essentially a perturbation of the problem of finite eigenstrains in an infinitely-long circular cylindrical bar, which was discussed in [24] for the special case of cylindrically-symmetric distribution of eigenstrains.

4.2.1 A nonlinear solid torus with finite eigenstrains and $\frac{r}{b}\ll 1$ and $\frac{R}{B}\ll 1$

In this section, we restrict our attention to the radially-symmetric dilatational eigenstrain distributions, for which $\Omega=\Omega\left(R\right)$. Moreover, we assume that $\frac{r}{b}$ and $\frac{R}{B}$ are sufficiently small and find the solutions to the first order in the thinness ratio $\varepsilon=\frac{R_o}{B}\ll 1$. For the zero-order problem ($\varepsilon\to 0$), the torus becomes a cylinder with the cylindrically-symmetric distribution of purely dilatational eigenstrains, for which, in the cylindrical coordinates, $r=r\left(R\right)$, $\phi=\Phi$, $z=\frac{b}{B}Z$, and $p=p\left(R\right)$. Therefore, we consider the following asymptotic expansions[2]

$$
\begin{aligned}
&r = r^{(0)}(R) + r^{(1)}(R,\Phi) + \mathcal{O}(\varepsilon^2), \qquad && \phi = \Phi + \phi^{(1)}(R,\Phi) + \mathcal{O}(\varepsilon^2),\\
&p = p^{(0)}(R) + p^{(1)}(R,\Phi) + \mathcal{O}(\varepsilon^2), \qquad && \sigma^{ij} = \sigma^{ij}_{(0)}(R) + \sigma^{ij}_{(1)}(R,\Phi) + \mathcal{O}\left(\varepsilon^2\right).
\end{aligned}
\tag{4.19}
$$

Substituting (4.19) into (4.6) and equating the zero and the first order terms on both sides one gets

$$
r^{(0)}\frac{dr^{(0)}}{dR} = \frac{BR}{b}e^{\frac{3}{2}\Omega(R)}\,, \tag{4.20}
$$

$$
\frac{dr^{(0)}}{dR}\phi^{(1)}_{,\Phi} + r^{(1)}_{,R} = \frac{BR}{br^{(0)}}e^{\frac{3}{2}\Omega(R)}\left[\left(\frac{R}{B} - \frac{r^{(0)}}{b}\right)\cos\Phi - \frac{r^{(1)}}{r^{(0)}}\right]. \tag{4.21}
$$

Similarly

$$
I_1 = I_1^{(0)}(R) + I_1^{(1)}(R,\Phi) + \mathcal{O}\left(\varepsilon^2\right), \qquad I_2 = I_2^{(0)}(R) + I_2^{(1)}(R,\Phi) + \mathcal{O}\left(\varepsilon^2\right). \tag{4.22}
$$

Denoting $\alpha = \frac{dr^{(0)}}{dR}$, one has

$$
I_1^{(0)} = \left[\alpha^2 + \frac{{r^{(0)}}^2}{R^2}\right]e^{-\Omega} + \frac{R^2e^{2\Omega}}{{r^{(0)}}^2\alpha^2}\,, \qquad I_2^{(0)} = \left[\frac{1}{\alpha^2} + \frac{R^2}{{r^{(0)}}^2}\right]e^{\Omega} + \frac{{r^{(0)}}^2\alpha^2}{R^2e^{2\Omega}}\,, \tag{4.23}
$$

$$
I_1^{(1)} = 2\left[\frac{{r^{(0)}}^2}{R^2e^{\Omega}} - \frac{R^2e^{2\Omega}}{{r^{(0)}}^2\alpha^2}\right]\left(\phi^{(1)}_{,\Phi} + \frac{r^{(1)}}{r^{(0)}}\right) + 2\left[\frac{\alpha}{e^{\Omega}} - \frac{R^2e^{2\Omega}}{{r^{(0)}}^2\alpha^3}\right]r^{(1)}_{,R}\,, \tag{4.24}
$$

$$
I_2^{(1)} = 2\left[\frac{{r^{(0)}}^2\alpha^2}{R^2e^{2\Omega}} - \frac{R^2e^{\Omega}}{{r^{(0)}}^2}\right]\left(\phi^{(1)}_{,\Phi} + \frac{r^{(1)}}{r^{(0)}}\right) + 2\left[\frac{{r^{(0)}}^2\alpha}{R^2e^{2\Omega}} - \frac{e^{\Omega}}{\alpha^3}\right]r^{(1)}_{,R}\,. \tag{4.25}
$$

[2] Note that $\sigma^{r\phi}_{(0)} = 0$ and $\hat{\sigma}^{\theta\theta} = \hat{\sigma}^{\theta\theta}_{(0)} + \hat{\sigma}^{\theta\theta}_{(1)} + \mathcal{O}(\varepsilon^2)$.

Assuming that the material is piecewise homogeneous[3] one has

$$W_{I_1} = W_{I_1}^{(0)} + W_{I_1 I_1}^{(0)} I_1^{(1)} + W_{I_1 I_2}^{(0)} I_2^{(1)}, \qquad W_{I_2} = W_{I_2}^{(0)} + W_{I_2 I_2}^{(0)} I_2^{(1)} + W_{I_1 I_2}^{(0)} I_1^{(1)}, \tag{4.26}$$

where $W_{{I_1}^{\alpha_1}{I_2}^{\alpha_2}}^{(0)} := \frac{\partial^{\alpha_1+\alpha_2} W}{\partial^{\alpha_1} I_1 \partial^{\alpha_2} I_2}(I_1^{(0)}, I_2^{(0)})$, for $(\alpha_1, \alpha_2 \in \{0,1,2\})$. Expanding (4.11) one obtains the following expressions for the non-zero Cauchy stress components

$$\sigma_{(0)}^{rr} = -p^{(0)} + 2e^{-\Omega}\left[\alpha^2 W_{I_1}^{(0)} + \frac{R^2 e^{2\Omega}}{{r^{(0)}}^2} W_{I_2}^{(0)}\right] + \frac{2{r^{(0)}}^2\alpha^2}{R^2 e^{2\Omega}} W_{I_2}^{(0)}\,, \tag{4.27a}$$

$$\begin{aligned}\sigma_{(1)}^{rr} &= 4r_{,R}^{(1)} e^{-\Omega}\alpha\left[W_{I_1}^{(0)} + \frac{{r^{(0)}}^2}{R^2 e^{\Omega}} W_{I_2}^{(0)}\right] + 2\left[\frac{{r^{(0)}}^2\alpha^2}{R^2 e^{2\Omega}} + \frac{R^2 e^{\Omega}}{{r^{(0)}}^2}\right]\left(W_{{I_2}^2}^{(0)} I_2^{(1)} + W_{I_1 I_2}^{(0)} I_1^{(1)}\right)\\ &\quad + 4\left(\phi_{,\Phi}^{(1)} + \frac{r^{(1)}}{r^{(0)}}\right)\left[\frac{{r^{(0)}}^2\alpha^2}{R^2 e^{2\Omega}} - \frac{R^2 e^{\Omega}}{{r^{(0)}}^2}\right] W_{I_2}^{(0)} + 2\alpha^2 e^{-\Omega}\left(W_{{I_1}^2}^{(0)} I_1^{(1)} + W_{I_1 I_2}^{(0)} I_2^{(1)}\right) - p^{(1)},\end{aligned} \tag{4.27b}$$

$$\sigma_{(1)}^{r\phi} = 2e^{-\Omega}\left[W_{I_1}^{(0)} + \frac{R^2 e^{2\Omega}}{{r^{(0)}}^2\alpha^2} W_{I_2}^{(0)}\right]\left(\alpha\phi_{,R}^{(1)} + \frac{r_{,\Phi}^{(1)}}{R^2}\right), \tag{4.27c}$$

$$\hat{\sigma}_{(0)}^{\theta\theta} = \frac{2b^2 e^{-\Omega}}{B^2}\left[W_{I_1}^{(0)} + e^{-\Omega}\left(\alpha^2 + \frac{{r^{(0)}}^2}{R^2}\right) W_{I_2}^{(0)}\right] - p^{(0)}, \tag{4.27d}$$

[3]For the sake of simplicity of calculations, here we do not consider the dependence of W on X, which would be needed in the case of an inhomogeneity. Instead, we model inhomogeneities by assuming different energy functions at different regions of the body.

$$\hat{\sigma}^{\theta\theta}_{(1)} = \frac{2b^2e^{-2\Omega}}{B^2}\left(\alpha^2+\frac{{r^{(0)}}^2}{R^2}\right)\left[W^{(0)}_{{I_2}^2}I_2^{(1)} + W^{(0)}_{I_1I_2}I_1^{(1)}\right]+\frac{4b^2e^{-2\Omega}}{B^2}\left[\alpha r^{(1)}_{,R} + \frac{{r^{(0)}}^2}{R^2}\left(\phi^{(1)}_{,\Phi} + \frac{r^{(1)}}{r^{(0)}}\right)\right]W^{(0)}_{I_2} - p^{(1)}$$

$$-\frac{4b^2e^{-\Omega}\cos\Phi}{B^2}\left(\frac{R}{B} - \frac{r^{(0)}}{b}\right)\left[W^{(0)}_{I_1} + e^{-\Omega}\left(\alpha^2 + \frac{{r^{(0)}}^2}{R^2}\right)W^{(0)}_{I_2}\right]+\frac{2b^2e^{-\Omega}}{B^2}\left[W^{(0)}_{{I_1}^2}I_1^{(1)} + W^{(0)}_{I_1I_2}I_2^{(1)}\right], \tag{4.27e}$$

$$\sigma^{\phi\phi}_{(0)} = -\frac{p^{(0)}}{{r^{(0)}}^2} + \frac{2e^{-\Omega}}{R^2}\left[W^{(0)}_{I_1} + \frac{R^2e^{2\Omega}}{{r^{(0)}}^2\alpha^2}W^{(0)}_{I_2}\right] + \frac{2\alpha^2}{R^2e^{2\Omega}}W^{(0)}_{I_2}, \tag{4.27f}$$

$$\sigma^{\phi\phi}_{(1)} = \frac{2e^{-\Omega}}{R^2}\left[W^{(0)}_{{I_1}^2}I_1^{(1)} + W^{(0)}_{I_1I_2}I_2^{(1)}\right]+2\left(\frac{e^{\Omega}}{{r^{(0)}}^2\alpha^2} + \frac{\alpha^2}{R^2e^{2\Omega}}\right)\left[W^{(0)}_{{I_2}^2}I_2^{(1)} + W^{(0)}_{I_1I_2}I_1^{(1)}\right]-\frac{4r^{(1)}e^{\Omega}}{{r^{(0)}}^3\alpha^2}W^{(0)}_{I_2}$$

$$-\frac{1}{{r^{(0)}}^2}\left[p^{(1)} - \frac{2r^{(1)}}{r^{(0)}}p^{(0)}\right] + \frac{4r^{(1)}_{,R}}{\alpha e^{2\Omega}}\left[\frac{\alpha^2}{R^2} - \frac{e^{3\Omega}}{{r^{(0)}}^2\alpha^2}\right]W^{(0)}_{I_2} + \frac{4\phi^{(1)}_{,\Phi}}{R^2e^{\Omega}}\left[W^{(0)}_{I_1} + \frac{\alpha^2}{e^{\Omega}}W^{(0)}_{I_2}\right]. \tag{4.27g}$$

Using (4.17) and (4.19), the non-trivial zero and first-order equilibrium equations are derived by expanding (4.15) and (4.16) as follows

$$\frac{d\sigma^{rr}_{(0)}}{dr^{(0)}} + \frac{\sigma^{rr}_{(0)}}{r^{(0)}} - r^{(0)}\sigma^{\phi\phi}_{(0)} = 0\,, \tag{4.28}$$

$$\frac{\sigma^{rr}_{(1),R}}{\alpha} - \frac{r^{(1)}_{,R}}{\alpha^2}\frac{d\sigma^{rr}_{(0)}}{dR} + \sigma^{r\phi}_{(1),\Phi} + \frac{\sigma^{rr}_{(1)}}{r^{(0)}} - \frac{r^{(1)}}{{r^{(0)}}^2}\sigma^{rr}_{(0)} + \frac{\cos\Phi}{b}\left(\sigma^{rr}_{(0)} - \hat{\sigma}^{\theta\theta}_{(0)}\right) - r^{(0)}\sigma^{\phi\phi}_{(1)} - r^{(1)}\sigma^{\phi\phi}_{(0)} = 0\,, \tag{4.29}$$

$$\frac{\sigma^{r\phi}_{(1),R}}{\alpha} - \frac{r^{(1)}_{,\Phi}}{\alpha}\frac{d\sigma^{\phi\phi}_{(0)}}{dR} + \sigma^{\phi\phi}_{(1),\Phi} + \frac{3}{r^{(0)}}\sigma^{r\phi}_{(1)} + \frac{r^{(0)}\sin\Phi}{b}\left(\frac{\hat{\sigma}^{\theta\theta}_{(0)}}{{r^{(0)}}^2} - \sigma^{\phi\phi}_{(0)}\right) = 0\,. \tag{4.30}$$

Assuming that $r(0) = 0$, (4.20) has the following solution

$$r^{(0)}(R) = \left(\frac{2B}{b} \int_0^R \zeta e^{\frac{3}{2}\Omega(\zeta)} d\zeta \right)^{\frac{1}{2}} . \tag{4.31}$$

It follows from (4.28) that $\frac{dp^{(0)}(R)}{dR} = h(R)$, where

$$\begin{aligned} h(R) = -\frac{2Re^{\Omega(R)}}{k{r^{(0)}}^2} \Bigg[\left\{ \frac{k^2 R^2 e^{\frac{3}{2}\Omega(R)}}{{r^{(0)}}^2} + \frac{{r^{(0)}}^2}{R^2 e^{\frac{3}{2}\Omega(R)}} - 2k \right\} \left(k^2 e^{\Omega(R)} W_{I_1}^{(0)}(R) + W_{I_2}^{(0)}(R) \right) - k^3 R e^{\Omega(R)} {W_{I_1}^{(0)}}'(R) \\ -k\Omega'(R) R \left(2k^2 e^{\Omega(R)} W_{I_1}^{(0)}(R) + \left(1 + \frac{k^2 {r^{(0)}}^2}{R^2} \right) W_{I_2}^{(0)}(R) \right) - kR \left(1 + \frac{k^2 {r^{(0)}}^2}{R^2} \right) {W_{I_2}^{(0)}}'(R) \Bigg] , \end{aligned} \tag{4.32}$$

and $k = \frac{B}{b}$. Note that

$$\begin{aligned} \frac{dW_{I_1}^{(0)}(R)}{dR} &= \frac{dI_1^{(0)}}{dR} W_{I_1 I_1}^{(0)} + \frac{dI_2^{(0)}}{dR} W_{I_1 I_2}^{(0)} , \\ \frac{dW_{I_2}^{(0)}(R)}{dR} &= \frac{dI_2^{(0)}}{dR} W_{I_2 I_2}^{(0)} + \frac{dI_1^{(0)}}{dR} W_{I_1 I_2}^{(0)} . \end{aligned} \tag{4.33}$$

Example: A toroidal inclusion with uniform pure dilatational eigenstrains in a neo-Hookean solid torus. Let us consider the following distribution of eigenstrains

$$\Omega(R) = \begin{cases} \Omega_o , & 0 \leq R < R_i \\ 0 , & R_i < R \leq R_o \end{cases} . \tag{4.34}$$

We assume that the torus is made of an incompressible homogeneous neo-Hookean solid, i.e., $W = \frac{\mu}{2}(I_1 - 3)$, where μ is the shear modulus at the ground state. Therefore, it follows from (4.31) that

$$r^{(0)}(R) = k^{\frac{1}{2}} \begin{cases} e^{\frac{3\Omega_o}{4}} R , & 0 \leq R \leq R_i \\ (R^2 + \gamma_o R_i^2)^{\frac{1}{2}} , & R_i \leq R \leq R_o \end{cases} , \tag{4.35}$$

where $\gamma_o = e^{\frac{3\Omega_o}{2}} - 1$. Using (4.32), we find the zero-order pressure field as

$$p^{(0)}(R) = \begin{cases} \mu c_i\,, & 0 \leq R < R_i \\ \mu c_o - \frac{k\mu}{2}\left[\frac{\gamma_o}{\gamma_o + \frac{R^2}{R_i^2}} + \ln\left(\frac{\frac{R^2}{R_i^2}}{\gamma_o + \frac{R^2}{R_i^2}}\right)\right], & R_i < R \leq R_o \end{cases}, \tag{4.36}$$

where c_o and c_i are constants to be determined after enforcing the boundary conditions (4.18) and the continuity of the traction vector on the inclusion-matrix interface. The continuity of the traction vector on the boundary of the inclusion implies that $\sigma^{rr}_{(0)}$ must be continuous at $R = R_i$. Therefore, c_o and c_i are computed as

$$c_o = k + \frac{k}{2}\ln\left(\frac{\frac{R_o^2}{R_i^2}}{\gamma_o + \frac{R_o^2}{R_i^2}}\right) - \frac{k\gamma_o}{2\left(\gamma_o + \frac{R_o^2}{R_i^2}\right)}, \quad c_i = c_o + \frac{3k\Omega_o}{4} - \frac{k}{2}\left(1 + e^{-\frac{3\Omega_o}{2}}\right) + ke^{\frac{\Omega_o}{2}}. \tag{4.37}$$

The zero-order stress components are simplified to read

$$\frac{\sigma^{rr}_{(0)}}{\mu} = \begin{cases} ke^{\frac{\Omega_o}{2}} - c_i\,, & 0 \leq R \leq R_i \\ k + \frac{k}{2}\ln\left(\frac{\frac{R^2}{R_i^2}}{\gamma_o + \frac{R^2}{R_i^2}}\right) - \frac{k\gamma_o}{2\left(\gamma_o + \frac{R^2}{R_i^2}\right)} - c_o\,, & R_i \leq R \leq R_o \end{cases}, \tag{4.38a}$$

$$\frac{\hat{\sigma}^{\theta\theta}_{(0)}}{\mu} = \begin{cases} \frac{e^{-\Omega_o}}{k^2} - c_i\,, & 0 \leq R < R_i \\ \frac{1}{k^2} - c_o + \frac{k}{2}\left[\frac{\gamma_o}{\gamma_o + \frac{R^2}{R_i^2}} + \ln\left(\frac{\frac{R^2}{R_i^2}}{\gamma_o + \frac{R^2}{R_i^2}}\right)\right], & R_i < R \leq R_o \end{cases}, \tag{4.38b}$$

$$\frac{R^2\sigma^{\phi\phi}_{(0)}}{\mu} = \begin{cases} e^{-\Omega_o}\left(1 - \frac{c_i}{k}e^{-\frac{\Omega_o}{2}}\right), & 0 \leq R < R_i \\ 1 - \frac{\frac{R^2}{R_i^2}}{k\left(\gamma_o + \frac{R^2}{R_i^2}\right)}\left\{c_o - \frac{k}{2}\left[\frac{\gamma_o}{\gamma_o + \frac{R^2}{R_i^2}} + \ln\left(\frac{\frac{R^2}{R_i^2}}{\gamma_o + \frac{R^2}{R_i^2}}\right)\right]\right\}, & R_i < R \leq R_o \end{cases}. \tag{4.38c}$$

Substituting (4.35) into (4.21), one obtains the following relations in the inclusion and the matrix

$$\phi^{(1)}_{,\Phi} + k^{-\frac{1}{2}} e^{-\frac{3\Omega_o}{4}} r^{(1)}_{,R} = \frac{R}{B}\left(1 - k^{\frac{3}{2}} e^{\frac{3\Omega_o}{4}}\right)\cos\Phi - \frac{r^{(1)}}{k^{\frac{1}{2}} e^{\frac{3\Omega_o}{4}} R}\,, \qquad 0 \le R \le R_i\,,$$
$$\phi^{(1)}_{,\Phi} + k^{-\frac{1}{2}}\left(1+\gamma_o\frac{R_i^2}{R^2}\right)^{\frac{1}{2}} r^{(1)}_{,R} = \frac{R}{B}\left(1 - k^{\frac{3}{2}}\left(1+\gamma_o\frac{R_i^2}{R^2}\right)^{\frac{1}{2}}\right)\cos\Phi - \frac{r^{(1)}}{k^{\frac{1}{2}} R\left(1+\gamma_o\frac{R_i^2}{R^2}\right)^{\frac{1}{2}}}\,, \quad R_i < R \le R_o\,. \tag{4.39}$$

The expressions (4.27) for the first-order stress components are simplified to read

$$\frac{\sigma^{rr}_{(1)}}{\mu} = \begin{cases} 2k^{\frac{1}{2}} e^{-\frac{\Omega_o}{4}} r^{(1)}_{,R} - \frac{p^{(1)}}{\mu}\,, & 0 \le R \le R_i \\ \frac{2k^{\frac{1}{2}} r^{(1)}_{,R}}{\left(1+\gamma_o\frac{R_i^2}{R^2}\right)^{\frac{1}{2}}} - \frac{p^{(1)}}{\mu}\,, & R_i < R \le R_o \end{cases}, \tag{4.40a}$$

$$\frac{\sigma^{r\phi}_{(1)}}{\mu} = \begin{cases} e^{-\Omega_o}\left(k^{\frac{1}{2}} e^{\frac{3\Omega_o}{4}} \phi^{(1)}_{,R} + \frac{r^{(1)}_{,\Phi}}{R^2}\right), & 0 \le R \le R_i \\ \frac{k^{\frac{1}{2}} \phi^{(1)}_{,R}}{\left(1+\gamma_o\frac{R_i^2}{R^2}\right)^{\frac{1}{2}}} + \frac{r^{(1)}_{,\Phi}}{R^2}\,, & R_i < R \le R_o \end{cases}, \tag{4.40b}$$

$$\frac{\hat{\sigma}^{\theta\theta}_{(1)}}{\mu} = \begin{cases} -\frac{2Re^{-\Omega_o}\cos\Phi}{Bk^2}\left(1 - k^{\frac{3}{2}} e^{\frac{3\Omega_o}{4}}\right) - \frac{p^{(1)}}{\mu}\,, & 0 \le R \le R_i \\ -\frac{2R\cos\Phi}{Bk^2}\left[1 - k^{\frac{3}{2}}\left(1+\gamma_o\frac{R_i^2}{R^2}\right)^{\frac{1}{2}}\right] - \frac{p^{(1)}}{\mu}\,, & R_i < R \le R_o \end{cases}, \tag{4.40c}$$

$$\frac{R^2\sigma^{\phi\phi}_{(1)}}{\mu} = \begin{cases} 2k^{-\frac{3}{2}} c_i e^{-\frac{9\Omega_o}{4}} \frac{r^{(1)}}{R} + 2e^{-\Omega_o}\phi^{(1)}_{,\Phi} - k^{-1} e^{-\frac{3\Omega_o}{2}} \frac{p^{(1)}}{\mu}, & 0 \le R \le R_i \\ 2\phi^{(1)}_{,\Phi} - \frac{p^{(1)}}{k\mu\left(1+\gamma_o\frac{R_i^2}{R^2}\right)} + \frac{2r^{(1)}}{k^{\frac{3}{2}} R\left(1+\gamma_o\frac{R_i^2}{R^2}\right)^{\frac{3}{2}}}\left(c_o - \frac{k}{2}\left[\frac{\gamma_o}{\gamma_o+\frac{R^2}{R_i^2}} + \ln\left(\frac{\frac{R^2}{R_i^2}}{\gamma_o+\frac{R^2}{R_i^2}}\right)\right]\right), & R_i < R \le R_o \end{cases}. \tag{4.40d}$$

We now seek a solution of the following form (see Appendix A.2 for a proof of this representation)[4]

$$r^{(1)} = \begin{cases} f_i(R)\cos\Phi\,, & 0 \le R \le R_i \\ f_o(R)\cos\Phi\,, & R_i \le R \le R_o \end{cases}\,,$$
$$\phi^{(1)} = \begin{cases} g_i(R)\sin\Phi\,, & 0 \le R \le R_i \\ g_o(R)\sin\Phi\,, & R_i \le R \le R_o \end{cases}\,, \tag{4.41}$$
$$\frac{p^{(1)}}{\mu} = \begin{cases} h_i(R)\cos\Phi\,, & 0 \le R < R_i \\ h_o(R)\cos\Phi\,, & R_i < R \le R_o \end{cases}\,.$$

One should also note that this solution is consistent with the symmetry of the problem and the governing equations, i.e., (4.39) and the equations found when (4.38) and (4.40) are substituted into (4.29) and (4.30). Substituting (4.41) into (4.39) gives

$$g_i(R) = \frac{R}{B}\left(1 - k^{\frac{3}{2}}e^{\frac{3\Omega_o}{4}}\right) - k^{-\frac{1}{2}}e^{-\frac{3\Omega_o}{4}}\left(f_i'(R) + \frac{f_i(R)}{R}\right)\,, \qquad 0 \le R \le R_i\,, \tag{4.42a}$$

$$g_o(R) = \frac{R}{B}\left[1 - k^{\frac{3}{2}}\left(1+\gamma_o\frac{R_i^2}{R^2}\right)^{\frac{1}{2}}\right] - k^{-\frac{1}{2}}\left(1+\gamma_o\frac{R_i^2}{R^2}\right)^{\frac{1}{2}} f_o'(R) - \frac{f_o(R)}{k^{\frac{1}{2}}R\left(1+\gamma_o\frac{R_i^2}{R^2}\right)^{\frac{1}{2}}}\,, \quad R_i \le R \le R_o\,. \tag{4.42b}$$

Substituting (4.41) into (4.29) and (4.30) one obtains

$$f_i''(R) + \frac{f_i'(R)}{R} - \frac{3f_i(R)}{2R^2} - \frac{e^{\frac{\Omega_o}{4}}h_i'(R)}{2k^{\frac{1}{2}}} + k^{\frac{1}{2}}e^{\frac{3\Omega_o}{4}}\left(\frac{g_i'(R)}{2} - \frac{g_i(R)}{R}\right) + \frac{1}{2Bk}\left(k^3e^{\frac{3\Omega_o}{2}} - 1\right) = 0\,, \tag{4.43a}$$

[4]Solutions with a similar form were discussed in [68, 69].

$$2R^3(R^2+\eta)f_o''(R)+(2R^4-\eta^2)f_o'(R)-R\left(\frac{R^4}{R^2+\eta}+2(R^2+\eta)\right)f_o(R)-k^{-\frac{1}{2}}R^2(R^2+\eta)^{\frac{3}{2}}h_o'(R)$$
$$+k^{\frac{1}{2}}R^4(R^2+\eta)^{\frac{1}{2}}g_o'(R)-2k^{\frac{1}{2}}R(R^2+\eta)^{\frac{3}{2}}g_o(R)+\frac{R^3}{Bk}\big(\left(k^3-1\right)R^2-\eta\big)=0\,, \quad \text{(4.43b)}$$

$$f_i'(R)+\frac{3f_i(R)}{R}-\frac{e^{\frac{\Omega_o}{4}}h_i(R)}{k^{\frac{1}{2}}}-k^{\frac{1}{2}}e^{\frac{3\Omega_o}{4}}\big(R^2g_i''(R)+3Rg_i'(R)-2g_i(R)\big)+\frac{R}{Bk}\left(k^3e^{\frac{3\Omega_o}{2}}-1\right)=0\,, \quad \text{(4.44a)}$$

$$2R(R^2+\eta)g_o(R)-R^2(3R^2+\eta)g_o'(R)-R^3(R^2+\eta)g_o''(R)-k^{-1}R^3h_o(R)+k^{-\frac{1}{2}}(R^2+\eta)^{\frac{3}{2}}f_o'(R)$$
$$+k^{-\frac{1}{2}}R\left(3(R^2+\eta)^{\frac{1}{2}}-\frac{\eta^2}{(R^2+\eta)^{\frac{3}{2}}}\right)f_o(R)+\frac{R(R^2+\eta)^{\frac{1}{2}}}{Bk^{\frac{3}{2}}}\big(\left(k^3-1\right)R^2+k^3\eta\big)=0\,, \quad \text{(4.44b)}$$

where $\eta = R_i^2\gamma_o$. Using (4.42), (4.43), and (4.44), one finds the following third-order linear ODEs for $f_i'(R)$ and $f_o'(R)$

$$f_i^{(4)}(R)+\frac{6f_i'''(R)}{R}+3\left(\frac{f_i''(R)}{R^2}-\frac{f_i'(R)}{R^3}\right)=0\,, \qquad 0\leq R\leq R_i\,, \quad \text{(4.45a)}$$

$$(R^2+\eta)^2\frac{f_o^{(4)}(R)}{R}+2(R^2+\eta)\left(3R^2-\eta\right)\frac{f_o'''(R)}{R^2}+\left(3R^4+2\eta^2\right)\frac{f_o''(R)}{R^3}-3f_o'(R)$$
$$=\frac{k^{\frac{1}{2}}\eta^2}{BR^2\left(R^2+\eta\right)^{\frac{1}{2}}}-\frac{k^2\eta^2\left(R^2+2\eta\right)}{BR^3\left(R^2+\eta\right)}\,, \qquad R_i\leq R\leq R_o\,. \quad \text{(4.45b)}$$

It then follows that (4.45a) has the following solution

$$f_i(R)=c_{i_1}R^2+\frac{c_{i_2}}{R^2}+c_{i_3}\ln R+c_{i_4}\,. \quad \text{(4.46)}$$

Enforcing $f_i(0) = 0$ implies that $c_{i_2} = c_{i_3} = c_{i_4} = 0$. $f_i(R)$ is now substituted into (4.42a) to obtain $g_i(R)$, from which $h_i(R)$ is calculated using (4.44a) as

$$g_i(R) = \frac{R}{B}\left(1 - k^{\frac{3}{2}} e^{\frac{3\Omega_o}{4}}\right) - 3c_{i_1} k^{-\frac{1}{2}} e^{-\frac{3\Omega_o}{4}} R\,, \tag{4.47}$$

$$h_i(R) = 8c_{i_1} k^{\frac{1}{2}} e^{-\frac{\Omega_o}{4}} R + \frac{R}{B}\left(2k^{\frac{5}{2}} e^{\frac{5\Omega_o}{4}} - k e^{\frac{\Omega_o}{2}} - k^{-\frac{1}{2}} e^{-\frac{\Omega_o}{4}}\right)\,. \tag{4.48}$$

After some simplifications, the boundary conditions (4.18) give one the following relations

$$f_o(R_o) = \frac{k^{\frac{1}{2}} \left(R_o^2 + \eta\right)^{\frac{3}{2}}}{R_o} g_o'(R_o)\,, \tag{4.49}$$

$$f_o'(R_o) = \frac{\left(R_o^2 + \eta\right)^{\frac{1}{2}}}{2k^{\frac{1}{2}} R_o} h_o(R_o)\,. \tag{4.50}$$

The continuity of the displacement field at the inclusion-matrix interface implies that

$$f_i(R_i) = f_o(R_i)\,, \qquad g_i(R_i) = g_o(R_i)\,. \tag{4.51}$$

The traction vector is defined as $\mathbf{t} = \langle\!\langle \boldsymbol{\sigma}, \boldsymbol{n} \rangle\!\rangle_{\mathbf{g}}$, which in components reads $t^a(x, \boldsymbol{n}) = \sigma^{ac} g_{bc} n^b$. The unit normal vector to the inclusion-matrix interface to the first order in ε reads $\boldsymbol{n} = \hat{\boldsymbol{r}} + n^{\phi}_{(1)} \hat{\boldsymbol{\phi}}$, where

$$n^{\phi}_{(1)} = \frac{f_i(R_i) \sin\Phi}{k^{\frac{1}{2}} e^{\frac{3\Omega_o}{4}} R_i}\,. \tag{4.52}$$

The continuity of the first-order terms of the traction vector on the inclusion-matrix boundary implies that $\hat{\sigma}^{rr}_{(1)}$ and $\hat{\sigma}^{r\phi}_{(1)} + n^{\phi}_{(1)} \hat{\sigma}^{\phi\phi}_{(0)}$ must be continuous at $R = R_i$, $-\pi \leq \Phi \leq \pi$. These conditions yield

$$h_o(R_i) - h_i(R_i) = 2k^{\frac{1}{2}} e^{-\frac{3\Omega_o}{4}} \left(f_o'(R_i) - e^{\frac{\Omega_o}{2}} f_i'(R_i)\right)\,, \tag{4.53}$$

$$k^{\frac{1}{2}} R_i^2 e^{\frac{9\Omega_o}{4}} \left(e^{\frac{\Omega_o}{2}} g_i'(R_i) - g_o'(R_i)\right) = f_i(R_i)\left(e^{2\Omega_o} - 1\right)\,. \tag{4.54}$$

We first find the homogeneous solution of (4.45b) using the power series expansion of $f_o'(R)$ centered at $R = 0$, under the assumption that $f_o'(R)$ can be analytically extended to the interval $[0, R_o]$. Note that depending on whether η is positive or negative, or equivalently, Ω_o is positive or negative, (4.45b) has different solutions. We have the following solutions for the homogeneous part of the differential equation, denoted by f_o^p and f_o^n for the positive and negative pure dilatational eigenstrain Ω_o, respectively, in the interval $[R_i, R_o]$ as follows[5]

$$f_o^p = c_{o_1}^p \frac{R}{\left(R^2+\eta\right)^{\frac{1}{2}}} + c_{o_2}^p \left\{ R\left(R^2+\eta\right)^{\frac{1}{2}} + \eta \sinh^{-1}\frac{R}{\eta^{\frac{1}{2}}} \right\} + c_{o_3}^p \left\{ R^2 - \frac{\eta R}{\left(R^2+\eta\right)^{\frac{1}{2}}} \sinh^{-1}\frac{R}{\eta^{\frac{1}{2}}} \right\} + c_{o_4}^p \,, \tag{4.55}$$

$$f_o^n = c_{o_1}^n \frac{R}{\left(R^2+\eta\right)^{\frac{1}{2}}} + c_{o_2}^n \left\{ R\left(R^2+\eta\right)^{\frac{1}{2}} + \eta \cosh^{-1}\frac{R}{(-\eta)^{\frac{1}{2}}} \right\} + c_{o_3}^n \left\{ R^2 - \frac{\eta R}{\left(R^2+\eta\right)^{\frac{1}{2}}} \cosh^{-1}\frac{R}{(-\eta)^{\frac{1}{2}}} \right\} + c_{o_4}^n \,. \tag{4.56}$$

We now use the method of variation of parameters to find the particular solution of (4.45b). After some calculations, the general solution of the differential equation for the positive and negative values of the pure dilatational eigenstrain are obtained as[6]

$$\begin{aligned} f_o^p(R) =& c_{o_1}^p \frac{R}{\left(R^2+\eta\right)^{\frac{1}{2}}} + c_{o_2}^p \left\{ R\left(R^2+\eta\right)^{\frac{1}{2}} + \eta \sinh^{-1}\frac{R}{\eta^{\frac{1}{2}}} \right\} + c_{o_3}^p \left\{ R^2 - \frac{\eta R}{\left(R^2+\eta\right)^{\frac{1}{2}}} \sinh^{-1}\frac{R}{\eta^{\frac{1}{2}}} \right\} + c_{o_4}^p \\ &+ \frac{k^{\frac{1}{2}}}{16B} \left\{ 2\eta^2 J_1^p(R) - \frac{2k^{\frac{3}{2}}\eta^3 R J_2^p(R)}{\left(R^2+\eta\right)^{\frac{1}{2}}} - \frac{\eta \sinh^{-1}\frac{R}{\eta^{\frac{1}{2}}}}{\left(R^2+\eta\right)^{\frac{1}{2}}} \left[\frac{k^{\frac{3}{2}}\eta}{R} + \left(k^{\frac{3}{2}} R + \left(R^2+\eta\right)^{\frac{1}{2}} \right) \ln\frac{R^2}{R^2+\eta} \right] \right. \\ &-2R\left(R^2+\eta\right)^{\frac{1}{2}} - 8k^{\frac{3}{2}}R^2 - \frac{R\eta}{\left(R^2+\eta\right)^{\frac{1}{2}}} \ln\frac{\eta}{R^4} + \left(k^{\frac{3}{2}}R - \left(R^2+\eta\right)^{\frac{1}{2}} \right) R \ln\frac{R^2}{R^2+\eta} \\ &\left. +2k^{\frac{3}{2}}\eta \ln\frac{\eta}{R^2+\eta} - 2\eta \ln\left(R + \left(R^2+\eta\right)^{\frac{1}{2}} \right) \right\} , \end{aligned} \tag{4.57}$$

[5]Note that one can easily verify that (4.55) and (4.56) are indeed the homogeneous solutions of (4.45b) for $R \in [R_i, R_o]$, and therefore, using the power series method is justified.

[6]The dilogarithm function is defined as: $\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} = -\int_0^z \ln(1-\zeta)\frac{d\zeta}{\zeta}$ for $|z| < 1$.

$$
\begin{aligned}
f_o^n(R) &= c_{o_1}^n \frac{R}{(R^2+\eta)^{\frac{1}{2}}} + c_{o_2}^n \left\{ R\left(R^2+\eta\right)^{\frac{1}{2}} + \eta \cosh^{-1}\frac{R}{(-\eta)^{\frac{1}{2}}} \right\} + c_{o_3}^n \left\{ R^2 - \frac{\eta R}{(R^2+\eta)^{\frac{1}{2}}} \cosh^{-1}\frac{R}{(-\eta)^{\frac{1}{2}}} \right\} \\
&+ \frac{k^{\frac{1}{2}}}{16B} \Bigg\{ 2\eta^2 J_1^n(R) - \frac{2k^{\frac{3}{2}}\eta^3 R J_2^n(R)}{(R^2+\eta)^{\frac{1}{2}}} - \frac{\eta\cosh^{-1}\frac{R}{(-\eta)^{\frac{1}{2}}}}{(R^2+\eta)^{\frac{1}{2}}} \left[\frac{k^{\frac{3}{2}}\eta}{R} + \left(k^{\frac{3}{2}}R + \left(R^2+\eta\right)^{\frac{1}{2}}\right) \ln\frac{R^2}{R^2+\eta} \right] \\
&-2R\left(R^2+\eta\right)^{\frac{1}{2}} - 8k^{\frac{3}{2}}R^2 - \frac{R\eta}{(R^2+\eta)^{\frac{1}{2}}} \ln\frac{-\eta}{R^4} + \left(k^{\frac{3}{2}}R - \left(R^2+\eta\right)^{\frac{1}{2}}\right) R \ln\frac{R^2}{R^2+\eta} \\
&+2k^{\frac{3}{2}}\eta \ln\frac{-\eta}{R^2+\eta} - 2\eta\ln\left(R + \left(R^2+\eta\right)^{\frac{1}{2}}\right) \Bigg\} + c_{o_4}^n\,,
\end{aligned} \tag{4.58}
$$

where

$$
J_1^p(R) = \int \frac{\sinh^{-1}\frac{\zeta}{\eta^{\frac{1}{2}}}}{\zeta\left(\zeta^2+\eta\right)} d\zeta = \frac{1}{2\eta}\left[\mathbf{Li}_2(-u(R)) - \mathbf{Li}_2(u(R)) + \ln\left(\frac{R^2}{R^2+\eta}\right) \sinh^{-1}\frac{R}{\eta^{\frac{1}{2}}} \right], \tag{4.59}
$$

$$
\begin{aligned}
J_2^p(R) = \int \frac{\sinh^{-1}\frac{\zeta}{\eta^{\frac{1}{2}}}}{\zeta^3\left(\zeta^2+\eta\right)} d\zeta = -\frac{1}{2\eta^2}\Bigg[&\frac{(R^2+\eta)^{\frac{1}{2}}}{R} + \left(\frac{\eta}{R^2} + \ln\frac{R^2}{R^2+\eta}\right) \sinh^{-1}\frac{R}{\eta^{\frac{1}{2}}} \\
&+ \mathbf{Li}_2(-u(R)) - \mathbf{Li}_2(u(R)) \Bigg],
\end{aligned} \tag{4.60}
$$

$$
J_1^n(R) = \int \frac{\cosh^{-1}\frac{\zeta}{(-\eta)^{\frac{1}{2}}}}{\zeta\left(\zeta^2+\eta\right)} d\zeta = \frac{1}{2\eta}\left[\mathbf{Li}_2(-u(R)) - \mathbf{Li}_2(u(R)) + \ln\left(\frac{R^2}{R^2+\eta}\right) \cosh^{-1}\frac{R}{(-\eta)^{\frac{1}{2}}} \right], \tag{4.61}
$$

$$
\begin{aligned}
J_2^n(R) = \int \frac{\cosh^{-1}\frac{\zeta}{(-\eta)^{\frac{1}{2}}}}{\zeta^3\left(\zeta^2+\eta\right)} d\zeta = -\frac{1}{2\eta^2}\Bigg[&\frac{(R^2+\eta)^{\frac{1}{2}}}{R} + \left(\frac{\eta}{R^2} + \ln\frac{R^2}{R^2+\eta}\right) \cosh^{-1}\frac{R}{(-\eta)^{\frac{1}{2}}} \\
&+ \mathbf{Li}_2(-u(R)) - \mathbf{Li}_2(u(R)) \Bigg],
\end{aligned} \tag{4.62}
$$

and $u(R) = \frac{\eta}{\left(R+(R^2+\eta)^{\frac{1}{2}}\right)^2}$.[7] The function $g_o(R)$ may now be calculated for the positive and negative values of the eigenstrain by substituting (4.57) and (4.58) for f_o, respectively, into (4.42b). We can then find $h_o(R)$ by substituting for f_o and g_o into (4.44b) (see Appendix A.1 for details). Using (4.40) and (4.41), along with the expressions (4.46), (4.47), and (4.48), the first-order physical components of the Cauchy stress are written as

$$\frac{\hat{\sigma}^{rr}_{(1)}}{\mu} = \begin{cases} -\frac{e^{-\frac{\Omega_o}{4}}}{Bk^{\frac{1}{2}}}\left(4Bkc_{i_1} - k^{\frac{3}{2}}e^{\frac{3\Omega_o}{4}} + 2k^3 e^{\frac{3\Omega_o}{2}} - 1\right) R\cos\Phi\,, & 0 \leq R \leq R_i \\ \left[\frac{2k^{\frac{1}{2}}Rf_o'(R)}{(R^2+\eta)^{\frac{1}{2}}} - h_o(R)\right]\cos\Phi\,, & R_i \leq R \leq R_o \end{cases}, \tag{4.63a}$$

$$\frac{\hat{\sigma}^{r\phi}_{(1)}}{\mu} = \begin{cases} -\frac{k^{\frac{1}{2}}e^{-\frac{\Omega_o}{4}}}{B}\left(4Bc_{i_1} + k^2 e^{\frac{3\Omega_o}{2}} - k^{\frac{1}{2}}e^{\frac{3\Omega_o}{4}}\right) R\sin\Phi\,, & 0 \leq R < R_i \\ k^{\frac{1}{2}}\left[k^{\frac{1}{2}}Rg_o'(R) - \frac{\left(R^2+\eta\right)^{\frac{1}{2}}f_o(R)}{R^2}\right]\sin\Phi\,, & R_i < R \leq R_o \end{cases}, \tag{4.63b}$$

$$\frac{\hat{\sigma}^{\theta\theta}_{(1)}}{\mu} = \begin{cases} -\frac{e^{-\Omega_o}}{Bk^2}\left(8Bk^{\frac{5}{2}}c_{i_1}e^{\frac{3\Omega_o}{4}} + 2k^{\frac{9}{2}}e^{\frac{9\Omega_o}{4}} - 3k^{\frac{3}{2}}e^{\frac{3\Omega_o}{4}} - k^3 e^{\frac{3\Omega_o}{2}} + 2\right) R\cos\Phi\,, & 0 \leq R < R_i \\ -\left[\frac{2R}{Bk^2}\left(1 - k^{\frac{3}{2}}\left(1+\frac{\eta}{R^2}\right)^{\frac{1}{2}}\right) + h_o(R)\right]\cos\Phi\,, & R_i < R \leq R_o \end{cases}, \tag{4.63c}$$

$$\frac{\hat{\sigma}^{\phi\phi}_{(1)}}{\mu} = \begin{cases} -\frac{e^{-\frac{\Omega_o}{4}}}{Bk^{\frac{1}{2}}}\left(12Bkc_{i_1} - 3k^{\frac{3}{2}}e^{\frac{3\Omega_o}{4}} + 4k^3 e^{\frac{3\Omega_o}{2}} - 1\right) R\cos\Phi\,, & 0 \leq R < R_i \\ \frac{1}{R^2}\left(2k^{\frac{1}{2}}\left(R^2+\eta\right)^{\frac{1}{2}} f_o(R) + 2k\left(R^2+\eta\right) g_o(R) - R^2 h_o(R)\right)\cos\Phi\,, & R_i < R \leq R_o \end{cases}. \tag{4.63d}$$

Remark 4.2.1. Note that when $R_i = R_o$, i.e., when the entire solid torus has a uniform pure dilatational eigenstrain, no residual stresses are generated. In this case, we recover the exact solution, for

[7] Note that $|u(R)| < 1$ for $R \in [R_i, R_o]$, and therefore, $\mathrm{Li}_2(\pm u(R))$ is well-defined.

which $\frac{r}{R} = \frac{b}{B} = e^{\frac{\Omega_o}{2}}$. Note that this is indeed the exact solution, as it is stress-free, and thus, the equilibrium equations are trivially satisfied. Also, it satisfies the incompressibility condition (4.6).

Remark 4.2.2. One can simply check that in the first-order approximation with respect to the thinness ratio the deformed shape of the outer boundaries of the inclusion and the matrix remain circular with their corresponding zero-order radii, but they become eccentric with eccentricity $E = f_o(R_o) - f_i(R_i)$. Note that the inclusion and the matrix outer boundary points rotate with respect to one another after deformation such that their relative rotation is $\Delta(\Phi) = (g_o(R_o) - g_i(R_i)) \sin \Phi$ for any pair of points located at an angle Φ in the initial configuration on the outer boundaries of the inclusion and the matrix.

Next, we proceed to numerically calculate the values of the constants k, c_{i_1}, c_{o_1}, c_{o_2}, c_{o_3}, and c_{o_4} using the expressions (4.49), (4.50), (4.51), (4.53), and (4.54) for a neo-Hookean solid torus with a given negative or positive pure dilatational eigentrain.

Numerical results. We now consider some numerical examples and examine the first-order residual stress field (4.63) for inclusions with different values of pure dilatational eigenstrains and various torus geometries. Figures 6.1 and 6.2 show the variation of the radial part of the first-order stress components for a torus with $\frac{R_o}{B} = 0.1$, containing inclusions with several values of $\frac{R_i}{R_o}$, and $\Omega_o = \pm 0.5$. Notice that all the first-order stress components vary linearly with the material radial coordinate in the inclusion. As expected all the stress components undergo a jump at the inclusion-matrix boundary except the radial stress component, which is continuous at the interface. For the positive eigenstrain case (Figure 6.1), the maximum shear stress in the inclusion and matrix is first increasing, then decreasing as $\frac{R_i}{R_o}$ increases from zero (a torus without eigenstrain) such that the maximum shear stress for a torus with $\frac{R_i}{R_o} = 0.4$ is greater than that of a torus with $\frac{R_i}{R_o} = 0.2$ and $\frac{R_i}{R_o} = 0.6$ in both the inclusion and the matrix. For the case of negative eigenstrain, the maximum shear stress in the torus increases with the increase in $\frac{R_i}{R_o}$ ratio from 0 to 0.8. After that, however, as $\frac{R_i}{R_o}$ increases the maximum shear stress decreases until it becomes zero when $\frac{R_i}{R_o} = 1$ (Figure 6.2). Note that $\frac{R_i}{R_o} = 1$ corresponds to the entire torus having a uniform pure dilatational eigenstrain distribution, which is stress-free as was discussed earlier in Remark 4.2.1.

The contour plots of the first-order residual stress components for a torus with $\frac{R_o}{B} = 0.1$ are depicted in Figures 4.5, 4.6, and 4.7. A torus with an inclusion with a negative pure dilatational eigenstrain and $\frac{R_i}{R_o} = 0.4$ is shown in Figure 4.5. One observes that the shear stress concentrates across the inclusion-matrix interface with its maximum attained at the top and the bottom. For the selected parameters, the first-order circumferential stress component $\hat{\sigma}_{(1)}^{\phi\phi}/\mu$ is negligible in the inclusion, and hence, the circumferential stress component remains uniform to the first order in the inclusion. This is also true for the positive eigenstrain cases $\Omega_o = 0.5$ and $\Omega_o = 0.7$ with $\frac{R_i}{R_o}$ ratio equal to 0.4 and 0.2, respectively (Figures 4.6 and 4.7).

Figure 4.8a illustrates the dependence of $\frac{b}{B}$ on the pure dilatational eigenstrain value Ω_o for different values of $\frac{R_i}{R_o}$ (Note that B and b represent the distance of the center of the inclusion from the origin in the initial and deformed configurations, respectively (see Figure 4.2).). For positive eigenstrain values, $\frac{b}{B}$ monotonically increases as Ω_o increases, and as expected, the higher the $\frac{R_i}{R_o}$ ratio, the more rapid the increase. For negative eigenstrains, nevertheless, $\frac{b}{B}$ reaches a minimum, which decreases as $\frac{R_i}{R_o}$ increases, and is attained at lower eigestrain values. As was mentioned earlier for both negative and positive eigenstains, as $\frac{R_i}{R_o}$ approaches 1, the $\frac{b}{B}$ curve gets closer to $e^{\frac{\Omega_o}{2}}$ (see Remark 4.2.1). The variation of the eccentricity ratio $\frac{E}{R_o}$, where $E = f_o(R_o) - f_i(R_i)$, with respect to Ω_o is shown in Figure 4.8b for several values of $\frac{R_i}{R_o}$. As Ω_o increases from 0, the eccentricity decreases until it reaches its minimum, which increases as $\frac{R_i}{R_o}$ increases, and is attained at lower values of Ω_o.[8] For $\Omega_o < 0$, the eccentricity ratio is first increasing, then decreasing as Ω_o decreases starting from zero. The maximum eccentricity corresponds to the lower values of eigesntrains as $\frac{R_i}{R_o}$ increases. Moreover, the maximum eccentricity first increases as $\frac{R_i}{R_o}$ increases, then it decreases. For instance, the maximum eccentricity for a torus with $\frac{R_i}{R_o} = 0.7$ is greater than that of a torus with $\frac{R_i}{R_o} = 0.5$ and $\frac{R_i}{R_o} = 0.9$ when $\Omega_o < 0$. As $\frac{R_i}{R_o}$ approaches 1, the eccentricity ratio tends to zero for any value of the eigenstrain Ω_o.

[8]Note that positive and negative eccentricity values correspond to the inclusion moving to the left and right relative to the matrix, respectively.

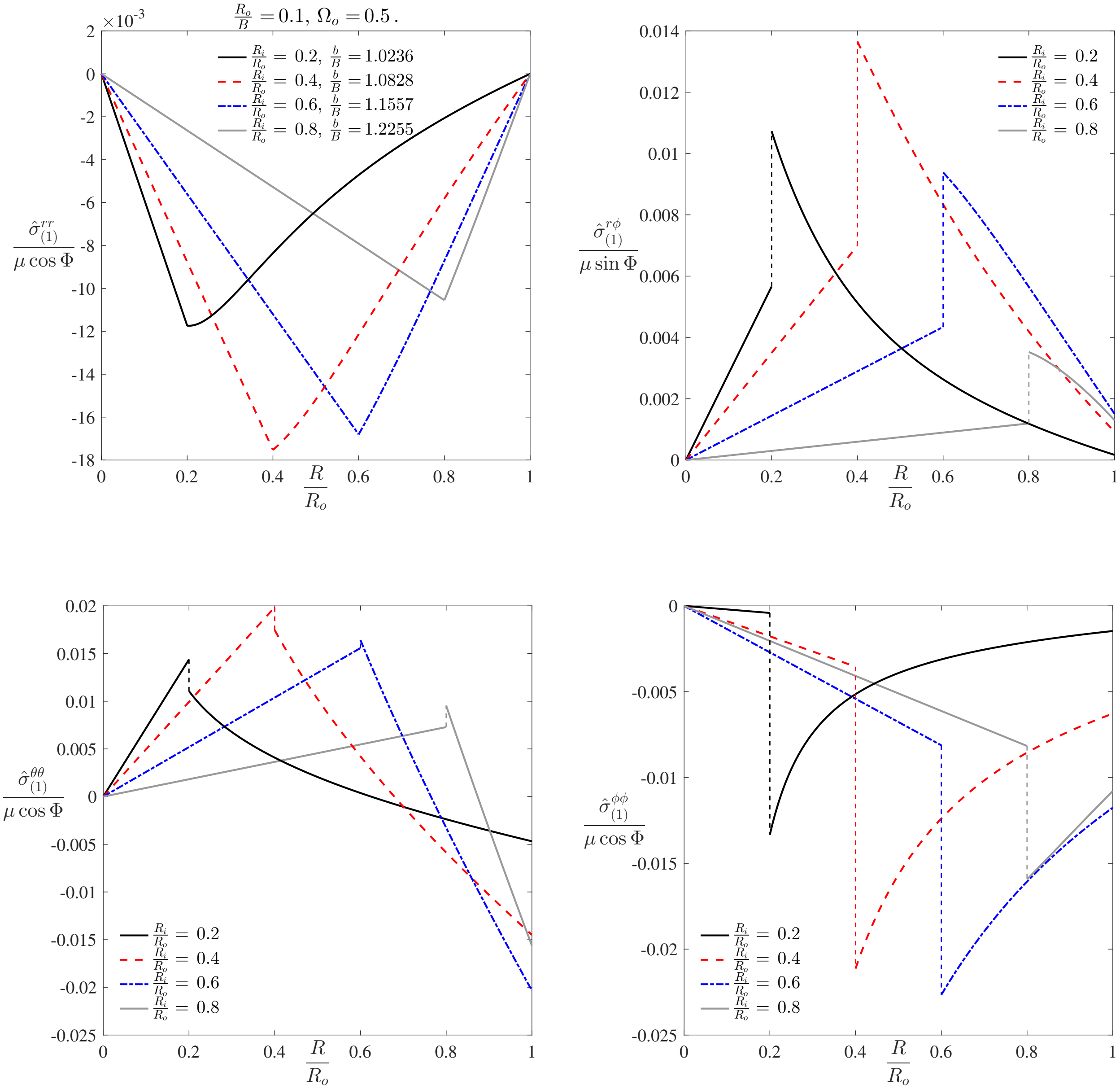


Figure 4.3: *The radial part of the first-order normalized components of the Cauchy stress tensor for* $\frac{R_o}{B} = 0.1$, $\Omega_o = 0.5$, *and different values of* $\frac{R_i}{R_o}$.

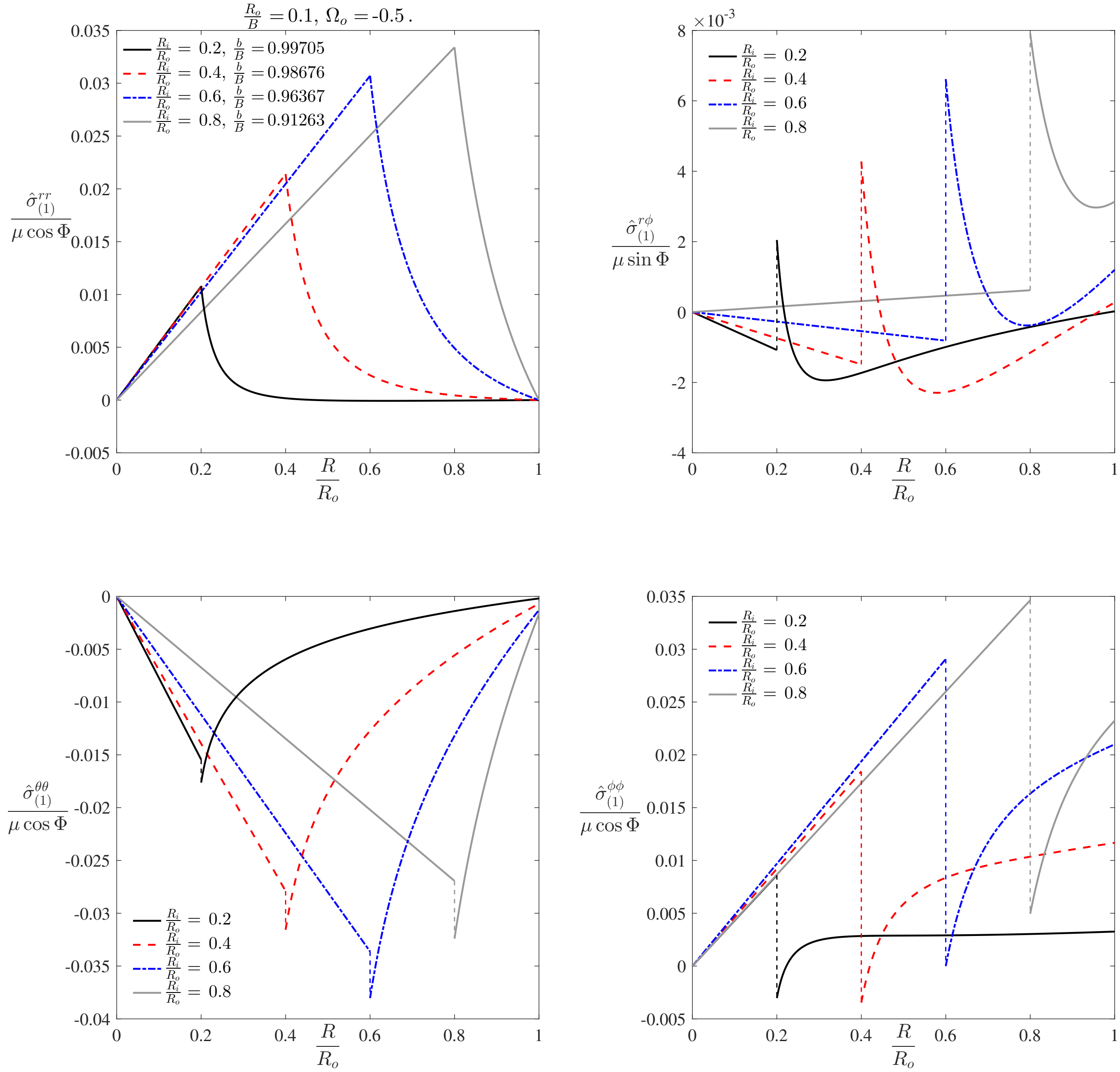


Figure 4.4: *The radial part of the first-order normalized components of the Cauchy stress tensor for* $\frac{R_o}{B} = 0.1$, $\Omega_o = -0.5$, *and different values of* $\frac{R_i}{R_o}$.

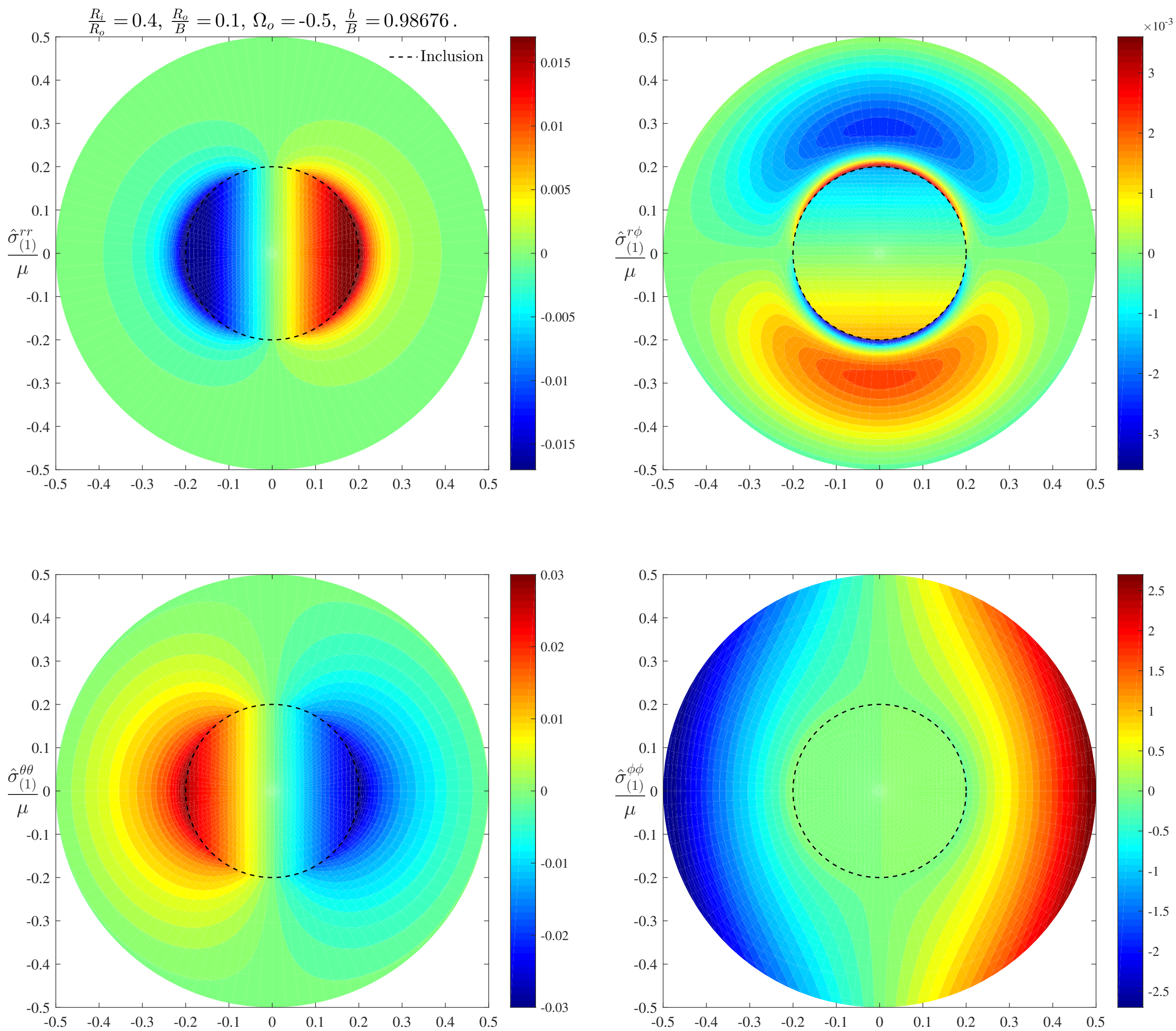


Figure 4.5: *The first-order physical components of the Cauchy stress in a torus having an inclusion with* $\frac{R_i}{R_o} = 0.4$, $\frac{R_o}{B} = 0.1$, *and constant pure dilatational eigenstrain distribution* $\Omega_o = -0.5$. *The ratio of the deformed major radius to the initial major radius of the torus is* $\frac{b}{B} = 0.98676$.

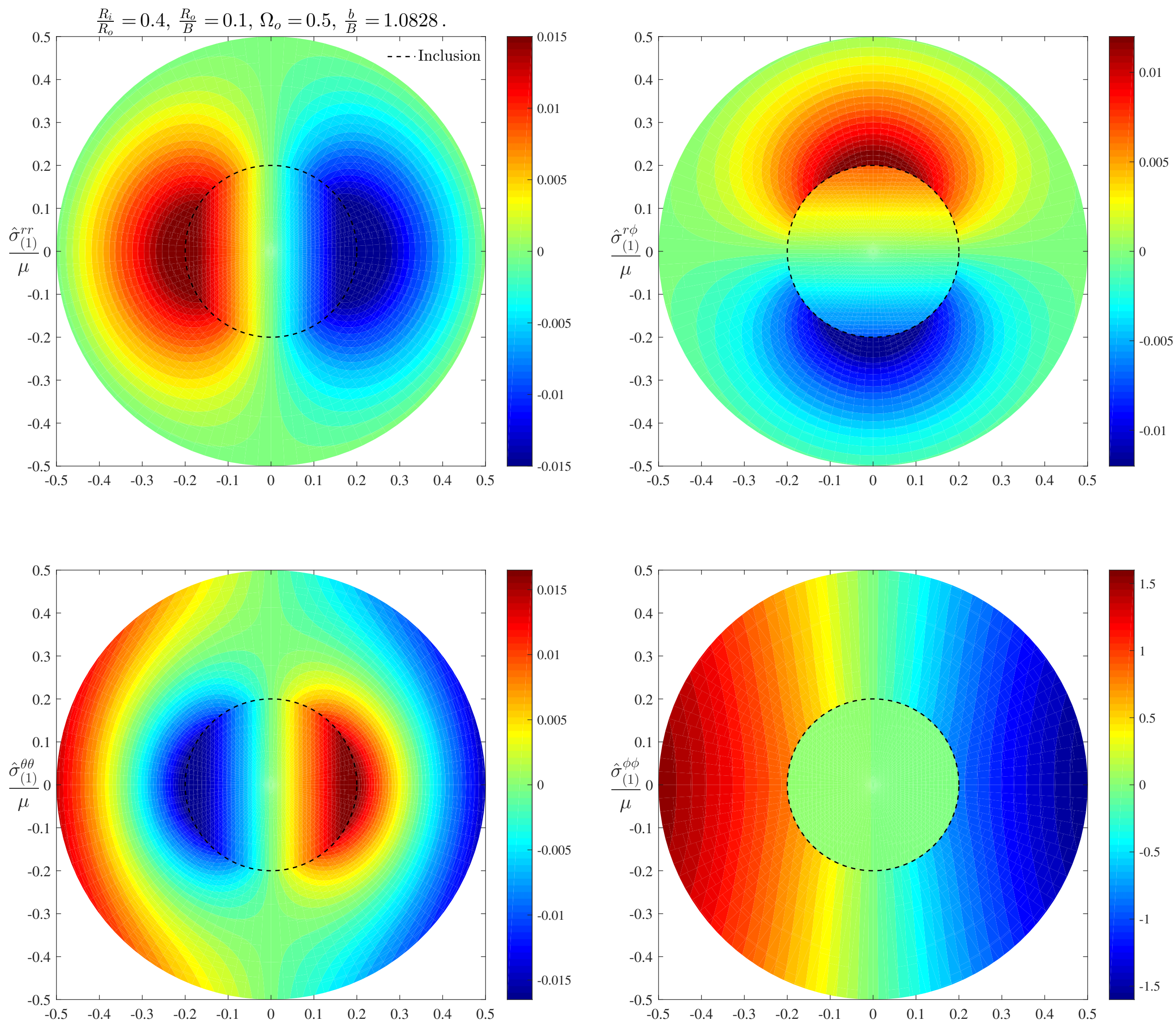


Figure 4.6: *The first-order physical components of the Cauchy stress in a torus having an inclusion with* $\frac{R_i}{R_o} = 0.4$, $\frac{R_o}{B} = 0.1$, *and constant pure dilatational eigenstrain distribution* $\Omega_o = 0.5$. *The ratio of the deformed major radius to the initial major radius of the torus is* $\frac{b}{B} = 1.0828$.

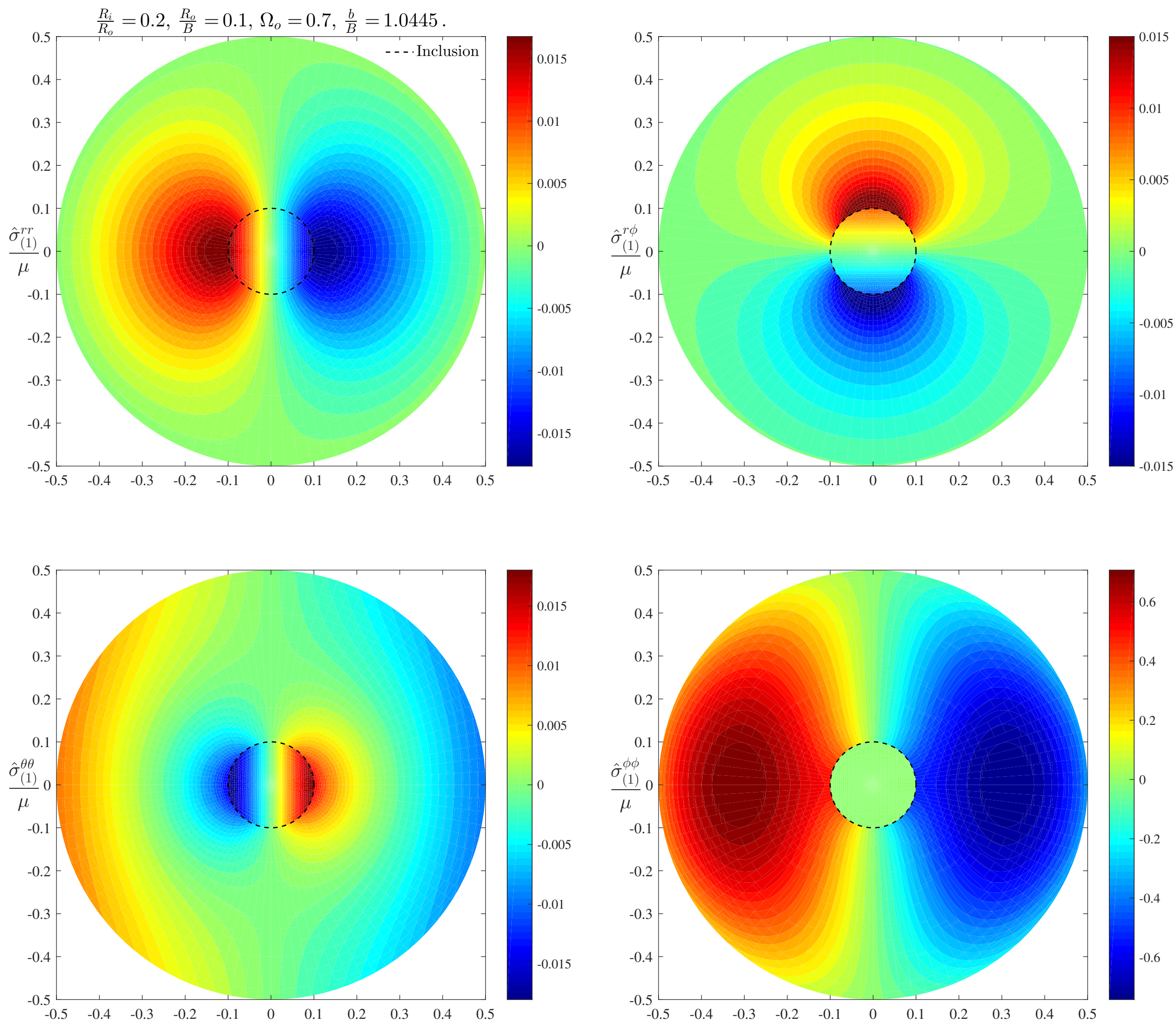
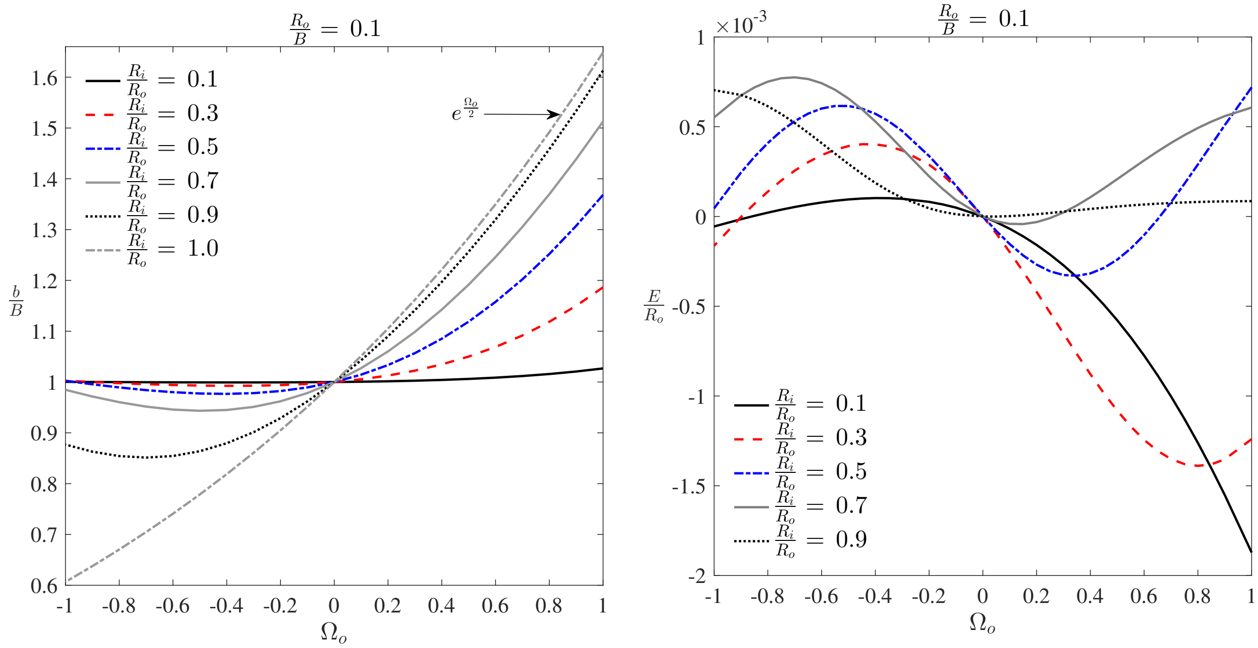


Figure 4.7: *The first-order physical components of the Cauchy stress in a torus having an inclusion with* $\frac{R_i}{R_o} = 0.2$, $\frac{R_o}{B} = 0.1$, *and constant pure dilatational eigenstrain distribution* $\Omega_o = 0.7$. *The ratio of the deformed major radius to the initial major radius of the torus is* $\frac{b}{B} = 1.0445$.



(a) (b)

Figure 4.8: *Variations of* $\frac{b}{B}$ *and the eccentricity ratio* $\frac{E}{R_o}$ *as functions of* Ω_o *for a torus with* $\frac{R_o}{B} = 0.1$ *and different values of* $\frac{R_i}{R_o}$*, shown in* (a) *and* (b)*, respectively.*

4.2.2 A linear elastic solid torus with small eigenstrains

In this section we derive the governing equations of a solid torus made of an incompressible linear elastic material that has a distribution of small eigenstrains. In geometric elasticity, in order to linearize one starts with a reference motion $\mathring{\varphi}$ and a one-parameter family of motions φ_ϵ such that $\varphi_{\epsilon=0} = \mathring{\varphi}$ [43, 82]. Let us consider a one-parameter family of motions φ_ϵ such that $\varphi_\epsilon(R, \Theta, \Phi) = (r_\epsilon(R, \Phi), \Theta, \phi_\epsilon(R, \Phi))$. We will linearize about the stress-free configuration $\mathring{\varphi}\,(R, \Theta, \Phi) = (R, \Theta, \Phi)$, i.e., $r_{\epsilon=0}(R, \Phi) = R$ and $\phi_{\epsilon=0}(R, \Phi) = \Phi$. The variation field is defined as

$$\delta\varphi(R, \Theta, \Phi) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \varphi_\epsilon(R, \Theta, \Phi) = (u(R, \Phi), 0, w(R, \Phi))\,, \tag{4.64}$$

where u and w are the non-zero displacement components.

Linearization of the incompressibility constraint. For any motion in the given one-parameter family we have

$$J_\epsilon = \frac{r_\epsilon(b_\epsilon + r_\epsilon \cos\phi_\epsilon)}{e^{\frac{3}{2}\Omega_\epsilon(R,\Phi)} R(B + R\cos\Phi)} \left(\frac{\partial r_\epsilon}{\partial R}\frac{\partial \phi_\epsilon}{\partial \Phi} - \frac{\partial r_\epsilon}{\partial \Phi}\frac{\partial \phi_\epsilon}{\partial R} \right) = 1. \tag{4.65}$$

Taking derivative with respect to ϵ of both sides and evaluating at $\epsilon = 0$, one obtains

$$u + \frac{R\left[\delta b + u\cos\Phi - wR\sin\Phi\right]}{B + R\cos\Phi} + R(u_{,R} + w_{,\Phi}) = \frac{3}{2}R\delta\Omega. \tag{4.66}$$

Similarly, from (4.9), one obtains the variation of I and β as

$$\delta I = -2\delta\beta = 2\left(-\delta\Omega + \frac{u}{R} + u_{,R} + w_{,\Phi}\right) . \tag{4.67}$$

Therefore, it follows from (4.67) that $\delta I_1 = \delta I_2 = 0$. To simplify the calculations, we assume that the material is piecewise homogeneous and use (4.11) and (4.67) to find the linearized components of the Cauchy stress tensor as

$$\delta\sigma^{rr} = -\delta p + 2\left(W_{I_1} + W_{I_2}\right)\left(2u_{,R} - \delta\Omega\right) , \tag{4.68a}$$

$$\delta\sigma^{r\phi} = 2\left(W_{I_1} + W_{I_2}\right)\left(w_{,R} + \frac{u_{,\Phi}}{R^2}\right) , \tag{4.68b}$$

$$\begin{aligned}\delta\sigma^{\theta\theta} &= -\frac{\delta p}{(B + R\cos\Phi)^2} + \frac{4\left(W_{I_1} + 2W_{I_2}\right)}{(B + R\cos\Phi)^3}\left[\delta b + u\cos\Phi - wR\sin\Phi\right] \\ &+ \frac{2}{(B + R\cos\Phi)^2}\left[2W_{I_2}\left(\frac{u}{R} + u_{,R} + w_{,\Phi}\right) - \delta\Omega\left(W_{I_1} + 4W_{I_2}\right)\right] ,\end{aligned} \tag{4.68c}$$

$$\delta\sigma^{\phi\phi} = -\frac{\delta p}{R^2} + \frac{4\left(W_{I_1} + W_{I_2}\right)}{R^2}\left(\frac{u}{R} + w_{,\Phi} - \frac{\delta\Omega}{2}\right) . \tag{4.68d}$$

Linearization of the equilibrium equations. Using (4.17), linearizing the equilibrium equations (4.15) and (4.16) one obtains

$$\delta\sigma^{rr}_{,R}+\delta\sigma^{r\phi}_{,\Phi}+\left(\frac{1}{R}+\frac{\cos\Phi}{B+R\cos\Phi}\right)\delta\sigma^{rr}-\cos\Phi\left(B+R\cos\Phi\right)\delta\sigma^{\theta\theta}-\frac{R\sin\Phi}{B+R\cos\Phi}\delta\sigma^{r\phi}-R\delta\sigma^{\phi\phi}=0\,, \tag{4.69}$$

$$\delta\sigma^{r\phi}_{,R}+\delta\sigma^{\phi\phi}_{,\Phi}+\left(\frac{3}{R}+\frac{\cos\Phi}{B+R\cos\Phi}\right)\delta\sigma^{r\phi}+\frac{\sin\Phi}{R}\left(B+R\cos\Phi\right)\delta\sigma^{\theta\theta}-\frac{R\sin\Phi}{B+R\cos\Phi}\delta\sigma^{\phi\phi}=0\,. \tag{4.70}$$

Substituting the linearized stress components given by (4.68) into the above equations one finds

$$\begin{aligned}&\delta p_{,R}+\frac{4\left(W_{I_1}+W_{I_2}\right)}{R}\left[\frac{u}{R}-u_{,R}-Ru_{,RR}-\frac{u_{,\Phi\Phi}}{2R}-\frac{Rw_{,R\Phi}}{2}+w_{,\Phi}+\frac{R\delta\Omega_{,R}}{2}\right]\\&+\frac{\cos\Phi}{B+R\cos\Phi}\left[-4W_{I_1}u_{,R}+4W_{I_2}\left(\frac{u}{R}+w_{,\Phi}\right)-6W_{I_2}\delta\Omega\right]+\frac{2R\left(W_{I_1}+W_{I_2}\right)\sin\Phi}{B+R\cos\Phi}\left(\frac{u_{,\Phi}}{R^2}+w_{,R}\right)\\&+\frac{4\left(W_{I_1}+2W_{I_2}\right)\cos\Phi}{\left(B+R\cos\Phi\right)^2}\left(\delta b+u\cos\Phi-Rw\sin\Phi\right)=0\,,\end{aligned} \tag{4.71}$$

$$\begin{aligned}&\frac{R\delta p_{,\Phi}}{2}+\frac{R\sin\Phi}{B+R\cos\Phi}\left[2W_{I_1}\left(u+Rw_{,\Phi}\right)+RW_{I_2}\left(3\delta\Omega-2u_{,R}\right)\right]\\&+\frac{2R^2\sin\Phi\left(W_{I_1}+2W_{I_2}\right)}{\left(B+R\cos\Phi\right)^2}\left[Rw\sin\Phi-u\cos\Phi-\delta b\right]-\frac{\left(W_{I_1}+W_{I_2}\right)\left(3B+4R\cos\Phi\right)}{B+R\cos\Phi}\left[u_{,\Phi}+R^2w_{,R}\right]\\&\quad-R\left(W_{I_1}+W_{I_2}\right)\left[u_{,R\Phi}+R^2w_{,RR}+2w_{,\Phi\Phi}-\delta\Omega_{,\Phi}\right]=0\,.\end{aligned} \tag{4.72}$$

Linearization of the boundary conditions. Similarly, boundary conditions (4.18) are linearized and are written as

$$2\left(W_{I_1}+W_{I_2}\right)\left(2u_{,R}-\delta\Omega\right)-\delta p=0\,,\qquad R=R_o\,,\quad -\pi\leq\Phi\leq\pi\,, \tag{4.73a}$$

$$w_{,R} + \frac{u_{,\Phi}}{R^2} = 0\,, \qquad R = R_o\,, \quad -\pi \leq \Phi \leq \pi\,. \tag{4.73b}$$

Example: A toroidal inclusion with uniform pure dilatational eigenstrains in an incompressible linear elastic solid torus. Let us consider the following distribution of eigenstrains in the torus

$$\delta\Omega(R) = \begin{cases} \delta\Omega_o\,, & 0 \leq R < R_i \\ 0\,, & R_i < R \leq R_o \end{cases}\,. \tag{4.74}$$

Pressure and displacement fields in the torus are written as

$$\begin{aligned} u(R,\Phi) &= \begin{cases} u_i(R,\Phi), & 0 \leq R < R_i \\ u_o(R,\Phi), & R_i < R \leq R_o \end{cases}\,, \\ w(R,\Phi) &= \begin{cases} w_i(R,\Phi), & 0 \leq R < R_i \\ w_o(R,\Phi), & R_i < R \leq R_o \end{cases}\,, \\ \delta p(R,\Phi) &= \begin{cases} \delta p_i(R,\Phi), & 0 \leq R < R_i \\ \delta p_o(R,\Phi), & R_i < R \leq R_o \end{cases}\,. \end{aligned} \tag{4.75}$$

Therefore, from (4.66) it follows that

$$u_i + \frac{R\left[\delta b + u_i \cos\Phi - w_i R \sin\Phi\right]}{B + R\cos\Phi} + R(u_{i,R} + w_{i,\Phi}) = \frac{3}{2} R\delta\Omega_o\,, \tag{4.76a}$$

$$u_o\left(B + R\cos\Phi\right) + R\left[\delta b + u_o \cos\Phi - w_o R \sin\Phi\right] + R(B + R\cos\Phi)(u_{o,R} + w_{o,\Phi}) = 0\,. \tag{4.76b}$$

From the continuity of the displacement field at the inclusion-matrix interface, we know that

$$u_i(R_i,\Phi) = u_o(R_i,\Phi)\,, \qquad w_i(R_i,\Phi) = w_o(R_i,\Phi)\,. \tag{4.77}$$

Also, we eliminate the rigid body motion by setting $u_i\,(0,\Phi) = 0$ and $w\,(R,0) = 0$.

We next show that for a torus made of an isotropic incompressible linear elastic solid with a toroidal inclusion having a non-zero uniform pure dilatational eigenstrain distribution, the stress field inside the inclusion cannot be uniform. Let us assume that the stress field inside the inclusion is uniform, i.e., each physical Cauchy stress component is constant. Thus

$$\delta\sigma^{rr} = c_1\,,\ \ R\delta\sigma^{r\phi} = c_2\,,\ \ (B+R\cos\Phi)^2\,\delta\sigma^{\theta\theta} = c_3\,,\ \ R^2\delta\sigma^{\phi\phi} = c_4\,, \tag{4.78}$$

where c_1 to c_4 are some constants. After some simplifications, it follows from (4.69) that

$$\frac{c_1-c_4}{R} + \frac{1}{B+R\cos\Phi}\,(c_1\cos\Phi - c_3\cos\Phi - c_2\sin\Phi) = 0\,, \tag{4.79}$$

which implies that $c_1 = c_3 = c_4 = C$ and $c_2 = 0$. Therefore (note that the equilibrium equation (4.70) has already been satisfied)

$$\delta\sigma^{rr} = (B+R\cos\Phi)^2\,\delta\sigma^{\theta\theta} = R^2\delta\sigma^{\phi\phi} = C\,, \quad 0 \le R < R_i\,, \tag{4.80}$$

$$\delta\sigma^{r\phi} = 0\,, \qquad 0 \le R < R_i\,. \tag{4.81}$$

From (4.68) and $\delta\sigma^{rr} - R^2\delta\sigma^{\phi\phi} = 0$, one obtains

$$\frac{u}{R} + w_{,\Phi} = u_{,R}. \tag{4.82}$$

Using the above relation in $\delta\sigma^{rr} - (B+R\cos\Phi)^2\,\delta\sigma^{\theta\theta} = 0$, one finds

$$\delta b + u\cos\Phi - Rw\sin\Phi = (B+R\cos\Phi)\,\frac{2u_{,R}\,(W_{I_1}-W_{I_2}) + 3W_{I_2}\delta\Omega_o}{2\,(W_{I_1}+2W_{I_2})}. \tag{4.83}$$

Similarly, (4.81) implies that

$$w_{,R} + \frac{u_{,\Phi}}{R^2} = 0\,. \tag{4.84}$$

We then use (4.83) and the incompressibility condition (4.76a), along with $u\,(0,\Phi) = 0$ to conclude that

$$u(R,\Phi) = \frac{\delta\Omega_o}{2} R. \tag{4.85}$$

Substituting the above relation into (4.82) and (4.84), one concludes that $w = 0$. Now going back to (4.83) one finally finds that

$$\delta b = \frac{\delta\Omega_o}{2} B\,. \tag{4.86}$$

This is a contradiction because δb has to depend on the radius of the inclusion R_i. In other words, the above relation is telling us that the change in the overall radius of the solid torus after deformation is independent of the size of the inclusion. This contradiction shows that the stress field inside the inclusion cannot be uniform.

CHAPTER 5
NONLINEAR ELASTIC INCLUSIONS IN ANISOTROPIC SOLIDS

5.1 Introduction

Willis [83] formulated the two-dimensional linear inclusion problem for an infinite anisotropic medium. He obtained explicit solutions for an elliptic inclusion in a medium with cubic symmetry. He showed that the stress field inside such an inclusion is uniform. In the setting of 3D linear elasticity, Li and Dunn [84] investigated the inclusion and inhomogeneity problem in an infinite anisotropic solid using Eshelby's approach. They found closed-form expressions for the Eshelby tensors in the case of transversely isotropic media containing cylindrical and thin-disk inclusions. Kinoshita and Mura [85] obtained the displacement and stress fields induced by an inclusion with a uniform distribution of eigenstrains in an infinitely extended homogeneous linear anisotropic elastic medium. Their expressions are valid for the general case of material anisotropy and different shapes of inclusions. In a series of papers [86, 87, 88, 89, 90], two-dimensional Eshelby's problem for linear polygonal inclusions in anisotropic full and half-planes were studied. Giordano *et al.* [91] investigated the elastic properties of composites consisting of isotropic spherical and cylindrical inhomogeneities embedded in a linear isotropic solid matrix. They obtained the elastic properties of the overall material in terms of the elastic constants of the constituents and their volume fractions under the simplifying assumptions of small strains for the body and small volume fractions of the embedded phase.

To our best knowledge, the problem of nonlinear inclusions in anisotropic solids has not been studied in the literature. In this chapter, we consider finite eigenstrains in transversely isotropic spherical balls and orthotropic cylindrical bars for both incompressible and compressible solids. We then determine conditions that guarantee that the stress field in spherical and cylindrical inclusions with uniform dilatational eigenstrains is uniform. In particular, we show that the results given in [24] for some special classes of compressible isotropic solids can be generalized to an arbitrary compressible isotropic solid. In the case of compressible transversely isotropic and orthotropic solids, we show that there are some nontrivial special cases for which uniform stress can be maintained in the inclusion when the radial and circumferential eigenstrains are not equal (or the axial stretch satisfies some conditions in the case of cylindrical bars). To investigate these cases, we employ the so called standard reinforcing model (see, e.g., [92]) and find the stress field in the inclusion in the case of compressible Mooney-Rivlin and Blatz-Ko materials for several reinforcement combinations.

This chapter is structured as follows. In §5.2.1 we consider finite eigenstrains in an incompressible transversely isotropic spherical ball. In §6.3.5 the corresponding problem in the case of compressible transversely isotropic and compressible isotropic solids is discussed. Finite eigenstrains in an incompressible orthotropic cylindrical bar is studied in §5.2.3. §5.2.4 is devoted to compressible orthotropic cylindrical bars with finite eigenstrains. Note that the results of this section have been previously reported in our published work [27].

5.2 Examples of Anisotropic Bodies with Finite Eigenstrains

In this section, we consider several examples of inclusions in transversely isotropic spherical balls and orthotropic cylindrical bars. We start with spherically and cylindrically symmetric distributions of finite dilatational eigenstrains in a spherical ball and a solid cylinder, respectively. We study the inclusion problem by considering uniform distribution of finite anisotropic eigenstrains in the inclusion region. We then investigate the conditions under which the stress inside the inclusion is uniform. We also identify those cases that exhibit stress singularities, depending on the values of the radial and circumferential eigenstrains, along with the axial eigenstrain in the case of cylindrical bars.

5.2.1 Finite Eigenstrains in an Incompressible Transversely Isotropic Spherical Ball

Consider a ball of radius R_o made of a nonlinear incompressible transversely isotropic material with a given spherically symmetric distribution of radial and circumferential eigenstrains. We assume that the material preferred direction is radial, i.e., $\mathbf{N} = \hat{\mathbf{R}}$, where $\hat{\mathbf{R}}$ is a unit vector in the radial direction. The material metric for the eigenstrain-free configuration in the spherical coordinates (R, Θ, Φ) reads

$\mathbf{G}_o = \operatorname{diag}\left(1, R^2, R^2\sin^2\Theta\right)$. To preserve the spherical symmetry, we require that the azimuthal and circumferential eigenstrains be equal. Therefore, the material metric for the ball with dilatational eigenstrains is written as[1]

$$\mathbf{G} = \begin{pmatrix} e^{2\omega_R(R)} & 0 & 0 \\ 0 & R^2 e^{2\omega_\Theta(R)} & 0 \\ 0 & 0 & e^{2\omega_\Theta(R)} R^2 \sin^2\Theta \end{pmatrix}, \tag{5.1}$$

where ω_R and ω_Θ describe the radial and circumferential eigenstrains, respectively. We endow the ambient space with the flat Euclidean metric $\mathbf{g} = \operatorname{diag}\left(1, r^2, r^2\sin^2\theta\right)$ in the spherical coordinates (r, θ, ϕ). We then assume an embedding of the material manifold into the ambient space with the form $(r, \theta, \phi) = (r(R), \Theta, \Phi)$, and hence, $\mathbf{F} = \operatorname{diag}(r'(R), 1, 1)$. Assuming incompressibility, i.e., $J = \sqrt{\frac{\det\mathbf{g}}{\det\mathbf{G}}}\det\mathbf{F} = 1$, one obtains

$$\frac{r^2(R) r'(R)}{R^2 e^{\omega_R(R) + 2\omega_\Theta(R)}} = 1\,. \tag{5.2}$$

Eliminating the rigid body translation by setting $r(0) = 0$ gives

$$r(R) = \left(\int_0^R 3\eta^2 e^{\omega_R(\eta) + 2\omega_\Theta(\eta)} d\eta\right)^{\frac{1}{3}}\,. \tag{5.3}$$

Therefore, the right Cauchy-Green deformation tensor is written as[2]

$$\mathbf{C} = \begin{pmatrix} \frac{R^4 e^{4\omega_\Theta(R)}}{r^4(R)} & 0 & 0 \\ 0 & \frac{r^2(R) e^{-2\omega_\Theta(R)}}{R^2} & 0 \\ 0 & 0 & \frac{r^2(R) e^{-2\omega_\Theta(R)}}{R^2} \end{pmatrix}. \tag{5.4}$$

[1]Similar constructions using nontrivial material manifolds with the explicit dependence of the material metric on the type of anelasticity were discussed in [20, 9, 24, 10, 25, 26].

[2]All the symbolic computations in this paper were performed using Mathematica [93].

Using (6.8), the invariants of the strain energy function are simplified to read[3]

$$I_1 = \text{tr}(\mathbf{C}) = \frac{2r^2(R)}{R^2}e^{-2\omega_\Theta(R)} + \frac{R^4}{r^4(R)}e^{4\omega_\Theta(R)}\,, \tag{5.5}$$

$$I_2 = \frac{1}{2}(\text{tr}(\mathbf{C}^2) - \text{tr}(\mathbf{C})^2) = \frac{2R^2}{r^2(R)}e^{2\omega_\Theta(R)} + \frac{r^4(R)}{R^4}e^{-4\omega_\Theta(R)}\,, \tag{5.6}$$

$$I_4 = \left(\frac{e^{\omega_\Theta(R)}R}{r(R)}\right)^4\,, \tag{5.7}$$

$$I_5 = \left(\frac{e^{\omega_\Theta(R)}R}{r(R)}\right)^8\,. \tag{5.8}$$

Following (6.14), the non-zero components of the Cauchy stress tensor read

$$\sigma^{rr} = -p + 2\left(\frac{e^{\omega_\Theta(R)}R}{r(R)}\right)^4 (W_{I_1} + W_{I_4}) + \left(\frac{2e^{\omega_\Theta(R)}R}{r(R)}\right)^2 W_{I_2} + 4W_{I_5}\left(\frac{e^{\omega_\Theta(R)}R}{r(R)}\right)^8\,, \tag{5.9}$$

$$\sigma^{\theta\theta} = \frac{2e^{-2\omega_\Theta(R)}W_{I_1}}{R^2} - \frac{p}{r^2(R)} + 2W_{I_2}\left(\frac{R^2e^{2\omega_\Theta(R)}}{r^4(R)} + \frac{r^2(R)}{R^4}e^{-4\omega_\Theta(R)}\right)\,, \tag{5.10}$$

$$\sigma^{\phi\phi} = \frac{1}{\sin^2\Theta}\sigma^{\theta\theta}\,. \tag{5.11}$$

Note that when the body is eigenstrain-free, $I_1 = I_2 = 3$ and $I_4 = I_5 = 1$. Assuming that the stress vanishes for this case, we obtain (similar conditions were derived in [92, 94])

$$(2W_{I_5} + W_{I_4})\,|_{I_1=I_2=3, I_4=I_5=1} = 0\,. \tag{5.12}$$

The physical components of the Cauchy stress tensor, i.e., $\hat{\sigma}^{ab} = \sigma^{ab}\sqrt{g_{aa}g_{bb}}$ (no summation) [67] are written as

$$\hat{\sigma}^{rr} = \sigma^{rr}\,, \quad \hat{\sigma}^{\theta\theta} = r^2(R)\sigma^{\theta\theta}\,, \quad \hat{\sigma}^{\phi\phi} = r^2(R)\sin^2\theta\,\sigma^{\phi\phi}\,. \tag{5.13}$$

[3]Note that $\hat{\mathbf{N}} = e^{-\omega_R(R)}\mathbf{E}_R$ is the unit vector defining the material preferred direction, where $\mathbf{E}_R = \frac{\partial}{\partial R}$ is a radial basis vector for $T_X\mathcal{B}$ such that $\langle\!\langle \mathbf{E}_R, \mathbf{E}_R \rangle\!\rangle_{\mathbf{G}} = G_{RR}$.

In the absence of body forces and inertial effects, the only non-trivial equilibrium equation is ${\sigma^{rb}}_{|b} = 0$ following (2.7). Note that $p = p(R)$ is implied from the other two equilibrium equations. Therefore

$$ {\sigma^{rr}}_{,r} + \frac{2}{r}\sigma^{rr} - r\sigma^{\theta\theta} - r\sin^2\theta\,\sigma^{\phi\phi} = 0\,. \tag{5.14} $$

Using (6.32), equation (5.14) is rewritten as

$$ \frac{1}{r'(R)}{\sigma^{rr}}_{,R} + \frac{2}{r}\sigma^{rr} - 2r\sigma^{\theta\theta} = 0\,. \tag{5.15} $$

Therefore, substituting (6.29) and (6.30) into (5.15), one obtains $p'(R) = h(R)$, where

$$ \begin{aligned} h(R) = -\frac{4e^{-2\omega_\Theta}}{R^3 r^{19}}\Big(& -8R^{18}W_{I_5I_5}r^3e^{18\omega_\Theta}\left(R\omega_\Theta'+1\right) + 8R^{17}\left(W_{I_4I_5}+W_{I_1I_5}\right)r^4e^{16\omega_\Theta+\omega_R} \\ & + 12R^{15}W_{I_2I_5}r^6e^{14\omega_\Theta+\omega_R} + 2R^{13}\left(3W_{I_5}+W_{I_4I_4}+2W_{I_1I_4}+W_{I_1I_1}\right)r^8e^{12\omega_\Theta+\omega_R} \\ & - 8R^{14}\left(W_{I_4I_5}+W_{I_1I_5}\right)r^7e^{14\omega_\Theta}\left(R\omega_\Theta'+1\right) - 12R^{12}W_{I_2I_5}r^9e^{12\omega_\Theta}\left(R\omega_\Theta'+1\right) \\ & + 2R^{11}\left(3W_{I_2I_4}-2W_{I_1I_5}+3W_{I_1I_2}\right)r^{10}e^{10\omega_\Theta+\omega_R} - 2R^5\left(W_{I_2I_4}+3W_{I_1I_2}\right)r^{16}e^{4\omega_\Theta+\omega_R} \\ & - 2R^8\left(3W_{I_2I_4}-2W_{I_1I_5}+3W_{I_1I_2}\right)r^{13}e^{8\omega_\Theta}\left(R\omega_\Theta'+1\right) + 8R^{21}W_{I_5I_5}e^{20\omega_\Theta+\omega_R} \\ & + 4W_{I_2I_2}r^{21}\left(R\omega_\Theta'+1\right) - 2R^6\left(W_{I_4}-2W_{I_2I_5}+2W_{I_2I_2}+W_{I_1}\right)r^{15}e^{6\omega_\Theta}\left(R\omega_\Theta'+1\right) \\ & - 2R^4\left(W_{I_2}-W_{I_1I_4}-W_{I_1I_1}\right)r^{17}e^{4\omega_\Theta}\left(R\omega_\Theta'+1\right) - R^3\left(4W_{I_2I_2}-W_{I_1}\right)r^{18}e^{2\omega_\Theta+\omega_R} \\ & - 2R^{10}\left(4W_{I_5}+W_{I_4I_4}+2W_{I_1I_4}+W_{I_1I_1}\right)r^{11}e^{10\omega_\Theta}\left(R\omega_\Theta'+1\right) + RW_{I_2}r^{20}e^{\omega_R} \\ & + 2R^2\left(W_{I_2I_4}+3W_{I_1I_2}\right)r^{19}e^{2\omega_\Theta(R)}\left(R\omega_\Theta'+1\right) + R^7\left(W_{I_2}-2\left(W_{I_1I_4}+W_{I_1I_1}\right)\right)r^{14}e^{6\omega_\Theta+\omega_R} \\ & + R^9\left(W_{I_4}-4W_{I_2I_5}+4W_{I_2I_2}+W_{I_1}\right)r^{12}e^{8\omega_\Theta+\omega_R}\Big)\,. \end{aligned} \tag{5.16} $$

If one assumes that the ball is subject to a uniform pressure p_∞ at its outer boundary, i.e, $\sigma^{rr}(R_o) = -p_\infty$, one obtains

$$p(R) = p_\infty + \int_{R_o}^{R} h(\zeta)d\zeta + 2\left(\frac{e^{\omega_\Theta(R_o)}R_o}{r(R_o)}\right)^4 (W_{I_1}|_{R=R_o} + W_{I_4}|_{R=R_o}) + \left(\frac{2e^{\omega_\Theta(R_o)}R_o}{r(R_o)}\right)^2 W_{I_2}|_{R=R_o} + 4W_{I_5}|_{R=R_o}\left(\frac{e^{\omega_\Theta(R_o)}R_o}{r(R_o)}\right)^8 . \quad (5.17)$$

Spherical inclusion in a transversely isotropic ball. Let us consider the following distributions of eigenstrains

$$\omega_R(R) = \begin{cases} \omega_1 , & 0 \leq R \leq R_i \\ 0 , & R_i \leq R \leq R_o \end{cases} , \quad \omega_\Theta(R) = \begin{cases} \omega_2 , & 0 \leq R \leq R_i \\ 0 , & R_i \leq R \leq R_o \end{cases} . \quad (5.18)$$

This corresponds to having an inclusion with radius R_i at the center of the ball. It follows from (5.2) that

$$r(R) = \begin{cases} e^{\frac{\omega_1}{3}+\frac{2\omega_2}{3}}R , & 0 \leq R \leq R_i \\ \left(R^3 + \left(e^{\omega_1+2\omega_2} - 1\right) R_i^3\right)^{\frac{1}{3}} , & R_i \leq R \leq R_o \end{cases} . \quad (5.19)$$

Using (5.16) and (5.19), one has $p'(R) = h_0/R$ in the inclusion ($0 \leq R \leq R_i$), where

$$h_0 = 4e^{-\frac{8\omega_1}{3}-\frac{4\omega_2}{3}}\Big(2e^{4\omega_2}W_{I_5} + e^{\frac{4\omega_1}{3}+\frac{8\omega_2}{3}}W_{I_4} - e^{4\omega_1}W_{I_2} + e^{2\omega_1+2\omega_2}W_{I_2} - e^{\frac{10\omega_1}{3}+\frac{2\omega_2}{3}}W_{I_1} + e^{\frac{4\omega_1}{3}+\frac{8\omega_2}{3}}W_{I_1}\Big)\Big|_{I_1=2I_a+I_a^{-2},I_2=2I_a^{-1}+I_a^2,I_4^2=I_5=I_a^{-4}} , \quad (5.20)$$

and $I_a = e^{\frac{2}{3}(\omega_1-\omega_2)}$. Moreover, in the matrix, $p'(R) = \hat{h}(R)$, where for $R_i \leq R \leq R_o$

$$\begin{aligned}\hat{h}(R) = -\frac{4}{R^3 r(R)^{19}}\Big(& -8R^{18}W_{I_5I_5}r^3 + 8R^{17}\left(W_{I_4I_5}+W_{I_1I_5}\right)r^4 - 8R^{14}\left(W_{I_4I_5}+W_{I_1I_5}\right)r^7 \\ & + 2R^{13}\left(3W_{I_5}+W_{I_4I_4}+2W_{I_1I_4}+W_{I_1I_1}\right)r^8 - 12R^{12}W_{I_2I_5}r^9 + 4W_{I_2I_2}r^{21} + 8R^{21}W_{I_5I_5} \\ & + 2R^{11}\left(3W_{I_2I_4}-2W_{I_1I_5}+3W_{I_1I_2}\right)r^{10} - 2R^{10}\left(4W_{I_5}+W_{I_4I_4}+2W_{I_1I_4}+W_{I_1I_1}\right)r^{11} \\ & + R^9\left(W_{I_4}-4W_{I_2I_5}+4W_{I_2I_2}+W_{I_1}\right)r^{12} - 2R^8\left(3W_{I_2I_4}-2W_{I_1I_5}+3W_{I_1I_2}\right)r^{13} + RW_{I_2}r^{20} \\ & + R^7\left(W_{I_2}-2\left(W_{I_1I_4}+W_{I_1I_1}\right)\right)r^{14} - 2R^6\left(W_{I_4}-2W_{I_2I_5}+2W_{I_2I_2}+W_{I_1}\right)r^{15} \\ & + 2R^4\left(W_{I_1I_4}+W_{I_1I_1}-W_{I_2}\right)r^{17} + R^3\left(W_{I_1}-4W_{I_2I_2}\right)r^{18} + 2R^2\left(W_{I_2I_4}+3W_{I_1I_2}\right)r^{19} \\ & + 12R^{15}W_{I_2I_5}r^6 - 2R^5\left(W_{I_2I_4}+3W_{I_1I_1}\right)r^{16}\Big). \end{aligned} \tag{5.21}$$

Therefore, the pressure field distribution is given by

$$p(R) = \begin{cases} h_0 \ln\left(\frac{R}{R_i}\right) - c_i\,, & 0 \leq R \leq R_i\,, \\ \int_{R_o}^{R} \hat{h}(\zeta)d\zeta - c_o\,, & R_i \leq R \leq R_o\,, \end{cases} \tag{5.22}$$

where c_i and c_o are constants of integration to be determined after imposing the boundary conditions. The physical components of the Cauchy stress have the following distributions

$$\hat{\sigma}^{rr}(R) = \begin{cases} h_0 \ln\left(\frac{R_i}{R}\right) + 2e^{\frac{2}{3}(\omega_2-4\omega_1)}\Big(2e^{2\omega_2}W_{I_5} + e^{\frac{2}{3}(2\omega_1+\omega_2)}W_{I_4} + 2e^{2\omega_1}W_{I_2} \\ \qquad + e^{\frac{2}{3}(2\omega_1+\omega_2)}W_{I_1}\Big)\Big|_{I_1=2I_a+I_a^{-2},I_2=2I_a^{-1}+I_a^2,I_4^2=I_5=I_a^{-4}} + c_i\,, & 0 \leq R \leq R_i\,, \\ c_o + \Big(\int_{R}^{R_o} \hat{h}(\zeta)d\zeta + 2\frac{R^4}{r^4}\left(W_{I_4}+W_{I_1}\right) + 4\frac{R^2}{r^2}W_{I_2} \\ \qquad + 4\frac{R^8}{r^8}W_{I_5}\Big)\Big|_{I_1=2I^{-2}(R)+I^4(R),I_2=2I^2(R)+I^{-4}(R),I_4^2=I_5=I^8(R)}\,, & R_i \leq R \leq R_o\,, \end{cases} \tag{5.23}$$

$$\hat{\sigma}^{\theta\theta}(R) = \begin{cases} h_0 \ln\left(\frac{R_i}{R}\right) + c_i + 2e^{-\frac{2}{3}(\omega_1+2\omega_2)}\Big[\left(e^{2\omega_1}+e^{2\omega_2}\right) W_{I_2} \\ \qquad\qquad + e^{\frac{2}{3}(2\omega_1+\omega_2)} W_{I_1}\Big]\Big|_{I_1=2I_a+I_a^{-2}, I_2=2I_a^{-1}+I_a^2, I_4^2=I_5=I_a^{-4}}\,, & 0 \le R \le R_i\,, \\ c_o + \Big[\int_R^{R_o} \hat{h}(\zeta)d\zeta + 2W_{I_2}\left(\frac{r^4}{R^4}+\frac{R^2}{r^2}\right) \\ \qquad\qquad + \frac{2W_{I_1} r^2}{R^2}\Big]\Big|_{I_1=2I^{-2}(R)+I^4(R), I_2=2I^2(R)+I^{-4}(R), I_4^2=I_5=I^8(R)}\,, & R_i \le R \le R_o\,, \end{cases} \tag{5.24}$$

where $I(R) = R/r(R)$, and note that $\hat{\sigma}^{\theta\theta}(R) = \hat{\sigma}^{\phi\phi}(R)$. The boundary condition $\sigma^{rr}(R_o) = -p_\infty$ gives us

$$\begin{aligned} c_o = -p_\infty - \Big(&\frac{2R_o^4}{r^4(R_o)}\left(W_{I_4}+W_{I_1}\right) + \frac{4R_o^2}{r^2(R_o)} W_{I_2} \\ &+ \frac{4R_o^8}{r^8(R_o)} W_{I_5}\Big)\Big|_{I_1=2I^{-2}(R_o)+I^4(R_o), I_2=2I^2(R_o)+I^{-4}(R_o), I_4^2=I_5=I^8(R_o)}\,. \end{aligned} \tag{5.25}$$

The continuity of the traction vector at the inclusion-matrix interface implies that σ^{rr} must be continuous at $R = R_i$. Using the expression for c_o in (5.25), this condition gives c_i as

$$\begin{aligned} c_i = &\int_{R_i}^{R_o} \hat{h}(\zeta)d\zeta + 2e^{-\frac{8}{3}(\omega_1+2\omega_2)}\Big(e^{\frac{4}{3}(\omega_1+2\omega_2)}\left(2e^{\frac{2}{3}(\omega_1+2\omega_2)} W_{I_2} + W_{I_4} + W_{I_1}\right) \\ &+ 2W_{I_5}\Big)\Big|_{I_1=2I_b^{-2}+I_b^4, I_2=2I_b^2+I_b^{-4}, I_4^2=I_5=I_b^8} - 2e^{\frac{2}{3}(\omega_2-4\omega_1)}\Big(2e^{2\omega_2} W_{I_5} + e^{\frac{2}{3}(2\omega_1+\omega_2)} W_{I_4} \\ &+ 2e^{2\omega_1} W_{I_2} + e^{\frac{2}{3}(2\omega_1+\omega_2)} W_{I_1}\Big)\Big|_{I_1=2I_a+I_a^{-2}, I_2=2I_a^{-1}+I_a^2, I_4^2=I_5=I_a^{-4}} - p_\infty - \Big(\frac{2R_o^4}{r^4(R_o)}\left(W_{I_4}+W_{I_1}\right) \\ &+ \frac{4R_o^2}{r^2(R_o)} W_{I_2} + \frac{4R_o^8}{r^8(R_o)} W_{I_5}\Big)\Big|_{I_1=2I^{-2}(R_o)+I^4(R_o), I_2=2I^2(R_o)+I^{-4}(R_o), I_4^2=I_5=I^8(R_o)}\,, \end{aligned} \tag{5.26}$$

where $I_b = I(R_i) = e^{-\frac{1}{3}(\omega_1+2\omega_2)}$.

Remark 5.2.1. Evidently, if $h_0 = 0$, from (5.23) and (5.24), the stress field in the inclusion will be uniform and hydrostatic. Note that when $\omega_1 = \omega_2$, one has $I_a = 1$, and hence, from (5.20) and (6.101)

$$h_0 = 4\Big(2W_{I_5} + W_{I_4}\Big)\Big|_{I_1=I_2=3, I_4=I_5=1} = 0\,. \tag{5.27}$$

Therefore, if $\omega_1 = \omega_2$, then $h_0 = 0$ for any nonlinear incompressible transversely isotropic solid. If $\omega_1 \neq \omega_2$, however, for h_0 to be zero the strain energy function must satisfy the following condition, which in turn puts a restriction on the energy function (cf. (5.20))

$$\begin{aligned}\Big(2e^{4\omega_2}W_{I_5} + e^{\frac{4\omega_1}{3}+\frac{8\omega_2}{3}}W_{I_4} - e^{4\omega_1}W_{I_2} + e^{2\omega_1+2\omega_2}W_{I_2} \\ - e^{\frac{10\omega_1}{3}+\frac{2\omega_2}{3}}W_{I_1} + e^{\frac{4\omega_1}{3}+\frac{8\omega_2}{3}}W_{I_1}\Big)\Big|_{I_1=2I_a+I_a^{-2}, I_2=2I_a^{-1}+I_a^2, I_4^2=I_5=I_a^{-4}} = 0\,.\end{aligned} \tag{5.28}$$

Therefore, we have proved the following proposition.

Proposition 5.2.2. *Consider a nonlinear incompressible transversely isotropic spherical ball such that the material preferred direction is radial. Suppose that the ball is subject to a uniform pressure on its boundary. Assume that the ball contains a spherical inclusion at its center with uniform radial and circumferential eigenstrains. The stress field inside the inclusion exhibits a logarithmic singularity at the center of the ball unless the radial and circumferential eigenstrains are equal or the energy function satisfies* (5.28)*. Moreover, the stress inside the inclusion is uniform and hydrostatic if the eigenstrains are pure dilatational.*

Remark 5.2.3. Given a nonlinear incompressible transversely isotropic spherical ball with the radial material preferred direction and a radially-symmetric distribution of radial and circumferential eigenstrains $e^{\omega_R(R)}$ and $e^{\omega_\Theta(R)}$, respectively, the stress exhibits a logarithmic singularity at the center of the ball unless $\omega_R(0) = \omega_\Theta(0)$. To see this, let $\omega_R(0) = \omega_1$ and $\omega_\Theta(0) = \omega_2$. Note that as $R \to 0$ (see also [65])

$$\omega_R(R) = \omega_1 + \mathcal{O}(R)\,, \qquad \omega_\Theta(R) = \omega_2 + \mathcal{O}(R)\,, \qquad r(R) = e^{\frac{\omega_1}{3}+\frac{2\omega_2}{3}}R + \mathcal{O}(R^2)\,. \tag{5.29}$$

Moreover

$$I_1(R) = 2I_a + I_a^{-2} + \mathcal{O}(R)\,,\;\; I_2(R) = 2I_a^{-1} + I_a^2 + \mathcal{O}(R)\,,\;\; I_4(R) = I_a^{-2} + \mathcal{O}(R)\,,\;\; I_5(R) = I_a^{-4} + \mathcal{O}(R)\,. \tag{5.30}$$

Thus

$$W_{I_i}(R) = W_{I_i}\big|_{I_1=2I_a+I_a^{-2},I_2=2I_a^{-1}+I_a^2,I_4^2=I_5=I_a^{-4}} + \mathcal{O}(R)\,, \quad i = 1,2,4,5\,. \tag{5.31}$$

Similarly

$$W_{I_iI_j}(R) = W_{I_iI_j}\big|_{I_1=2I_a+I_a^{-2},I_2=2I_a^{-1}+I_a^2,I_4^2=I_5=I_a^{-4}} + \mathcal{O}(R)\,, \quad i,j = 1,2,4,5\,. \tag{5.32}$$

Therefore, using the above asymptotic expansions, from (5.16), one obtains

$$h(R) = \frac{h_0}{R} + \mathcal{O}(1)\,. \tag{5.33}$$

Hence, $p(R) = h_0 \ln R + \mathcal{O}(R)$, i.e., the stress field has a logarithmic singularity at the origin only if $\omega_R(0) \neq \omega_\Theta(0)$.

5.2.2 Finite Eigenstrains in a Compressible Transversely Isotropic Spherical Ball

Next, we consider a compressible transversely isotropic material with a radial material preferred direction. Given an embedding of the form $(r,\theta,\phi) = (r\,(R)\,,\Theta,\Phi)$, the right Cauchy-Green deformation tensor reads

$$\mathbf{C} = \begin{pmatrix} r'(R)^2 e^{-2\omega_R(R)} & 0 & 0 \\ 0 & \frac{r^2(R)e^{-2\omega_\Theta(R)}}{R^2} & 0 \\ 0 & 0 & \frac{r^2(R)e^{-2\omega_\Theta(R)}}{R^2} \end{pmatrix}. \tag{5.34}$$

The Jacobean is written as

$$J = \sqrt{\frac{\det\mathbf{g}}{\det\mathbf{G}}}\det\mathbf{F} = \frac{r^2(R)r'(R)}{R^2 e^{\omega_R(R)+2\omega_\Theta(R)}}\,. \tag{5.35}$$

The invariants are found using (6.8) and read

$$I_1 = \text{tr}(\mathbf{C}) = r'(R)^2 e^{-2\omega_R(R)} + \frac{2r^2(R)e^{-2\omega_\Theta(R)}}{R^2}\,, \tag{5.36}$$

$$I_2 = \frac{1}{2}(\text{tr}(\mathbf{C}^2) - \text{tr}(\mathbf{C})^2) = \frac{r^4(R)}{R^4} r'(R)^2 e^{-2(2\omega_\Theta(R)+\omega_R(R))}\left(\frac{e^{2\omega_R(R)}}{r'(R)^2} + \frac{2R^2 e^{2\omega_\Theta(R)}}{r(R)^2}\right)\,, \tag{5.37}$$

$$I_3 = \det(\mathbf{C}) = \frac{r^4(R)}{R^4} r'(R)^2 e^{-2(2\omega_\Theta(R)+\omega_R(R))}\,, \tag{5.38}$$

$$I_4 = e^{-2\omega_R(R)} r'(R)^2\,, \tag{5.39}$$

$$I_5 = e^{-4\omega_R(R)} r'(R)^4\,. \tag{5.40}$$

The non-zero components of the Cauchy stress tensor are written as

$$\sigma^{rr} = \frac{2r'(R)e^{-2\omega_\Theta(R)-3\omega_R(R)}}{R^2 r^2(R)}\Big[R^4 e^{4\omega_\Theta(R)}\left(e^{2\omega_R(R)}\left(W_{I_1} + W_{I_4}\right) + 2W_{I_5} r'(R)^2\right) + 2R^2 W_{I_2} r^2(R) e^{2(\omega_\Theta(R)+\omega_R(R))} + W_{I_3} r^4(R) e^{2\omega_R(R)}\Big]\,, \tag{5.41}$$

$$\sigma^{\theta\theta} = \frac{2e^{-\omega_R(R)}}{r'(R)r^2(R)}\left[W_{I_1} e^{2\omega_R(R)} + W_{I_2} r'(R)^2 + \frac{e^{-2\omega_\Theta(R)} r^2(R)}{R^2}\left(W_{I_2} e^{2\omega_R(R)} + W_{I_3} r'(R)^2\right)\right]\,, \tag{5.42}$$

$$\sigma^{\phi\phi} = \frac{1}{\sin^2\Theta}\sigma^{\theta\theta}\,. \tag{5.43}$$

When the body is eigenstrain-free, we assume that the stress vanishes. Therefore

$$(W_{I_4} + 2W_{I_5})\,|_{I_1=I_2=3, I_3=I_4=I_5=1} = 0\,, \quad \text{and} \quad (W_{I_1} + 2W_{I_2} + W_{I_3})\,|_{I_1=I_2=3, I_3=I_4=I_5=1} = 0\,. \tag{5.44}$$

Substituting the stress components into (5.14), the simplified radial equilibrium equation is given in Appendix B.1.

Next, we consider the eigenstrain distribution (5.18) and solve the problem of a spherical inclusion with uniform anisotropic eigenstrains in a compressible transversely isotropic spherical ball. We then explore conditions under which the induced stress field in the inclusion is uniform. These conditions would impose some restrictions on the energy function, in general. Let us assume that the stress field in the inclusion is uniform, i.e., $\hat{\sigma}^{rr} = C_1$ and $\hat{\sigma}^{\theta\theta} = C_2$, where C_1 and C_2 are constants. It then follows from (5.41) and (5.42) for $0 \leq R \leq R_i$ that:

$$\begin{aligned} C_1 = \frac{2e^{-2\omega_2-3\omega_1} r'(R)}{R^2 r(R)^2} \Big[& R^4 e^{4\omega_2} \left(e^{2\omega_1} \left(W_{I_1} + W_{I_4} \right) + 2W_{I_5} r'(R)^2 \right) \\ & + 2R^2 e^{2(\omega_1+\omega_2)} W_{I_2} r(R)^2 + e^{2\omega_1} W_{I_3} r(R)^4 \Big]\,, \end{aligned} \tag{5.45}$$

and

$$C_2 = \frac{2e^{-\omega_1-2\omega_2}}{R^2 r'(R)} \left[R^2 e^{2(\omega_1+\omega_2)} W_{I_1} + r(R)^2 \left(e^{2\omega_1} W_{I_2} + W_{I_3} r'(R)^2 \right) + R^2 e^{2\omega_2} W_{I_2} r'(R)^2 \right]. \tag{5.46}$$

The first-order[4] nonlinear ODEs (5.45) and (5.46) are subject to the boundary condition $r(0) = 0$. We note that for $r(R) = \beta R$ in the inclusion, with β a constant, all the invariants of deformation are constant in the inclusion, and so are the partial derivatives of the energy function with respect to the invariants. Therefore, one can immediately see that $r(R) = \beta R$ is a solution of both initial-value problems (IVPs).[5] That is, the stress field in the inclusion is uniform if $r(R) = \beta R$ for $0 \leq R \leq R_i$.

[4]Note that the invariants of deformation, and thus, the energy function and its partial derivatives with respect to the invariants depend on the first and not higher order derivatives of r.

[5]Note that it is straightforward to show that there are no other solutions of the form $r(R) = \beta R^\alpha$, $\alpha > 1$ to these IVPs.

Note that when the stress in the inclusion is uniform, it then immediately follows from the equilibrium equation (5.15) that the stress is hydrostatic as well, i.e., $C_1 = C_2$. Now, we examine the conditions that guarantee that $r(R) = \beta R$ satisfies the radial equilibrium equation ($C_1 = C_2$). Using (5.45) and (5.46), one obtains the following condition in the inclusion.

$$\Big[e^{2\omega_1}\left(e^{2\omega_2}-e^{2\omega_1}\right)\left(\beta^2 W_{I_2}+e^{2\omega_2}W_{I_1}\right)+e^{4\omega_2}\left(2\beta^2 W_{I_5}\right.\\ \left.+e^{2\omega_1}W_{I_4}\right)\Big]\Big|_{I_1=\beta^2\left(e^{-2\omega_1}+2e^{-2\omega_2}\right),I_2=\beta^4e^{-4\omega_2}\left(2e^{2\omega_2-2\omega_1}+1\right),I_3=\beta^6e^{-2(\omega_1+2\omega_2)},I_4^2=I_5=e^{-4\omega_1}\beta^4}=0\,. \tag{5.47}$$

Note that when the radial and circumferential eigenstrains are equal ($\omega_1 = \omega_2$), the above condition is satisfied without imposing any restrictions on the energy function or β if the material is compressible and isotropic, i.e., $W = W(I_1, I_2, I_3)$, and hence, $W_{I_4} = W_{I_5} = 0$. This observation suggests that if the material is compressible and isotropic, and the inclusion has a uniform distribution of pure dilatational eigenstrains, then the stress inside the inclusion is uniform and hydrostatic. This generalizes the result of Yavari and Goriely [24] that was proved for harmonic solids and class II and III materials according to Carroll [95]. For compressible isotropic solids, (B.1) gives us the following second-order nonlinear ODE in the matrix (for $R_i \leq R \leq R_o$)

$$\begin{aligned}
&R^5\left(R^4W_{I_1}r''+2R^2r^2W_{I_2}r''+r^4W_{I_3}r''-2R^2rW_{I_1}-2r^3W_{I_2}\right)\\
&+4rr'^3\left(Rr'-r\right)\left[R^4r^2\left(2W_{I_2I_2}+W_{I_1I_3}\right)+3R^2r^4W_{I_2I_3}+r^6W_{I_3I_3}+R^6W_{I_1I_2}\right]\\
&-r'\left[4R^6r^2W_{I_1I_1}+2R^4r^4\left(6W_{I_1I_2}+W_{I_3}\right)+4R^2r^6\left(2W_{I_2I_2}+W_{I_1I_3}\right)+4r^8W_{I_2I_3}-2R^8W_{I_1}\right]\\
&+2Rr'^2\Big\{R^8W_{I_1I_1}r''+4R^6r^2W_{I_1I_2}r''+2R^4r^4\left(2W_{I_2I_2}+W_{I_1I_3}\right)r''+4R^2r^6W_{I_2I_3}r''+r^8W_{I_3I_3}r''\\
&+2r^7W_{I_2I_3}+R^6r\left(2W_{I_1I_1}+W_{I_2}\right)+R^4r^3\left(6W_{I_1I_2}+W_{I_3}\right)+2R^2r^5\left(2W_{I_2I_2}+W_{I_1I_3}\right)\Big\}=0\,,
\end{aligned} \tag{5.48}$$

for which we need two boundary conditions, and given that β is also an unknown, we need three boundary conditions in total. These are given by continuity of $r(R)$ and the traction vector at $R = R_i$, and the boundary condition $\hat{\sigma}^{rr}(R_o) = -p_\infty$. Therefore, we have proved the following proposition.

Proposition 5.2.4. *Consider a spherical ball made of a compressible isotropic solid subject to a uniform pressure on its boundary sphere. Assume that the ball contains a spherical inclusion at its center with uniform radial and circumferential eigenstrains. The stress field in the inclusion is uniform and hydrostatic if the eigenstrains are pure dilatational.*

Remark 5.2.5. Consider the conditions in Proposition 5.2.4 for compressible isotropic solids and assume that the stress field inside the inclusion is uniform. We observed that $r(R) = \beta R$, where $0 \leq R \leq R_i$ is a solution for (5.45) and (5.46) subject to the boundary condition $r(0) = 0$. Therefore, the simplified equilibrium equation (5.47) implies that the radial and circumferential eigenstrains must be equal. Otherwise, from (5.47) the energy function and β must satisfy the following relation

$$\left[\beta^2 W_{I_2} + e^{2\omega_2} W_{I_1}\right]\Big|_{I_1=\beta^2\left(e^{-2\omega_1}+2e^{-2\omega_2}\right), I_2=\beta^4 e^{-4\omega_2}\left(2e^{2\omega_2-2\omega_1}+1\right), I_3=\beta^6 e^{-2(\omega_1+2\omega_2)}} = 0\,. \tag{5.49}$$

The boundary conditions and the above relation in turn put a restriction on the energy function.

For a compressible transversely isotropic material if the radial and circumferential eigenstrains are equal ($\omega_1 = \omega_2 = \omega$), from (5.47), we obtain

$$\left(W_{I_4} + 2a^2 W_{I_5}\right)\big|_{I_1=3a^2, I_2=3a^4, I_3=a^6, I_4^2=I_5=a^4} = 0\,, \tag{5.50}$$

where $a = \beta e^{-\omega}$. Clearly, from the first equation in (5.44), $a = 1$ is a trivial solution of the above equation, which is stress-free and volume preserving ($I_3 = 1$). If we assume that the traction in the fiber direction is tensile for extension ($a > 1$) and compressive for contraction ($a < 1$), e.g., see [96], then $a = 1$ is the only solution of (5.50). This result simply suggests that for compressible transversely isotropic materials the induced stress field inside the inclusion with uniform pure dilatational eigenstrains is uniform in the trivial case $R_i = R_o$, i.e., when the entire ball has a uniform distribution of pure dilatational eigenstrains, which is stress-free.

Nonetheless, there are some nontrivial cases that can only occur if the radial and circumferential eigenstrains are different ($\omega_1 \neq \omega_2$). Such cases are special in the sense that a specific pressure must be applied on the boundary to maintain a uniform hydrostatic stress field inside the inclusion, or for a given pressure applied on the outer boundary, the ratio R_i/R_o is determined. This is because β is determined from (5.47) when ($\omega_1 \neq \omega_2$), and as the equilibrium equation in the matrix is a nonlinear second-order ODE, we only need two boundary conditions to find its solution. These are given by the continuity of $r(R)$ and the traction vector at $R = R_i$. To see this, we note that when β is determined from (5.47), the stress and deformation fields in the inclusion will be fully known. Therefore, the two boundary conditions of the equilibrium equation in the matrix are written as

$$r(R_i^+) = \beta R_i \,, \qquad \hat{\sigma}^{rr}(R_i^+) = \hat{\sigma}^{rr}(R_i^-) \,. \tag{5.51}$$

Hence, one may fix R_i/R_o and find the pressure that must be applied on the outer boundary using the relation $\hat{\sigma}^{rr}(R_o) = -p_\infty$. Alternatively, using this relation, one can find R_i/R_o by prescribing the pressure p_∞.

Next, we consider some specific strain energy functions to explore (5.47), where a choice of energy function determines β when $\omega_1 \neq \omega_2$. In doing so, we employ the so called *standard reinforcing model* for compressible materials, defined as [92, 97]

$$W = W\left(I_1, I_2, I_3, I_4, I_5\right) = W_{\text{iso}}\left(I_1, I_2, I_3\right) + W_{\text{fib}}\left(I_4, I_5\right) \,, \tag{5.52}$$

where the first term denotes the isotropic base material, whereas the second term represents the anisotropic effects due to the fiber reinforcement. Let us consider the following strain energy functions (see, e.g., [97]):

i) Compressible Mooney-Rivlin reinforced model (I_4 reinforcement) for which

$$W(I_1, I_2, I_3, I_4) = C_1\left(I_1 - 3\right) + C_2\left(I_2 - 3\right) - \left(C_1 + 2C_2\right)\left(I_3 - 1\right) + \frac{\mu}{2}\left(I_4 - 1\right)^2 \,, \tag{5.53}$$

where C_1, C_2, and μ are constants, and $\mu > 0$ is an anisotropy parameter describing the reinforcement property. Therefore, from (5.47), we have

$$\beta = e^{\omega_1+\omega_2}\left[\frac{C_1\delta+\mu e^{2\omega_2}}{\mu e^{4\omega_2}-C_2e^{2\omega_1}\delta}\right]^{\frac{1}{2}}, \tag{5.54}$$

where $\delta = e^{2\omega_1} - e^{2\omega_2}$. We need to have the following constraint on μ for β to be a real positive number.

$$\begin{aligned}
&\mu > C_2 e^{2(\omega_1-\omega_2)}\left(e^{2(\omega_1-\omega_2)}-1\right), \quad \text{for} \quad \omega_1 > \omega_2\,, \\
&\mu > C_1\left(1-e^{2(\omega_1-\omega_2)}\right), \qquad\qquad \text{for} \quad \omega_1 < \omega_2\,.
\end{aligned} \tag{5.55}$$

As expected, the stress field in the inclusion is uniform and hydrostatic, i.e., $\hat{\sigma}^{rr} = \hat{\sigma}^{\theta\theta} = \hat{\sigma}^{\phi\phi} = \sigma_o$, where

$$\begin{aligned}
\sigma_o = -&\frac{2\delta e^{\omega_2}}{\left(\mu e^{4\omega_2}-C_2\delta e^{2\omega_1}\right)^2}\sqrt{\frac{\mu e^{4\omega_2}-C_2\delta e^{2\omega_1}}{C_1\delta+\mu e^{2\omega_2}}}\Big[C_1\left(C_2\mu e^{2\omega_2}\left(\delta+4e^{2\omega_1}\right)+C_2^2\delta e^{2\omega_1}+\mu^2e^{4\omega_2}\right) \\
&+2C_1^2e^{2\omega_1}\left(C_2\delta+\mu e^{2\omega_2}\right)+C_2\mu\left(C_2e^{2\omega_1}\left(\delta+2e^{2\omega_2}\right)+\mu e^{4\omega_2}\right)+C_1^3\delta e^{2\omega_1}\Big].
\end{aligned} \tag{5.56}$$

ii) Compressible Mooney-Rivlin reinforced model (I_5 reinforcement) that has the following energy function

$$W(I_1,I_2,I_3,I_5) = C_1\left(I_1-3\right)+C_2\left(I_2-3\right)-\left(C_1+2C_2\right)\left(I_3-1\right)+\frac{\mu}{2}\left(I_5-1\right)^2. \tag{5.57}$$

Substituting (5.57) into (5.47) gives us

$$2\beta^2\mu e^{4\omega_2}\left(\beta^4e^{-4\omega_1}-1\right)-e^{2\omega_1}\left(e^{2\omega_1}-e^{2\omega_2}\right)\left(C_1e^{2\omega_2}+\beta^2C_2\right)=0\,. \tag{5.58}$$

Therefore

$$\beta = \frac{e^{-2\omega_2}}{6^{\frac{1}{3}}\mu^{\frac{1}{2}}}\left(\frac{6^{\frac{1}{3}}\mu e^{4\omega_2}}{\Delta}\left(C_2\delta e^{2\omega_1}+2\mu e^{4\omega_2}\right)+e^{4\omega_1}\Delta\right)^{\frac{1}{2}}, \tag{5.59}$$

where Δ is defined as[6]

$$\Delta = e^{2(\omega_2-\omega_1)}\left[\sqrt{3}\mu^{3/2}\sqrt{27C_1^2\delta^2\mu e^{8\omega_2}-2\left(C_2\delta e^{2\omega_1}+2\mu e^{4\omega_2}\right)^3}+9C_1\delta\mu^2 e^{4\omega_2}\right]^{\frac{1}{3}}. \quad (5.60)$$

The value of the hydrostatic stress in the inclusion is

$$\sigma_o = \frac{e^{-7\omega_1-2\omega_2}}{\beta}\left[2C_1 e^{6\omega_1}\left(e^{4\omega_2}-\beta^4\right)+4\beta^2\left\{C_2 e^{6\omega_1}\left(e^{2\omega_2}-\beta^2\right)-\mu e^{4\omega_2}\left(e^{4\omega_1}-\beta^4\right)\right\}\right]. \quad (5.61)$$

iii) Blatz-Ko reinforced model (I_4 reinforcement) for which the energy function is written as

$$W(I_2,I_3,I_4) = \frac{\mu_o}{2}\left(\frac{I_2}{I_3}+2I_3^{\frac{1}{2}}-5\right)+\frac{\mu}{2}\left(I_4-1\right)^2, \quad (5.62)$$

where $\mu_1,\mu_2>0$. From (5.47), we have

$$\mu_o e^{4\omega_1}\delta+2\beta^4\mu\left(e^{2\omega_1}-\beta^2\right)=0. \quad (5.63)$$

Hence

$$\beta = \frac{1}{\sqrt{3}}\left(\frac{2^{2/3}\mu e^{4\omega_1}}{\eta}+\frac{\eta}{2^{2/3}\mu}+e^{2\omega_1}\right)^{\frac{1}{2}}, \quad (5.64)$$

where η is given by

$$\eta = e^{\frac{4\omega_1}{3}}\mu^{\frac{2}{3}}\left[3\sqrt{3\mu_o}\sqrt{27\delta^2\mu_o+8\mu\delta e^{2\omega_1}}+27\delta\mu_o+4\mu e^{2\omega_1}\right]^{\frac{1}{3}}. \quad (5.65)$$

For β to be physical, i.e., $\beta\in\mathbb{R}^+$, it can be shown that one must have $\omega_1>\omega_2$. In that case, the hydrostatic stress in the inclusion reads

$$\sigma_o = \frac{e^{2\omega_2}}{\beta^5}\left[\mu_o\left(\beta^5 e^{-2\omega_2}-e^{3\omega_1}\right)-2e^{-3\omega_1}\beta^4\mu\left(e^{2\omega_1}-\beta^2\right)\right]. \quad (5.66)$$

[6]Note that $\frac{6^{\frac{1}{3}}\mu e^{4\omega_2}}{\Delta}\left(C_2\delta e^{2\omega_1}+2\mu e^{4\omega_2}\right)+e^{4\omega_1}\Delta>0$ puts a constraint on the elastic constants.

5.2.3 Finite Eigenstrains in a Finite Incompressible Orthotropic Cylindrical Bar

Let us consider a finite circular cylindrical bar of radius R_o made of a nonlinear incompressible orthotropic solid with a cylindrically-symmetric distribution of radial and circumferential eigenstrains in the reference configuration. Assume that the material orthotropic axes are in the R, Θ, and Z directions in the cylindrical coordinates (R, Θ, Z). Given the eigenstrain-free material metric, i.e., $\mathbf{G}_o = \mathrm{diag}\,(1, R^2, 1)$, the material metric for the bar with eigenstrains is written as

$$\mathbf{G} = \begin{pmatrix} e^{2\omega_R(R)} & 0 & 0 \\ 0 & R^2 e^{2\omega_\Theta(R)} & 0 \\ 0 & 0 & e^{2\omega_Z(R)} \end{pmatrix}, \tag{5.67}$$

where ω_R, ω_Θ, and ω_Z are some functions describing the radial, circumferential, and axial eigenstrains, respectively. The ambient space is endowed with the Euclidean metric $\mathbf{g} = (1, r^2, 1)$. We embed the material manifold into the ambient space by looking for mappings of the form $(r, \theta, z) = (r(R), \Theta, \alpha Z)$, where α is a constant representing the axial stretch of the bar that depends on the axial boundary conditions.[7] Therefore, the deformation gradient reads $\mathbf{F} = \mathrm{diag}\,(r'(R), 1, \alpha)$. Incompressibility constraint is written as

$$J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}}\det \mathbf{F} = \frac{\alpha r(R) r'(R)}{R e^{\omega_R(R)+\omega_\Theta(R)+\omega_Z(R)}} = 1\,. \tag{5.68}$$

Requiring $r(0) = 0$, one obtains

$$r(R) = \left(\int_0^R \frac{2\eta}{\alpha} e^{\omega_R(\eta)+\omega_\Theta(\eta)+\omega_Z(\eta)} d\eta\right)^{\frac{1}{2}}. \tag{5.69}$$

[7] Note that mappings of this form correspond to the bar being subject to a displacement control loading with the axial stretch α.

The right Cauchy-Green deformation tensor reads

$$\mathbf{C} = \begin{pmatrix} \frac{e^{2\omega_Z(R)+2\omega_\Theta(R)}R^2}{\alpha^2 r^2(R)} & 0 & 0 \\ & & \\ 0 & \frac{e^{-2\omega_\Theta(R)}r^2(R)}{R^2} & 0 \\ & & \\ 0 & 0 & e^{-2\omega_Z(R)}\alpha^2 \end{pmatrix} . \tag{5.70}$$

Let us denote the orthotropic axes by $\mathbf{N}_1 = \hat{\mathbf{R}}$, $\mathbf{N}_2 = \hat{\mathbf{Z}}$, and $\mathbf{N}_3 = \hat{\boldsymbol{\Theta}}$, where $\hat{\mathbf{R}}$, $\hat{\mathbf{Z}}$, and $\hat{\boldsymbol{\Theta}}$ denote the unit vectors in the radial, longitudinal, and circumferential directions, respectively. Thus, $\mathbf{N}_1 = e^{-\omega_R(R)}\mathbf{E}_R$, $\mathbf{N}_2 = e^{-\omega_Z(R)}\mathbf{E}_Z$, and $\mathbf{N}_3 = e^{-\omega_\Theta(R)}\mathbf{E}_\Theta/R$, where $\mathbf{E}_R = \partial/\partial R$, $\mathbf{E}_Z = \partial/\partial Z$, and $\mathbf{E}_\Theta = \partial/\partial\Theta$ form a basis for $T_X\mathcal{B}$. In light of (6.16), the invariants are written as

$$I_1 = \frac{r^2(R)e^{-2\omega_\Theta(R)}}{R^2} + \frac{R^2 e^{2\omega_\Theta(R)+2\omega_Z(R)}}{\alpha^2 r^2(R)} + \alpha^2 e^{-2\omega_Z(R)}\,, \tag{5.71}$$

$$I_2 = \frac{R^2 e^{2\omega_\Theta(R)}}{r^2(R)} + \frac{\alpha^2 r^2(R)e^{-2\omega_\Theta(R)-2\omega_Z(R)}}{R^2} + \frac{e^{2\omega_Z(R)}}{\alpha^2}\,, \tag{5.72}$$

$$I_4 = \left(\frac{Re^{\omega_\Theta(R)+\omega_Z(R)}}{\alpha r(R)}\right)^2, \tag{5.73}$$

$$I_5 = \left(\frac{Re^{\omega_\Theta(R)+\omega_Z(R)}}{\alpha r(R)}\right)^4, \tag{5.74}$$

$$I_6 = e^{-2\omega_Z(R)}\alpha^2\,, \tag{5.75}$$

$$I_7 = e^{-4\omega_Z(R)}\alpha^4\,. \tag{5.76}$$

The Cauchy stress components given by (6.20) read

$$\sigma^{rr} = \frac{2R^2 e^{2\omega_\Theta(R)+2\omega_Z(R)}}{\alpha^2 r^2(R)} \left(W_{I_1} + W_{I_4}\right) + 2W_{I_2} \left(\frac{R^2 e^{2\omega_\Theta(R)}}{r^2(R)} + \frac{e^{2\omega_Z(R)}}{\alpha^2}\right) + 4W_{I_5} \left(\frac{R e^{\omega_\Theta(R)+\omega_Z(R)}}{\alpha r(R)}\right)^4 - p\,, \tag{5.77}$$

$$\sigma^{\theta\theta} = \frac{2e^{-2(\omega_\Theta(R)+\omega_Z(R))}}{R^2} \left(W_{I_1} e^{2\omega_Z(R)} + \alpha^2 W_{I_2}\right) + \frac{2W_{I_2} e^{2\omega_Z(R)}}{\alpha^2 r^2(R)} - \frac{p}{r^2(R)}\,, \tag{5.78}$$

$$\sigma^{zz} = 2e^{-4\omega_Z(R)} \alpha^2 \left(\left(W_{I_1} + W_{I_6}\right) e^{2\omega_Z(R)} + 2\alpha^2 W_{I_7}\right) + \frac{2R^2 W_{I_2} e^{2\omega_\Theta(R)}}{r^2(R)} + 2W_{I_2} \left(\frac{\alpha r(R)}{e^{\omega_\Theta(R)+\omega_Z(R)} R}\right)^2 - p\,. \tag{5.79}$$

Assuming that the eigenstrain-free body is stress-free gives the following conditions (see also [92, 94] for more details)

$$\left(W_{I_4} + 2W_{I_5}\right)|_{I_1=I_2=3, I_4=I_5=I_6=I_7=1} = 0\,, \quad \text{and} \quad \left(W_{I_6} + 2W_{I_7}\right)|_{I_1=I_2=3, I_4=I_5=I_6=I_7=1} = 0\,. \tag{5.80}$$

From (2.7), the only nontrivial equilibrium equation is written as

$$\sigma^{rr}{}_{,r} + \frac{\sigma^{rr}}{r} - r\sigma^{\theta\theta} = 0\,. \tag{5.81}$$

Therefore, after some simplifications, $p'(R) = k(R)$, where the expression for $k(R)$ is given in Appendix B.2. Assuming that the bar is subject to a uniform pressure on its boundary cylinder, i.e., $\sigma^{rr}(R_o) = -p_\infty$, gives

$$p(R) = p_\infty + \int_{R_o}^{R} k(\zeta) d\zeta + \frac{2R_o^2 e^{2\omega_\Theta(R_o)+2\omega_Z(R_o)}}{\alpha^2 r^2(R_o)} \left(W_{I_1} + W_{I_4}\right)|_{R=R_o} + 2\left(\frac{R_o^2 e^{2\omega_\Theta(R_o)}}{r^2(R_o)} + \frac{e^{2\omega_Z(R_o)}}{\alpha^2}\right) W_{I_2}|_{R=R_o} + 4\left(\frac{R_o e^{\omega_\Theta(R_o)+\omega_Z(R_o)}}{\alpha r(R_o)}\right)^4 W_{I_5}|_{R=R_o}\,. \tag{5.82}$$

A cylindrical inclusion in a finite orthotropic cylindrical bar. We next consider the following distribution of eigenstrains in a cylindrical bar, corresponding to a cylindrical inclusion with radius

R_i along the axis of the bar.

$$\omega_R(R) = \begin{cases} \omega_1\,, & 0 \leq R \leq R_i \\ 0\,, & R_i \leq R \leq R_o \end{cases} \,, \quad \omega_\Theta(R) = \begin{cases} \omega_2\,, & 0 \leq R \leq R_i \\ 0\,, & R_i \leq R \leq R_o \end{cases} \,, \quad \omega_Z(R) = \begin{cases} \omega_3\,, & 0 \leq R \leq R_i \\ 0\,, & R_i \leq R \leq R_o \end{cases} \,. \tag{5.83}$$

Using (5.68), one finds

$$r(R) = \frac{1}{\alpha^{\frac{1}{2}}} \begin{cases} e^{\frac{1}{2}(\omega_1+\omega_2+\omega_3)} R\,, & 0 \leq R \leq R_i \\ \left(R^2 + \left(e^{\omega_1+\omega_2+\omega_3} - 1\right) R_i^2\right)^{\frac{1}{2}}\,, & R_i \leq R \leq R_o \end{cases} \,. \tag{5.84}$$

Simplifying (B.2), it follows that in the inclusion $p'(R) = k_0/R$, where

$$k_0 = \frac{2e^{-2\omega_1-\omega_2-\omega_3}}{\alpha^2} \Big[\alpha e^{\omega_1} \left(e^{2(\omega_2+\omega_3)} W_{I_4} - \left(e^{2\omega_1} - e^{2\omega_2}\right)\left(\alpha^2 W_{I_2} + e^{2\omega_3} W_{I_1}\right)\right) + 2e^{3(\omega_2+\omega_3)} W_{I_5}\Big]\Big|_{I_1=a^{-1}+b^{-1}+ab,\, I_2=a+b+(ab)^{-1},\, I_4^2=I_5=a^{-2},\, I_6^2=I_7=a^2b^2} \,, \tag{5.85}$$

and $a = e^{\omega_1-\omega_2-\omega_3}\alpha$ and $b = e^{\omega_2-\omega_1-\omega_3}\alpha$. Also, in the matrix, $p'(R) = \hat{k}(R)$, where for $R_i \leq R \leq R_o$

$$\begin{aligned} \hat{k}(R) = \frac{2}{\alpha^9 r^{10} R^3} \Big(& -8R^{10} W_{I_5 I_5}\left(R^2 - \alpha r^2\right) + 8\alpha^2 r^2 R^8 W_{I_4 I_5}\left(\alpha r^2 - R^2\right) - \alpha^8 r^6 W_{I_2}\left(R^3 - \alpha r^2 R\right)^2 \\ & -\alpha^6 r^6 W_{I_1}\left(R^3 - \alpha r^2 R\right)^2 + \alpha^6 r^6 R^4 W_{I_4}\left(2\alpha r^2 - R^2\right) + 2\alpha^4 r^4 R^6 W_{I_4 I_4}\left(\alpha r^2 - R^2\right) + 2\alpha^4 r^4 R^6 W_{I_5}\left(4\alpha r^2 - 3R^2\right) \\ & -2\alpha^6 r^4 W_{I_2 I_2}\left(R^2 - \alpha r^2\right)^2 \left[\alpha r^4 + \left(\alpha^3 + 1\right) r^2 R^2 + \alpha^2 R^4\right] - 2\alpha^4 r^4 W_{I_1 I_2}\left(R^2 - \alpha r^2\right)^2 \left(\alpha r^4 + \left(2\alpha^3 + 1\right) r^2 R^2 \right. \\ & \left. +2\alpha^2 R^4\right) - 2\alpha^4 r^4 W_{I_1 I_1}\left(\alpha r^2 + R^2\right)\left(R^3 - \alpha r^2 R\right)^2 - 4\alpha^2 r^2 R^4 W_{I_1 I_5}\left(\alpha^3 r^6 - \alpha^2 r^4 R^2 - 2\alpha r^2 R^4 + 2R^6\right) \\ & -4\alpha^2 r^2 R^4 W_{I_2 I_5}\left(\alpha^5 r^6 - \alpha\left(\alpha^3 + 1\right) r^4 R^2 + \left(1 - 2\alpha^3\right) r^2 R^4 + 2\alpha^2 R^6\right) - 2\alpha^4 r^4 R^2 W_{I_1 I_4}\Big\{\alpha^3 r^6 - \alpha^2 r^4 R^2 \\ & - 2\alpha r^2 R^4 + 2R^6\Big\} - 2\alpha^4 r^4 R^2 W_{I_2 I_4}\left(\alpha^5 r^6 - \alpha\left(\alpha^3 + 1\right) r^4 R^2 + \left(1 - 2\alpha^3\right) r^2 R^4 + 2\alpha^2 R^6\right) \Big) \,. \end{aligned} \tag{5.86}$$

Therefore, the pressure field is given by

$$p(R) = \begin{cases} k_0 \ln\left(\frac{R}{R_i}\right) - p_i\,, & 0 \leq R \leq R_i\,, \\ \int_{R_o}^{R} \hat{k}(\zeta) d\zeta - p_o\,, & R_i \leq R \leq R_o\,, \end{cases} \tag{5.87}$$

where p_i and p_o are integration constants to be determined. The physical components of the Cauchy stress read

$$\hat{\sigma}^{rr} = \begin{cases} k_0 \ln\left(\frac{R_i}{R}\right) + p_i + \frac{2e^{-2\omega_1-\omega_3}}{\alpha^2}\Big(\alpha e^{\omega_1+\omega_2+2\omega_3}\left(W_{I_1}+W_{I_4}\right) + \alpha^3 e^{\omega_1+\omega_2} W_{I_2} + e^{2\omega_1+3\omega_3} W_{I_2} \\ \qquad + 2e^{2\omega_2+3\omega_3} W_{I_5}\Big)\Big|_{I_1=a^{-1}+b^{-1}+ab, I_2=a+b+(ab)^{-1}, I_4^2=I_5=a^{-2}, I_6^2=I_7=a^2b^2}\,, & 0 \leq R \leq R_i\,, \\ p_o + \Big[\int_{R}^{R_o} \hat{k}(\zeta) d\zeta + \frac{2}{\alpha^4 r^4}\Big\{\alpha^2 R^2 r^2 \left(\alpha^2 W_{I_2} + W_{I_1} + W_{I_4}\right) + 2R^4 W_{I_5} \\ \qquad + \alpha^2 W_{I_2} r^4\Big\}\Big]\Big|_{I_1=\alpha^2+I_R^{-2}+\alpha^{-2}I_R^2, I_2=\alpha^{-2}+I_R^2+\alpha^2 I_R^{-2}, I_4^2=I_5=\alpha^{-4}I_R^4, I_6^2=I_7=\alpha^4}\,, & R_i \leq R \leq R_o\,, \end{cases} \tag{5.88}$$

$$\hat{\sigma}^{\theta\theta} = \begin{cases} k_0 \ln\left(\frac{R_i}{R}\right) + p_i + \Big[\left(\frac{2e^{2\omega_3}}{\alpha^2} + 2\alpha e^{\omega_1-\omega_2-\omega_3}\right) W_{I_2} \\ \qquad + \frac{2e^{\omega_1-\omega_2+\omega_3}}{\alpha} W_{I_1}\Big]\Big|_{I_1=a^{-1}+b^{-1}+ab, I_2=a+b+(ab)^{-1}, I_4^2=I_5=a^{-2}, I_6^2=I_7=a^2b^2}\,, & 0 \leq R \leq R_i\,, \\ p_o + \Big[\int_{R}^{R_o} \hat{k}(\zeta) d\zeta + \frac{2r^2}{R^2}\left(W_{I_1} + \alpha^2 W_{I_2}\right) \\ \qquad + \frac{2}{\alpha^2} W_{I_2}\Big]\Big|_{I_1=\alpha^2+I_R^{-2}+\alpha^{-2}I_R^2, I_2=\alpha^{-2}+I_R^2+\alpha^2 I_R^{-2}, I_4^2=I_5=\alpha^{-4}I_R^4, I_6^2=I_7=\alpha^4}\,, & R_i \leq R \leq R_o\,, \end{cases} \tag{5.89}$$

$$\hat{\sigma}^{zz} = \begin{cases} k_0 \ln\left(\frac{R_i}{R}\right) + p_i + 2\alpha e^{-4\omega_3}\Big\{\alpha e^{2\omega_3}\left(W_{I_1}+W_{I_6}\right) + e^{\omega_1-\omega_2+3\omega_3} W_{I_2} + e^{-\omega_1+\omega_2+3\omega_3} W_{I_2} \\ \qquad + 2\alpha^3 W_{I_7}\Big\}\Big|_{I_1=a^{-1}+b^{-1}+ab, I_2=a+b+(ab)^{-1}, I_4^2=I_5=a^{-2}, I_6^2=I_7=a^2b^2}\,, & 0 \leq R \leq R_i\,, \\ p_o + \Big[\int_{R}^{R_o} \hat{k}(\zeta) d\zeta + 2\alpha^2\Big(W_{I_1} + \frac{R^2}{\alpha^2 r^2} W_{I_2} + \frac{r^2}{R^2} W_{I_2} + W_{I_6} \\ \qquad + 2\alpha^2 W_{I_7}\Big)\Big]\Big|_{I_1=\alpha^2+I_R^{-2}+\alpha^{-2}I_R^2, I_2=\alpha^{-2}+I_R^2+\alpha^2 I_R^{-2}, I_4^2=I_5=\alpha^{-4}I_R^4, I_6^2=I_7=\alpha^4}\,, & R_i \leq R \leq R_o\,, \end{cases} \tag{5.90}$$

where $I_R = R/r(R)$. Imposing uniform pressure on the boundary cylinder, $\hat{\sigma}^{rr}(R_o) = -p_\infty$, gives

$$p_o = -p_\infty - \Big[\frac{2}{\alpha^4 r^4(R_o)}\Big\{\alpha^2 R_o^2 r^2(R_o)\left(\alpha^2 W_{I_2} + W_{I_1} + W_{I_4}\right) + 2R_o^4 W_{I_5} + \alpha^2 W_{I_2} r^4(R_o)\Big\}\Big]\Big|_{I_1=\alpha^2+I_{R_o}^{-2}+\alpha^{-2}I_{R_o}^2, I_2=\alpha^{-2}+I_{R_o}^2+\alpha^2 I_{R_o}^{-2}, I_4^2=I_5=\alpha^{-4}I_{R_o}^4, I_6^2=I_7=\alpha^4}. \tag{5.91}$$

The continuity of the traction vector at the inclusion-matrix interface requires that σ^{rr} be continuous at $R = R_i$. Therefore, p_i is calculated as

$$\begin{aligned}
p_i &= -p_\infty + \int_{R_i}^{R_o} \hat{k}(\zeta)d\zeta + \frac{2e^{-(\omega_1+\omega_2+\omega_3)}}{\alpha^2}\Big[\alpha\left(W_{I_1}+W_{I_4}\right) + \left(\alpha^3 + e^{\omega_1+\omega_2+\omega_3}\right)W_{I_2} \\
&+ 2e^{-(\omega_1+\omega_2+\omega_3)}W_{I_5}\Big]\Big|_{I_1=\alpha^2+I_{R_i}^{-2}+\alpha^{-2}I_{R_i}^2, I_2=\alpha^{-2}+I_{R_i}^2+\alpha^2 I_{R_i}^{-2}, I_4^2=I_5=\alpha^{-4}I_{R_i}^4, I_6^2=I_7=\alpha^4} \\
&- \frac{2e^{-2\omega_1-\omega_3}}{\alpha^2}\Big(\alpha e^{\omega_1+\omega_2+2\omega_3}\left(W_{I_1}+W_{I_4}\right) + \alpha^3 e^{\omega_1+\omega_2}W_{I_2} + e^{2\omega_1+3\omega_3}W_{I_2} \\
&+ 2e^{2\omega_2+3\omega_3}W_{I_5}\Big)\Big|_{I_1=a^{-1}+b^{-1}+ab, I_2=a+b+(ab)^{-1}, I_4^2=I_5=a^{-2}, I_6^2=I_7=a^2b^2} - \Big[\frac{2}{\alpha^4 r^4(R_o)}\Big\{2R_o^4 W_{I_5} + \alpha^2 W_{I_2} r^4(R_o) \\
&+ \alpha^2 R_o^2 r^2(R_o)\left(\alpha^2 W_{I_2} + W_{I_1} + W_{I_4}\right)\Big\}\Big]\Big|_{I_1=\alpha^2+I_{R_o}^{-2}+\alpha^{-2}I_{R_o}^2, I_2=\alpha^{-2}+I_{R_o}^2+\alpha^2 I_{R_o}^{-2}, I_4^2=I_5=\alpha^{-4}I_{R_o}^4, I_6^2=I_7=\alpha^4}.
\end{aligned} \tag{5.92}$$

Remark 5.2.6. For the stress to be uniform in the inclusion, k_0 must be zero (cf. (5.88), (5.89), and (5.90)). If $\omega_1 \neq \omega_2$, k_0 is zero only if the energy function satisfies the following condition

$$\Big[\alpha e^{\omega_1}\left(e^{2(\omega_2+\omega_3)}W_{I_4} - \left(e^{2\omega_1} - e^{2\omega_2}\right)\left(\alpha^2 W_{I_2} + e^{2\omega_3}W_{I_1}\right)\right) + 2e^{3(\omega_2+\omega_3)}W_{I_5}\Big]\Big|_{I_1=a^{-1}+b^{-1}+ab, I_2=a+b+(ab)^{-1}, I_4^2=I_5=a^{-2}, I_6^2=I_7=a^2b^2} = 0\,. \tag{5.93}$$

However, if $\omega_1 = \omega_2$, then $a = b = e^{-\omega_3}\alpha$, and (5.85) implies that k_0 is zero if

$$k_0 = 2a^{-1}\left(W_{I_4} + 2a^{-1}W_{I_5}\right)|_{I_1=2a^{-1}+a^2, I_2=2a+a^{-2}, I_4^2=I_5=a^{-2}, I_6^2=I_7=a^4} = 0\,. \tag{5.94}$$

In addition, if we assume that the traction in the radial fiber direction is tensile for extension $a < 1$

and compressive for contraction $a > 1$, then (5.94) implies that $a = b = 1$, or $\alpha = e^{\omega_3}$, and hence, for any nonlinear incompressible orthotropic material, $k_0 = 2\left(W_{I_4} + 2W_{I_5}\right)|_{I_1=I_2=3, I_4=I_5=I_6=I_7=1} = 0$, from (5.80). Therefore, we have proved the following proposition.

Proposition 5.2.7. *Consider a finite incompressible orthotropic elastic solid cylinder such that the material orthotropic axes are in the radial, circumferential, and longitudinal directions of the cylinder. Assume that the bar is subject to a uniform pressure on its boundary cylinder and contains an inclusion along its axis with uniform radial, circumferential, and longitudinal eigenstrains. The Cauchy stress exhibits a logarithmic singularity at the centerline of the cylinder unless the radial and circumferential eigenstrains are equal and the axial stretch α is equal to e^{ω_3}, or the energy function satisfies* (5.93)*. If the radial and circumferential eigenstrains are equal and $\alpha = e^{\omega_3}$, then the stress inside the inclusion is uniform and hydrostatic.*

Note that Proposition 5.2.7 holds for a cylindrical bar made of any incompressible transversely isotropic solid with material preferred directions along the radial and circumferential directions as well. If the material preferred direction is longitudinal, then we do not need the condition $\alpha = e^{\omega_3}$ for the results of the proposition to hold.

5.2.4 Finite Eigenstrains in a Finite Compressible Orthotropic Cylindrical Bar

In this section, we release the incompressibility constraint of the problem of a bar with a finite cylindrically-symmetric eigenstrain distribution and consider a compressible orthotropic solid. Assuming that the material manifold is embedded into the ambient space using the mappings of the form $(r, \theta, z) = (r(R), \Theta, \alpha Z)$, the right Cauchy-Green deformation tensor is written as

$$\mathbf{C} = \begin{pmatrix} r'(R)^2 e^{-2\omega_R(R)} & 0 & 0 \\ 0 & \frac{e^{-2\omega_\Theta(R)} r^2(R)}{R^2} & 0 \\ 0 & 0 & e^{-2\omega_Z(R)}\alpha^2 \end{pmatrix}. \tag{5.95}$$

The Jacobean reads

$$J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}} \det \mathbf{F} = \frac{\alpha r(R) r'(R)}{R e^{\omega_R(R)+\omega_\Theta(R)+\omega_Z(R)}} \,. \tag{5.96}$$

Using (6.16), one obtains the invariants of deformation as follows

$$\begin{aligned}
I_1 =& \mathrm{tr}(\mathbf{C}) = \frac{r^2(R) e^{-2\omega_\Theta(R)}}{R^2} + r'(R)^2 e^{-2\omega_R(R)} + \alpha^2 e^{-2\omega_Z(R)} \,, & (5.97)\\
I_2 =& \frac{1}{2}(\mathrm{tr}(\mathbf{C}^2) - \mathrm{tr}(\mathbf{C})^2) = \frac{\alpha^2 r^2(R) r'(R)^2}{e^{2(\omega_R(R)+\omega_\Theta(R)+\omega_Z(R))} R^2} \left[\frac{R^2 e^{2\omega_\Theta(R)}}{r^2(R)} + \frac{e^{2\omega_R(R)}}{r'(R)^2} + \frac{e^{2\omega_Z(R)}}{\alpha^2} \right] \,, & (5.98)\\
I_3 =& \frac{\alpha^2 r^2(R) r'(R)^2}{R^2} e^{-2(\omega_R(R)+\omega_\Theta(R)+\omega_Z(R))} \,, & (5.99)\\
I_4^2 =& I_5 = e^{-4\omega_R(R)} r'(R)^4 \,, & (5.100)\\
I_6^2 =& I_7 = e^{-4\omega_Z(R)} \alpha^4 \,. & (5.101)
\end{aligned}$$

Noting that the Jacobean is given by (5.96), the components of the Cauchy stress are written as

$$\begin{aligned}
\sigma^{rr} = \frac{2r'(R) e^{-(\omega_\Theta(R)+3\omega_R(R)+\omega_Z(R))}}{\alpha R r(R)} \Big[& R^2 e^{2\omega_\Theta(R)} \Big\{ e^{2\omega_Z(R)} \left(e^{2\omega_R(R)} \left(W_{I_1} + W_{I_4} \right) + 2 W_{I_5} r'(R)^2 \right) \\
& + \alpha^2 W_{I_2} e^{2\omega_R(R)} \Big\} + r(R)^2 e^{2\omega_R(R)} \left(W_{I_2} e^{2\omega_Z(R)} + \alpha^2 W_{I_3} \right) \Big] \,, \qquad (5.102)
\end{aligned}$$

$$\sigma^{\theta\theta} = \frac{e^{-(\omega_\Theta(R)+\omega_R(R)+\omega_Z(R))}}{\alpha R r(R) r'(R)} \left[2 e^{2\omega_R(R)} \left(W_{I_1} e^{2\omega_Z(R)} + \alpha^2 W_{I_2} \right) + 2 r'(R)^2 \left(W_{I_2} e^{2\omega_Z(R)} + \alpha^2 W_{I_3} \right) \right] \,, \tag{5.103}$$

$$\begin{aligned}
\sigma^{zz} = \frac{2\alpha e^{-(\omega_\Theta(R)+\omega_R(R)+3\omega_Z(R))}}{R r(R) r'(R)} \Big[& R^2 e^{2\omega_\Theta(R)} \left\{ e^{2\omega_Z(R)} \left(e^{2\omega_R(R)} \left(W_{I_1} + W_{I_6} \right) + W_{I_2} r'(R)^2 \right) + 2\alpha^2 W_{I_7} e^{2\omega_R(R)} \right\} \\
& + r(R)^2 e^{2\omega_Z(R)} \left(W_{I_2} e^{2\omega_R(R)} + W_{I_3} r'(R)^2 \right) \Big] \,. \qquad (5.104)
\end{aligned}$$

We need to have the following conditions in order for the eigenstrain-free body to be stress-free.

$$
\begin{aligned}
&(W_{I_1}+2W_{I_2}+W_{I_3})\,|_{I_1=I_2=3,I_3=I_4=I_5=I_6=I_7=1}=0\,,\\
(W_{I_4}+2W_{I_5})\,|_{I_1=I_2=3,I_3=I_4=I_5=I_6=I_7=1}=0\,,\quad &(W_{I_6}+2W_{I_7})\,|_{I_1=I_2=3,I_3=I_4=I_5=I_6=I_7=1}=0\,.
\end{aligned}
\tag{5.105}
$$

Substituting for the stress components into (5.81) using (6.49) and (5.103), the radial equilibrium equation is simplified and is given in Appendix B.3.

We next consider the eigenstrain distribution (5.83) and solve the problem of a cylindrical inclusion with uniform anisotropic eigenstrains in a finite compressible orthotropic cylindrical bar. Following the same procedure that was explained in 6.3.5, we first assume that the stress field inside the inclusion is uniform, i.e., $\hat{\sigma}^{rr}=C_1$, $\hat{\sigma}^{\theta\theta}=C_2$, $\hat{\sigma}^{zz}=C_3$, where C_i, $i=1,2,3$ are constants. Using (6.49), (5.103), and (5.104), we have the following three first-order ODEs for $0\leq R\leq R_i$ subject to the boundary condition $r(0)=0$:

$$
\begin{aligned}
C_1=\frac{2e^{-(3\omega_1+\omega_2+\omega_3)}r'(R)}{\alpha R r(R)}\Big[&R^2e^{2\omega_2}\left\{e^{2\omega_3}\left(e^{2\omega_1}\left(W_{I_1}+W_{I_4}\right)+2W_{I_5}r'(R)^2\right)+\alpha^2W_{I_2}e^{2\omega_1}\right\}\\
&+r(R)^2e^{2\omega_1}\left(W_{I_2}e^{2\omega_3}+\alpha^2W_{I_3}\right)\Big]\,,
\end{aligned}
\tag{5.106}
$$

$$
C_2=\frac{e^{-(\omega_1+\omega_2+\omega_3)}r(R)}{\alpha R r'(R)}\Big[2e^{2\omega_1}\left(W_{I_1}e^{2\omega_3}+\alpha^2W_{I_2}\right)+2r'(R)^2\left(W_{I_2}e^{2\omega_3}+\alpha^2W_{I_3}\right)\Big]\,,
\tag{5.107}
$$

$$
\begin{aligned}
C_3=\frac{2e^{-(\omega_1+\omega_2+3\omega_3)}\alpha}{R r(R) r'(R)}\Big[&R^2e^{2\omega_2}\left\{e^{2\omega_3}\left(e^{2\omega_1}\left(W_{I_1}+W_{I_6}\right)+W_{I_2}r'(R)^2\right)+2\alpha^2W_{I_7}e^{2\omega_1}\right\}\\
&+r(R)^2e^{2\omega_3}\left(W_{I_2}e^{2\omega_1}+W_{I_3}r'(R)^2\right)\Big]\,.
\end{aligned}
\tag{5.108}
$$

Note that $r(R)=\beta R$, with β a constant, is a solution of all the above IVPs, i.e., the stress inside the inclusion is uniform if $r(R)=\beta R$ for $0\leq R\leq R_i$. From the radial equilibrium equation (5.81), it follows that $C_1 = C_2$ when the stress field in the inclusion is assumed to be uniform. Therefore, from (5.106) and (5.107), the equilibrium equation in the inclusion ($C_1 = C_2$) for $r(R) = \beta R$ implies that

$$\begin{aligned}&\Big[e^{2(\omega_2+\omega_3)}\left(2\beta^2 W_{I_5} + e^{2\omega_1} W_{I_4}\right) - e^{2\omega_1}\left(e^{2\omega_1} - e^{2\omega_2}\right)\left(\alpha^2 W_{I_2}\right.\\ &\left.+e^{2\omega_3} W_{I_1}\right)\Big]\Big|_{I_1=\alpha^2 e^{-2\omega_3}+\beta^2\left[e^{-2\omega_1}+e^{-2\omega_2}\right], I_2=\kappa\beta^2\left[\alpha^2\left(e^{2\omega_1}+e^{2\omega_2}\right)+e^{2\omega_3}\beta^2\right], I_3=\kappa\alpha^2\beta^4, I_4^2=I_5=e^{-4\omega_1}\beta^4, I_6^2=I_7=e^{-4\omega_3}\alpha^4} = 0\,,\end{aligned} \tag{5.109}$$

where $\kappa = e^{-2(\omega_1+\omega_2+\omega_3)}$. If the material is compressible and isotropic, i.e., $W = W(I_1, I_2, I_3)$, then $W_{I_4} = W_{I_5} = 0$, and (5.109) is clearly satisfied without restricting the longitudinal stretch α or the strain energy function if $\omega_1 = \omega_2$. In this case, in the matrix, we have the following second-order nonlinear ODE from (B.3) for $R_i \leq R \leq R_o$:

$$\begin{aligned}&rR^3\Big[-r'^2\left(2\left\{\alpha^2\left(r'^2\left(\alpha^2 W_{I_2I_3} + W_{I_1I_3}\right) + W_{I_2I_2}\left(\alpha^2 + r'^2\right)\right) + W_{I_1I_2}\left(2\alpha^2 + r'^2\right) + W_{I_1I_1}\right\} + \alpha^2 W_{I_3}\right)\\ &+W_{I_2}\left(\alpha^2 - r'^2\right) + W_{I_1}\Big] + 2\alpha^4 r^4 r'^3 W_{I_3I_3} + 2\alpha^2 r^4 r' W_{I_1I_3} + 2r^4 r' W_{I_1I_2} - 2r^3 R r'^2\Big[\alpha^4 r'^2 W_{I_3I_3} + \alpha^2 W_{I_2I_3}\left(\alpha^2 + 2r'^2\right)\\ &+W_{I_2I_2}\left(\alpha^2 + r'^2\right) + \alpha^2 W_{I_1I_3} + W_{I_1I_2}\Big] - R^4\Big(\alpha^2 W_{I_2}\left(Rr'' + r'\right) + 2Rr'^2 r''\left(\alpha^4 W_{I_2I_2} + 2\alpha^2 W_{I_1I_2} + W_{I_1I_1}\right)\\ &\quad + W_{I_1}\left(Rr'' + r'\right)\Big) + r^2R^2\Big[\alpha^2 W_{I_3}\left(r' - Rr''\right) + 2r'\Big\{\alpha^2\Big(r'\left(\alpha^2 W_{I_2I_3} + W_{I_1I_3}\right)\left(r' - 2Rr''\right)\\ &\quad + W_{I_2I_2}\left(\alpha^2 + r'^2 - 2Rr'r''\right)\Big) + W_{I_1I_2}\left(2\alpha^2 + r'^2 - 2Rr'r''\right) + W_{I_1I_1}\Big\} + W_{I_2}\left(r' - Rr''\right)\Big]\\ &-2\alpha^4 r^4 R r'^2 r'' W_{I_3I_3} + 2\alpha^2 r^4 r' W_{I_2I_3}\left(\alpha^2 + 2r'^2 - 2Rr'r''\right) + 2r^4 r' W_{I_2I_2}\left(\alpha^2 + r'^2 - Rr'r''\right) = 0\,.\end{aligned} \tag{5.110}$$

The boundary conditions for the ODE (5.110) and determining the unknown β are given by continuity of $r(R)$ and the traction vector at the inclusion-matrix interface, i.e., $r(R)|_{R=R_i^+} = \beta R_i$ and $\sigma^{rr}|_{R=R_i^+} = \sigma^{rr}|_{R=R_i^-}$, respectively, along with the boundary condition $\hat{\sigma}^{rr}(R_o) = -p_\infty$. Thus, we have proved the following proposition.

Proposition 5.2.8. *Consider a cylindrical bar made of a compressible isotropic solid subject to a uniform pressure on its boundary cylinder. Assume that the bar contains a cylindrical inclusion along* *its axis with uniform radial, circumferential, and axial eigenstrains. The stress field in the inclusion is uniform if the radial and circumferential eigenstrains are equal.*

Returning to the compressible orthotropic solid case, if the radial and circumferential eigenstrains are equal $\omega_1 = \omega_2 = \omega$, (5.109) implies that

$$\left(2b^2 W_{I_5} + W_{I_4}\right)\Big|_{I_1=a^2+2b^2, I_2=b^2(2a^2+b^2), I_3=a^2b^4, I_4^2=I_5=b^4, I_6^2=I_7=a^4} = 0\,, \tag{5.111}$$

where $a = \alpha e^{-\omega_3}$ and $b = \beta e^{-\omega}$. Note that if $\alpha = e^{\omega_3}$, i.e., $a = 1$, then $b = 1$ is trivially a solution of (5.111) from (5.105). If we further assume that the traction in the radial fiber direction is tensile for extension $b > 1$ and compressive for contraction $b < 1$, then $b = 1$ is the only solution of (5.111). This corresponds to the trivial stress-free case, where the entire bar has a uniform distribution of dilatational eigenstrains such that the radial and circumferential eigenstrains are equal, and the axial stretch is equal to e^{ω_3}. However, there are some nontrivial cases for which uniform stress can be maintained in the inclusion with uniform dilatational eigenstrains such that $\omega_1 \neq \omega_2$ or $\alpha \neq e^{\omega_3}$. As was already mentioned in 6.3.5, these cases are special because a choice of energy function, in general, fully determines β from (5.109), which in turn specifies the kinematics and the stress field in the inclusion.

We next assume some specific energy functions analogous to (6.36) of the following form

$$W = W\left(I_1, I_2, I_3, I_4, I_5, I_6, I_7\right) = W_{\text{iso}}\left(I_1, I_2, I_3\right) + W_{\text{fib}}^R\left(I_4, I_5\right) + W_{\text{fib}}^Z\left(I_6, I_7\right)\,, \tag{5.112}$$

where the isotropic base material with the strain energy function W_{iso} is augmented by a fiber reinforcing model such that W_{fib}^R and W_{fib}^Z represent the reinforcing effects in the radial and longitudinal directions, respectively.

i) Compressible Mooney-Rivlin reinforced model (I_4, I_6, I_7 reinforcement) with the energy function

$$W(I_1, I_2, I_3, I_4, I_6, I_7) = C_1 (I_1 - 3) + C_2 (I_2 - 3) - (C_1 + 2C_2) (I_3 - 1) \\ + \frac{\mu}{2} (I_4 - 1)^2 + \frac{\mu_1}{2} (I_6 - 1)^2 + \frac{\mu_2}{2} (I_7 - 1)^2 \ , \tag{5.113}$$

where C_1 and C_2 are the constants of the Mooney-Rivlin base material and $\mu > 0$ is a material constant describing the strength of the reinforcement in the radial direction, while μ_1 and μ_2 are positive material constants pertaining to the reinforcement strength in the axial direction. Substituting (5.113) into (5.109) one gets

$$\beta = e^{\omega_1} \left[\frac{\delta}{\mu} \left(\alpha^2 e^{-2\omega_3} C_2 + C_1 \right) e^{-2\omega_2} + 1 \right]^{\frac{1}{2}} \ , \tag{5.114}$$

where $\delta = e^{2\omega_1} - e^{2\omega_2}$. For $\omega_1 \geq \omega_2$, there is no condition imposed on the material parameters for β to be positive, whereas for $\omega_1 < \omega_2$, one needs to have the following condition

$$\mu > \left(C_1 + \alpha^2 e^{-2\omega_3} C_2 \right) \left[1 - e^{2(\omega_1 - \omega_2)} \right] , \qquad \text{for} \quad \omega_1 < \omega_2 \, . \tag{5.115}$$

The stress field in the inclusion is uniform and has the following non-zero components (cf. (6.49), (5.103), and (5.104))

$$\hat{\sigma}^{rr} = \hat{\sigma}^{\theta\theta} = \frac{2e^{-3\omega_1 - \omega_2 - \omega_3}}{\alpha} \Big[C_1 e^{2\omega_1} \left(e^{2(\omega_2 + \omega_3)} - \alpha^2 \beta^2 \right) + C_2 e^{2\omega_1} \left(\alpha^2 e^{2\omega_2} + \beta^2 e^{2\omega_3} - 2\alpha^2 \beta^2 \right) \\ - \mu e^{2(\omega_2 + \omega_3)} \left(e^{2\omega_1} - \beta^2 \right) \Big] , \tag{5.116a}$$

$$\hat{\sigma}^{zz} = \frac{2\alpha e^{-\omega_1 - \omega_2 - 7\omega_3}}{\beta^2} \Big[\beta^2 C_2 e^{6\omega_3} \left(e^{2\omega_1} + e^{2\omega_2} - 2\beta^2 \right) + C_1 e^{6\omega_3} \left(e^{2(\omega_1 + \omega_2)} - \beta^4 \right) \\ - e^{2(\omega_1 + \omega_2)} \left(e^{2\omega_3} - \alpha^2 \right) \left\{ 2\alpha^2 \mu_2 \left(\alpha^2 + e^{2\omega_3} \right) + \mu_1 e^{4\omega_3} \right\} \Big] . \tag{5.116b}$$

ii) Compressible Mooney-Rivlin reinforced model (I_5, I_6, I_7 reinforcement) for which the energy function is written as

$$
\begin{aligned}
W(I_1, I_2, I_3, I_5, I_6, I_7) = C_1 (I_1 - 3) + C_2 (I_2 - 3) - (C_1 + 2C_2)(I_3 - 1) \\
+ \frac{\mu}{2}(I_5 - 1)^2 + \frac{\mu_1}{2}(I_6 - 1)^2 + \frac{\mu_2}{2}(I_7 - 1)^2 \, .
\end{aligned}
\tag{5.117}
$$

Using (6.56) and (5.109), one obtains

$$
\beta = \frac{e^{\omega_1 - \omega_2 - \omega_3}\left(2\mu^2 6^{\frac{1}{3}} e^{4(\omega_2+\omega_3)} + \left[\Delta^{\frac{1}{2}} + 9\delta\mu^2 e^{4(\omega_2+\omega_3)}\left(e^{2\omega_3}C_1 + \alpha^2 C_2\right)\right]^{\frac{2}{3}}\right)^{\frac{1}{2}}}{6^{\frac{1}{3}}\mu^{\frac{1}{2}}\left[\Delta^{\frac{1}{2}} + 9\delta\mu^2 e^{4(\omega_2+\omega_3)}\left(e^{2\omega_3}C_1 + \alpha^2 C_2\right)\right]^{\frac{1}{6}}} \, , \tag{5.118}
$$

where[8]

$$
\Delta = 3\mu^4 e^{8(\omega_2+\omega_3)}\left[27\delta^2\left(\alpha^4 C_2^2 + C_1^2 e^{4\omega_3}\right) + 54\delta^2\alpha^2 C_1 C_2 e^{2\omega_3} - 16\mu^2 e^{4(\omega_2+\omega_3)}\right] \, . \tag{5.119}
$$

The non-zero components of the Cauchy stress in the inclusion read

$$
\begin{aligned}
\hat{\sigma}^{rr} = \hat{\sigma}^{\theta\theta} = \frac{2e^{-7\omega_1-\omega_2-\omega_3}}{\alpha}\Big[2\beta^2\mu e^{2(\omega_2+\omega_3)}\left(\beta^4 - e^{4\omega_1}\right) + C_1 e^{6\omega_1}\left(e^{2(\omega_2+\omega_3)} - \alpha^2\beta^2\right) \\
+ C_2 e^{6\omega_1}\left(\alpha^2 e^{2\omega_2} + \beta^2 e^{2\omega_3} - 2\alpha^2\beta^2\right)\Big] \, ,
\end{aligned}
\tag{5.120a}
$$

$$
\begin{aligned}
\hat{\sigma}^{zz} = \frac{2\alpha e^{-\omega_1-\omega_2-7\omega_3}}{\beta^2}\Big[\beta^2 C_2 e^{6\omega_3}\left(e^{2\omega_1} + e^{2\omega_2} - 2\beta^2\right) + C_1 e^{6\omega_3}\left(e^{2(\omega_1+\omega_2)} - \beta^4\right) \\
- e^{2(\omega_1+\omega_2)}\left(e^{2\omega_3} - \alpha^2\right)\left\{2\alpha^2\mu_2\left(\alpha^2 + e^{2\omega_3}\right) + \mu_1 e^{4\omega_3}\right\}\Big] \, .
\end{aligned}
\tag{5.120b}
$$

[8]Note that $\beta \in \mathbb{R}^+$ puts a constraint on the elastic constants.

iii) Blatz-Ko reinforced model (I_4, I_6, I_7 reinforcement) with the following energy function

$$W(I_2, I_3, I_4, I_6, I_7) = \frac{\mu_o}{2}\left(\frac{I_2}{I_3} + 2I_3^{\frac{1}{2}} - 5\right) + \frac{\mu}{2}(I_4 - 1)^2 + \frac{\mu_1}{2}(I_6 - 1)^2 + \frac{\mu_2}{2}(I_7 - 1)^2 \,, \tag{5.121}$$

where μ_o is a positive constant of the Blatz-Ko base material. From (5.121) and (5.109), one has

$$\mu_o e^{4\omega_1}\delta + 2\beta^4\mu\left(e^{2\omega_1} - \beta^2\right) = 0\,, \tag{5.122}$$

which is the same as (5.63), obtained in the case of a spherical ball made of the same material. Therefore, β is given by (5.64), and one can show that β is physical (real and positive) only if $\omega_1 > \omega_2$. Note that in this case β is determined independently of the longitudinal stretch α and ω_3. The stress field in the inclusion is uniform with the non-zero stress components given as

$$\hat{\sigma}^{rr} = \hat{\sigma}^{\theta\theta} = \mu_o\left(1 - \frac{e^{3\omega_1+\omega_2+\omega_3}}{\alpha\beta^4}\right) - \frac{2\mu}{\alpha}\left(e^{2\omega_1} - \beta^2\right)e^{-3\omega_1+\omega_2+\omega_3}\,, \tag{5.123a}$$

$$\begin{aligned}\hat{\sigma}^{zz} = -\frac{e^{\omega_1+\omega_2-7\omega_3}}{\alpha^3\beta^2}\Big[&4\alpha^6\mu_2\left(e^{4\omega_3} - \alpha^4\right) + 2\alpha^4\mu_1 e^{4\omega_3}\left(e^{2\omega_3} - \alpha^2\right) \\ &+ e^{7\omega_3}\mu_o\left(e^{3\omega_3} - \alpha^3\beta^2 e^{-\omega_1-\omega_2}\right)\Big]\,.\end{aligned} \tag{5.123b}$$

CHAPTER 6

LINE AND POINT DEFECTS IN NONLINEAR ANISOTROPIC SOLIDS

6.1 Introduction

In anelasticity, any measure of strain has both an elastic and a non-elastic part. Given a pair of thermodynamically conjugate stress and strain, locally a non-vanishing strain does not necessarily correspond to a non-vanishing stress. *Elastic* strain refers to the part of strain that is locally related to the corresponding stress. The remaining part is referred to as *eigenstrain*, a term that was first used by Mura [1].[1] Defects are one source of anelasticity. Vito Volterra, in his seminal work [98], pioneered the mathematical study of defects many years before the first experimental observations of defects in solids. He classified line defects into six types, three of which are now called dislocations or translational defects, and the other three are called disclinations or rotational defects. Kondo [99, 100] and Bilby *et al.* [101] independently explored the profound connections between the mechanics of defects and non-Riemannian geometries in the 1950s. Kondo [99, 100] discovered that the reference configuration of a solid is not necessarily Euclidean in the presence of defects. He realized that the curvature and the torsion of the reference manifold are measures of incompatibility and the density of dislocations, respectively. Defects due to plastic deformations naturally occur in most of the known problems in mechanics and tribology, e.g., contact mechanics [102, 103, 104, 105], mechanical impact [106], and dislocation-boundary interactions [107, 108]. Other examples of anelastic sources include swelling and cavitation [11, 14, 15], bulk and surface growth [19, 20, 21], thermal strains [109, 9, 10], and the presence of inclusions and inhomogeneities [24, 25, 26, 27]. There have been some theoretical investigations on the effects of eigenstrains in linear anisotropic media, e.g., [83, 84, 85, 91], and references therein.

[1]One should, however, note that this term had been used by German researchers in its original German version *Eigenspannungen* long before Mura's work. Apparently, Mura decided not to translate *Eigen*; he only translated *Spannung* = Strain. We are grateful to an anonymous reviewer who pointed this out to us.

Very little is known about the effects of material anisotropies on the stress field and energetics of defects in solids. Eshelby [110] investigated infinitely-long straight edge dislocations in linear anisotropic solids. He extended Nabarro's calculation of the width of a dislocation to the anisotropic case. His results are limited to edge dislocations with an axis that is an infinite straight line, but there is no restriction on the type of anisotropy of the medium. Eshelby *et al.* [111] developed the general solution for the induced displacement fields of dislocations in homogeneous linear anisotropic solids for the special case of the elastic state being independent of one of the three Cartesian coordinates. The dynamical response of uniformly moving dislocations in linear anisotropic media was studied by Teutonico [112]. It was observed that both edge and screw dislocations are prone to exhibiting anomalous dynamical behavior such that the interaction force between two parallel dislocations (on the same slip plane) changes sign when dislocation velocity increases. Head [113] predicted instabilities of dislocations in some anisotropic metallic crystals. It was found that a straight dislocation may decrease its energy if it changes to a zig-zag shape, i.e., a straight dislocation may be unstable. In the setting of the linear theory of elasticity, Willis [114] analyzed dislocations in anisotropic media (see also [115]). Particularly, the displacement fields of infinite straight dislocations and plane curvilinear dislocation loops were obtained. Schaefer and Kronmüller [116] investigated the elastic interaction of point defects in linear isotropic and anisotropic cubic media using Green's function approach. They specifically discussed the differences between the interactions in isotropic and anisotropic materials and the effects of anisotropy on the interaction potential. Some basic developments in the linear theory of dislocations in anisotropic media were presented in [117, 118]. Methods for obtaining the induced linear elastic fields of defects in transversely isotropic bimaterials and orthotropic bicrystals (in 2D) were proposed in [119] and [120], respectively. In particular, some closed-form solutions for the elastic fields of inclusions and dislocations were presented.

A successive approximation method was proposed in [121] to study the nonlinear screw dislocation problem using the linear elasticity solution. Nonetheless, the method fails to find the correct solution near the dislocation axis. Only a handful of exact solutions for defects in nonlinear elastic solids exist in the literature, and they are all restricted to isotropic materials. We should mention [122, 123, 124, 125, 126, 44] for dislocations, [124, 127, 128] for disclinations, and [63, 64, 129] for point defects and discombinations.

To the best of our knowledge, despite the known importance of the anisotropic behavior of solids, especially at finite strains, the study of defects in the setting of nonlinear elasticity has been limited to isotropic solids. In this chapter we study several examples of line and point defects in nonlinear anisotropic solids and present some analytical solutions for their stress fields. We consider an arbitrary cylindrically-symmetric distribution of parallel screw dislocations in orthotropic and monoclinic media, along with a parallel cylindrically-symmetric distribution of wedge disclinations in an infinite orthotropic medium. As the geometry of the material manifold explicitly depends on the distribution of defects, the material preferred directions (that identify the type of anisotropy) in the reference configuration explicitly depend on the defect distribution as well, and, in general, are different from those of the material in its current configuration. For instance, for the distributed screw dislocations that we consider, the assumption that the dislocated body is orthotropic in the reference (current) configuration implies that the body is monoclinic in the current (reference) configuration.

In this chapter, the boundedness of the stress on the dislocation and disclination axes will be discussed. In particular, for an arbitrary cylindrically-symmetric distribution of parallel screw dislocations the stress exhibits a logarithmic singularity on the dislocation axis unless the axial deformation is suppressed. Note that these singularities arise due to the presence of anisotropy, e.g., radial fiber-reinforcement, and in particular, they do not occur when the material is isotropic. Exploiting the so-called standard reinforcing model (see, e.g., [92]), we obtain conditions under which the energy per unit length and the resultant longitudinal force of a single screw dislocation for a fiber-reinforced material are finite provided that the isotropic base material has a finite axial force and a finite energy per unit length. Employing Cartan's moving frames approach, for a given distribution of edge dislocations we will construct the material manifold and obtain explicit solutions for the stress field when the medium is orthotropic. We will also consider a spherically-symmetric distribution of point defects in a finite transversely isotropic spherical ball. We will show that for an arbitrary incompressible transversely isotropic material with the radial material preferred direction a uniform point defect distribution induces a uniform hydrostatic stress inside the region the distribution is supported.

The rest of the chapter is structured as follows. In §6.2 we tersely review some fundamentals of geometric nonlinear anisotropic elasticity and some related topics on nonlinear defect mechanics. We consider a cylindrically-symmetric distribution of parallel screw dislocations in orthotropic and monoclinic media in §6.3.1 and §6.3.2, respectively. A cylindrically-symmetric distribution of parallel wedge disclinations in an orthotropic medium is studied in §6.3.3. In §6.3.4 edge dislocations in an orthotropic medium are considered. In §6.3.5 we calculate the residual stresses due to a spherically-symmetric distribution of point defects in a transversely isotropic ball. Note that the results of this section have been previously reported in our published work [40].

6.2 Geometric Anelasticity for Anisotropic Solids

In this section we briefly review some fundamental elements of the geometric theory of nonlinear elasticity for anisotropic solids. For more detailed discussions, see [43, 46].

Kinematics. A body $\mathcal{B}$ is identified with a Riemannian manifold $(\mathcal{B}, \mathbf{G})$, and a configuration of $\mathcal{B}$ is a smooth embedding $\varphi : \mathcal{B} \to \mathcal{S}$, where $(\mathcal{S}, \mathbf{g})$ is a Riemannian manifold —the ambient space. An affine connection ∇ on a smooth manifold $\mathcal{B}$ is a linear mapping $\nabla : \mathcal{X}(\mathcal{B}) \times \mathcal{X}(\mathcal{B}) \to \mathcal{X}(\mathcal{B})$, where $\mathcal{X}(\mathcal{B})$ represents the set of all smooth vector fields on $\mathcal{B}$, such that the following properties are satisfied $\forall\, \mathbf{X}, \mathbf{Y}, \mathbf{X}_1, \mathbf{X}_2, \mathbf{Y}_1, \mathbf{Y}_2 \in \mathcal{X}(\mathcal{B}), \forall\, f, f_1, f_2 \in C^\infty(\mathcal{B}), \forall\, a_1, a_2 \in \mathbb{R}$ (see [47, 48] for more details): a) $\nabla_{f_1\mathbf{X}_1+f_2\mathbf{X}_2}\mathbf{Y} = f_1\nabla_{\mathbf{X}_1}\mathbf{Y} + f_2\nabla_{\mathbf{X}_2}\mathbf{Y}$, b) $\nabla_{\mathbf{X}}(a_1\mathbf{Y}_1 + a_2\mathbf{Y}_2) = a_1\nabla_{\mathbf{X}}(\mathbf{Y}_1) + a_2\nabla_{\mathbf{X}}(\mathbf{Y}_2)$, c) $\nabla_{\mathbf{X}}(f\mathbf{Y}) = f\nabla_{\mathbf{X}}\mathbf{Y} + (\mathbf{X}f)\mathbf{Y}$. It can be shown that there is a unique torsion-free and compatible affine connection associated with any Riemannian manifold that is called a Riemannian connection. Let us denote the Levi-Civita connection associated with the Riemannian manifolds $(\mathcal{B}, \mathbf{G})$ and $(\mathcal{S}, \mathbf{g})$ by $\nabla^{\mathbf{G}}$ and $\nabla^{\mathbf{g}}$, respectively. We denote the set of all configurations of $\mathcal{B}$ by $\mathcal{C}$. A motion is a curve $c : \mathbb{R}^+ \to \varphi_t \in \mathcal{C}$ such that φ_t assigns a spatial point $x = \varphi_t(X) = \varphi(X, t) \in \mathcal{S}$ to every material point $X \in \mathcal{B}$ at any time t. The body is assumed to be stress-free in its reference configuration, which may have a nontrivial geometry, in general, e.g., in the presence of eigenstrains. The deformation gradient $\mathbf{F}$ is the tangent map of φ defined as $\mathbf{F}(X,t) = d\varphi_t(X) : T_X\mathcal{B} \to T_{\varphi_t(X)}\mathcal{S}$. The adjoint of $\mathbf{F}$ is defined as $\mathbf{F}^\mathsf{T}(X,t) : T_{\varphi_t(X)}\mathcal{S} \to T_X\mathcal{B}$, $\mathbf{g}\left(\mathbf{F}\mathbf{V},\mathbf{v}\right) = \mathbf{G}\left(\mathbf{V},\mathbf{F}^\mathsf{T}\mathbf{v}\right), \forall\, \mathbf{V} \in T_X\mathcal{B},\ \mathbf{v} \in T_{\varphi_t(X)}\mathcal{S}$. The right Cauchy-Green deformation tensor is defined as $\mathbf{C}(X,t) = \mathbf{F}^\mathsf{T}(X,t)\mathbf{F}(X,t) : T_X\mathcal{B} \to T_X\mathcal{B}$. The Finger deformation tensor is defined as $\mathbf{b}(x,t) = \mathbf{F}(X,t)\mathbf{F}^\mathsf{T}(X,t) : T_x\varphi\left(\mathcal{B}\right) \to T_x\varphi\left(\mathcal{B}\right)$, and in components, $b^{ab} = F^a{}_A F^b{}_B G^{AB}$. Another measure of strain is the Lagrangian strain tensor $\mathbf{E} = \frac{1}{2}\left(\varphi_t^*\mathbf{g} - \mathbf{G}\right)$, where $\varphi_t^*\mathbf{g}$ is the pull-back of the spatial metric (in components $(\varphi_t^*\mathbf{g})_{AB} = F^a{}_A F^b{}_B g_{ab}$). The Jacobian of deformation J relates the Riemannian volume element of the material manifold $dV(X,\mathbf{G})$ to that of the spatial manifold $dv(\varphi_t(X),\mathbf{g})$, written as

$$J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}}\det \mathbf{F}\,, \quad dv(x,\mathbf{g}) = J\, dV(X,\mathbf{G})\,. \tag{6.1}$$

Equilibrium Equations. The localized balance of linear momentum in spatial and material forms are written as

$$\operatorname{div}\boldsymbol{\sigma} + \rho\mathbf{b} = \rho\mathbf{a}, \quad \operatorname{Div}\mathbf{P} + \rho_0\mathbf{B} = \rho_0\mathbf{A}, \tag{6.2}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress and $\mathbf{P}$ is the first Piola-Kirchhoff stress, and ρ_0, $\mathbf{A}$, and $\mathbf{B}$ are the material mass density, material acceleration, and material body force, respectively, and ρ, $\mathbf{a}$, and $\mathbf{b}$ are their corresponding spatial counterparts. Note that the material and spatial divergence operators in components are given as

$$\begin{aligned}(\operatorname{div}\boldsymbol{\sigma})^a &= \sigma^{ab}{}_{|b} = \frac{\partial\sigma^{ab}}{\partial x^b} + \sigma^{ac}\gamma^b{}_{cb} + \sigma^{cb}\gamma^a{}_{cb}\,,\\ (\operatorname{Div}\mathbf{P})^a &= P^{aA}{}_{|A} = \frac{\partial P^{aA}}{\partial X^A} + P^{aB}\Gamma^A{}_{AB} + P^{cA}F^b{}_A\gamma^a{}_{bc}\,,\end{aligned} \tag{6.3}$$

where $\gamma^a{}_{bc}$ and $\Gamma^A{}_{BC}$ denote the Christoffel symbols of the connections $\nabla^\mathbf{g}$ and $\nabla^\mathbf{G}$, respectively. Note that in the local coordinate charts $\{x^a\}$ and $\{X^A\}$, one has $\nabla^\mathbf{g}{}_{\partial_b}\partial_c = \gamma^a{}_{bc}\partial_a$ and $\nabla^\mathbf{G}{}_{\partial_B}\partial_C = \Gamma^A{}_{BC}\partial_A$.

Material Symmetry Group. In the case of a *simple* material, the response function at any material point depends only on the first deformation gradient (and its evolution) at that point [130]. Consider an elastic body made of a simple material with the response function W at a material point X. We assume that the response function is the energy function. A response function may be any measure of stress as well. The material symmetry group $\mathcal{G}_X$ associated with the body at the point X with respect to the reference configuration $(\mathcal{B}, \mathbf{G})$ is defined as

$$W(X, \mathbf{FK}, \mathbf{G}, \mathbf{g}) = W(\mathbf{F}, \mathbf{G}, \mathbf{g})\,, \qquad \forall\ \mathbf{K} \in \mathcal{G}_X\,, \tag{6.4}$$

for all deformation gradients $\mathbf{F}$, where $\mathbf{K} : T_X\mathcal{B} \to T_X\mathcal{B}$ is an invertible linear transformation. For hyperelastic solids, objectivity implies that the energy function depends on the deformation at a referential point X through the right Cauchy-Green deformation tensor $\mathbf{C}^\flat$, i.e., $W = W(X, \mathbf{C}^\flat, \mathbf{G})$. Thus, the material symmetry group $\mathcal{G}_X$ for a hyperelastic solid is defined to be the subgroup of $\mathbf{G}$-orthogonal transformations Orth$(\mathbf{G})$ such that [131]

$$W(X, \mathbf{Q}^{-\star}\mathbf{C}^\flat\mathbf{Q}^{-1}, \mathbf{G}) = W(X, \mathbf{C}^\flat, \mathbf{G})\,, \qquad \forall\ \mathbf{Q} \in \mathcal{G}_X \leqslant \mathrm{Orth}\,(\mathbf{G})\ , \tag{6.5}$$

where Orth$(\mathbf{G}) = \left\{\mathbf{Q} : T_X\mathcal{B} \to T_X\mathcal{B} \quad | \quad \mathbf{Q}^\top = \mathbf{Q}^{-1}\right\}$, and we use the notation $\mathscr{G} \leqslant \mathscr{H}$ when $\mathscr{G}$ is a subgroup of $\mathscr{H}$. Note that the set of orthogonal transformations is explicitly metric dependent. In other words, if the material metric $\mathbf{G}$ changes, Orth$(\mathbf{G})$, and hence, $\mathcal{G}_X$ changes as well. The symmetry group can be equivalently characterized using a finite collection of structural tensors $\boldsymbol{\zeta}_i$ of order μ_i, $i = 1, \ldots, n$, forming a basis for the space of tensors that are invariant under the action of $\mathcal{G}$ as follows (also, see [49, 51, 132])

$$\mathbf{Q} \in \mathcal{G} \leqslant \mathrm{Orth}\,(\mathbf{G}) \iff \langle \mathbf{Q} \rangle_{\mu_1} \boldsymbol{\zeta}_1 = \boldsymbol{\zeta}_1\,, \ldots\,, \langle \mathbf{Q} \rangle_{\mu_n} \boldsymbol{\zeta}_n = \boldsymbol{\zeta}_n\,, \tag{6.6}$$

where $\langle \mathbf{Q} \rangle_\mu$ is the μ-th power Kronecker product of a $\mathbf{G}$-orthogonal transformation $\mathbf{Q}$ defined for any μ-th order tensor $\boldsymbol{\zeta}$ as $(\langle \mathbf{Q} \rangle_\mu \zeta)^{\bar{A}_1 \ldots \bar{A}_\mu} = Q^{\bar{A}_1}{}_{A_1} \ldots Q^{\bar{A}_\mu}{}_{A_\mu} \zeta^{A_1 \ldots A_\mu}$. Note that (6.6) suggests that the material symmetry group $\mathcal{G}$ is the invariance group of the set of the structural tensors $\boldsymbol{\zeta}_i$, $i = 1, \ldots, n$.

Remark 6.2.1. We define the material symmetry group at a material point in its natural (stress-free) state. This has the following physical interpretation. One is given a body with a distribution of defects that is residually-stressed in its current configuration. Now imagine that the body is partitioned into a large number of small elements and each is allowed to relax. The symmetry group of a material point in this locally relaxed configuration is the same as its symmetry group in the Riemannian material manifold $(\mathcal{B}, \mathbf{G})$.

Remark 6.2.2. In the so-called theory of "material uniformity" of Noll [133] and Wang [134] one characterizes uniformity of the mechanical response of a body that may be residually stressed, although Noll and Wang did not explicitly mention residual stresses (see also [135]). Their developments are essentially based on the multiplicative decomposition of the deformation gradient into elastic and plastic parts: $\mathbf{F} = \overset{\mathrm{e}}{\mathbf{F}}\overset{\mathrm{p}}{\mathbf{F}}$. They call $\overset{\mathrm{p}}{\mathbf{F}}$ and $\overset{\mathrm{e}}{\mathbf{F}}$ a "local configuration", and a "local deformation", respectively. A body is "materially uniform" if its energy function (or any response function) depends only on $\overset{\mathrm{e}}{\mathbf{F}}$ and not on the material point X. In our formulation of anelasticity, the deformation gradient is purely elastic; the plastic (defect) part is buried into the material metric. Therefore, our symmetry group $\mathcal{G}_X$ is what Noll and Wang call the "isotropy group relative" to $\overset{\mathrm{p}}{\mathbf{F}}$. Their "isotropy group relative" to $\overset{\mathrm{p}}{\mathbf{F}}$ explicitly depends on $\overset{\mathrm{p}}{\mathbf{F}}$ while our material symmetry group explicitly depends on the material metric $\mathbf{G}$. What Wang [134] calls an "intrinsic" Riemannian metric is our material metric $\mathbf{G} = \overset{\mathrm{p}}{\mathbf{F}}{}^{\mathsf{T}}\overset{\mathrm{p}}{\mathbf{F}}$. However, Noll and Wang did not use the concept of natural distances and a stress-free reference configuration; their main interest was the material symmetry group. In particular, for isotropic solids the symmetry group is preserved under a uniform scaling of the "intrinsic" Riemannian metric, i.e., this metric is unique for isotropic solids up to a constant factor [134]. However, note that a scaling of $\mathbf{G}$ changes the natural distances. In other words, two material metrics related by a uniform scaling are not equivalent in our geometric theory of anelasticity as they do not correspond to equivalent reference configurations. More specifically, if the body is stress-free in $(\mathcal{B}, \mathbf{G})$, in general, it is not stress-free in $(\mathcal{B}, \alpha^2\mathbf{G})$, if for example, the boundary has prescribed displacements.

Constitutive Equations. In this chapter our calculations are restricted to incompressible transversely isotropic, orthotropic, and monoclinic solids. To establish a materially covariant strain energy density function, *structural tensors* corresponding to the symmetry group of the material are used. For detailed discussions on structural tensors and the determination of the integrity basis and the corresponding invariants of a set of tensors, see [49, 50, 51, 52, 53].

Transverse Isotropy. Let us assume a compressible transversely isotropic material such that the unit vector $\mathbf{N}(\mathrm{X})$ identifies the material preferred direction at a point X in the reference configuration. The strain energy density per unit volume of the reference configuration is given as (see, e.g., [56, 50, 53]) $W = W(\mathrm{X}, \mathbf{G}, \mathbf{C}^\flat, \mathbf{A})$, where $\mathbf{A} = \mathbf{N} \otimes \mathbf{N}$ is a structural tensor representing the transverse isotropy of the material symmetry group, and $(.)^\flat$ denotes the flat operator for lowering tensor indices. The second Piola-Kirchhoff stress tensor is given by

$$\mathbf{S} = 2\frac{\partial W}{\partial \mathbf{C}^\flat}\,. \tag{6.7}$$

The energy function W depends on the following five independent invariants defined as

$$I_1 = \operatorname{tr}\mathbf{C}\,, \quad I_2 = \det\mathbf{C}\operatorname{tr}\mathbf{C}^{-1}\,, \quad I_3 = \det\mathbf{C}\,, \quad I_4 = \mathbf{N}\cdot\mathbf{C}\cdot\mathbf{N}\,, \quad I_5 = \mathbf{N}\cdot\mathbf{C}^2\cdot\mathbf{N}\,. \tag{6.8}$$

In components they read

$$I_1 = C^A{}_A\,,\ I_2 = \det(C^A{}_B)(C^{-1})^D{}_D\,,\ I_3 = \det(C^A{}_B)\,,\ I_4 = N^A N^B C_{AB}\,,\ I_5 = N^A N^B C_{BQ} C^Q{}_A\,. \tag{6.9}$$

Using (6.7), one obtains[2]

$$\mathbf{S} = \sum_{n=1}^{5} 2W_{I_n}\frac{\partial I_n}{\partial \mathbf{C}^\flat}\,, \qquad W_{I_n} := \frac{\partial W}{\partial I_n}\,, \quad n = 1, \ldots, 5\,. \tag{6.10}$$

[2]For the sake of brevity, we do not assume an explicit dependence of W on X, which in the case of inhomogeneous bodies is needed. We suppose instead that the material is piece-wise homogeneous and model an inhomogeneity using different energy functions in different regions of the body.

Note that

$$\frac{\partial I_1}{\partial \mathbf{C}^\flat} = \mathbf{G}^\sharp\,, \;\; \frac{\partial I_2}{\partial \mathbf{C}^\flat} = I_2\mathbf{C}^{-1} - I_3\mathbf{C}^{-2}\,, \;\; \frac{\partial I_3}{\partial \mathbf{C}^\flat} = I_3\mathbf{C}^{-1}\,, \;\; \frac{\partial I_4}{\partial \mathbf{C}^\flat} = \mathbf{N}\otimes\mathbf{N}\,, \quad \frac{\partial I_5}{\partial \mathbf{C}^\flat} = \mathbf{N}\otimes\mathbf{C}\cdot\mathbf{N} + \mathbf{N}\cdot\mathbf{C}\otimes\mathbf{N}\,, \tag{6.11}$$

where $(.)^\sharp$ is the sharp operator for raising tensor indices. Thus, from (6.10) and (7.32), one obtains the following representation for the second Piola-Kirchhoff stress tensor

$$\mathbf{S} = 2\left\{W_{I_1}\mathbf{G}^\sharp + W_{I_2}\left(I_2\mathbf{C}^{-1} - I_3\mathbf{C}^{-2}\right) + W_{I_3}I_3\mathbf{C}^{-1} + W_{I_4}\left(\mathbf{N}\otimes\mathbf{N}\right) + W_{I_5}\left(\mathbf{N}\otimes\mathbf{C}\cdot\mathbf{N} + \mathbf{N}\cdot\mathbf{C}\otimes\mathbf{N}\right)\right\}\,. \tag{6.12}$$

If the material is incompressible, then $I_3 = 1$, and thus, $W = W\left(\mathrm{X}, I_1, I_2, I_4, I_5\right)$. Therefore, from (6.12), $\mathbf{S}$ is expressed as

$$\mathbf{S} = 2\left\{W_{I_1}\mathbf{G}^\sharp + W_{I_2}\left(I_2\mathbf{C}^{-1} - \mathbf{C}^{-2}\right) + W_{I_4}\left(\mathbf{N}\otimes\mathbf{N}\right) + W_{I_5}\left(\mathbf{N}\otimes\mathbf{C}\cdot\mathbf{N} + \mathbf{N}\cdot\mathbf{C}\otimes\mathbf{N}\right)\right\} - p\mathbf{C}^{-1}\,, \tag{6.13}$$

in which p is the Lagrange multiplier associated with the incompressibility condition $J = 1$. The Cauchy stress tensor $\sigma^{ab} = \frac{1}{J}F^a{}_A F^b{}_B S^{AB}$ is represented in component form as[3]

$$\sigma^{ab} = 2F^a{}_A F^b{}_B\left[(W_{I_1} + I_1 W_{I_2})G^{AB} - W_{I_2}C^{AB} + W_{I_4}N^A N^B + W_{I_5}\left(N^Q N^A C^B{}_Q + N^P N^B C_P{}^A\right)\right] - pg^{ab}. \tag{6.14}$$

Orthotropy. Next, we consider a compressible orthotropic material with three $\mathbf{G}$-orthonormal vectors $\mathbf{N}_1(\mathrm{X})$, $\mathbf{N}_2(\mathrm{X})$, and $\mathbf{N}_3(\mathrm{X})$ specifying the orthotropic axes in the reference configuration at a point X. A choice of structural tensors is given by $\mathbf{A}_1 = \mathbf{N}_1\otimes\mathbf{N}_1$, $\mathbf{A}_2 = \mathbf{N}_2\otimes\mathbf{N}_2$, and $\mathbf{A}_3 = \mathbf{N}_3\otimes\mathbf{N}_3$, where only two of which are independent as $\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3 = \mathbf{I}$. Hence, the energy function is given as [56, 50, 53]

$$W = W(\mathrm{X}, \mathbf{G}, \mathbf{C}^\flat, \mathbf{A}_1, \mathbf{A}_2)\,. \tag{6.15}$$

[3] Note that one can use the Cayley-Hamilton theorem to obtain $\frac{\partial I_2}{\partial \mathbf{C}^\flat} = I_2(\mathbf{C}^{-1})^\sharp - I_3(\mathbf{C}^{-2})^\sharp = I_1\mathbf{G}^\sharp - \mathbf{C}^\sharp$.

The energy function W is represented in terms of the following seven independent invariants

$$\begin{gathered} I_1 = \operatorname{tr}\mathbf{C}\,, \quad I_2 = \det\mathbf{C}\operatorname{tr}\mathbf{C}^{-1}\,, \quad I_3 = \det\mathbf{C}\,, \quad I_4 = \mathbf{N}_1\cdot\mathbf{C}\cdot\mathbf{N}_1\,, \\ I_5 = \mathbf{N}_1\cdot\mathbf{C}^2\cdot\mathbf{N}_1\,, \quad I_6 = \mathbf{N}_2\cdot\mathbf{C}\cdot\mathbf{N}_2\,, \quad I_7 = \mathbf{N}_2\cdot\mathbf{C}^2\cdot\mathbf{N}_2\,. \end{gathered} \tag{6.16}$$

Using (6.7), one writes

$$\mathbf{S} = \sum_{n=1}^{7} 2W_{I_n}\frac{\partial I_n}{\partial \mathbf{C}^\flat}\,, \qquad W_{I_n} := \frac{\partial W}{\partial I_n}\,, \quad n = 1,\ldots,7\,. \tag{6.17}$$

Substituting (7.32) into (6.17), the second Piola-Kirchhoff stress tensor is given by

$$\begin{aligned} \mathbf{S} = 2\Big\{ & W_{I_1}\mathbf{G}^\sharp + W_{I_2}\left(I_2\mathbf{C}^{-1} - I_3\mathbf{C}^{-2}\right) + W_{I_3}I_3\mathbf{C}^{-1} + W_{I_4}\left(\mathbf{N}_1\otimes\mathbf{N}_1\right) + W_{I_5}\left(\mathbf{N}_1\otimes\mathbf{C}\cdot\mathbf{N}_1 + \mathbf{N}_1\cdot\mathbf{C}\otimes\mathbf{N}_1\right) \\ & + W_{I_6}\left(\mathbf{N}_2\otimes\mathbf{N}_2\right) + W_{I_7}\left(\mathbf{N}_2\otimes\mathbf{C}\cdot\mathbf{N}_2 + \mathbf{N}_2\cdot\mathbf{C}\otimes\mathbf{N}_2\right)\Big\}\,. \end{aligned} \tag{6.18}$$

In the case of incompressible solids $I_3 = 1$ and $W = W(\mathrm{X}, I_1, I_2, I_4, I_5, I_6, I_7)$. Therefore, using (6.18), one obtains the following representation for the second Piola-Kirchhoff stress tensor

$$\begin{aligned} \mathbf{S} = 2\Big\{ & W_{I_1}\mathbf{G}^\sharp + W_{I_2}\left(I_2\mathbf{C}^{-1} - \mathbf{C}^{-2}\right) + W_{I_4}\left(\mathbf{N}_1\otimes\mathbf{N}_1\right) + W_{I_5}\left(\mathbf{N}_1\otimes\mathbf{C}\cdot\mathbf{N}_1 + \mathbf{N}_1\cdot\mathbf{C}\otimes\mathbf{N}_1\right) \\ & + W_{I_6}\left(\mathbf{N}_2\otimes\mathbf{N}_2\right) + W_{I_7}\left(\mathbf{N}_2\otimes\mathbf{C}\cdot\mathbf{N}_2 + \mathbf{N}_2\cdot\mathbf{C}\otimes\mathbf{N}_2\right)\Big\} - p\mathbf{C}^{-1}\,. \end{aligned} \tag{6.19}$$

In components, the Cauchy stress tensor is given as

$$\begin{aligned} \sigma^{ab} = 2F^a{}_A F^b{}_B\Big[& (W_{I_1} + I_1 W_{I_2})G^{AB} - W_{I_2}C^{AB} + W_{I_4}N_1{}^A N_1{}^B + W_{I_5}\left(N_1{}^Q N_1{}^A C^B{}_Q + N_1{}^P N_1{}^B C_P{}^A\right) \\ & + W_{I_6}N_2{}^A N_2{}^B + W_{I_7}\left(N_2{}^S N_2{}^A C^B{}_S + N_2{}^K N_2{}^B C_K{}^A\right)\Big] - p g^{ab}\,. \end{aligned} \tag{6.20}$$

Monoclinic Symmetry. One of the preferred directions of a material with a monoclinic symmetry (say $\mathbf{N}_3(\mathrm{X})$) is perpendicular to the plane of the other two (denoted by $\mathbf{N}_1(\mathrm{X})$ and $\mathbf{N}_2(\mathrm{X})$), which are not orthogonal. As an example one can consider an isotropic base material reinforced with two families of fibers such that the fibers are not at right angles, nor are they mechanically equivalent. In this case, the energy function is similar to that of orthotropic materials given by (6.15), where $\mathbf{A}_1 = \mathbf{N}_1 \otimes \mathbf{N}_1$ and $\mathbf{A}_2 = \mathbf{N}_2 \otimes \mathbf{N}_2$. Nonetheless, an extra invariant $I_8 = (\mathbf{N}_1 \cdot \mathbf{N}_2)\mathbf{N}_1 \cdot \mathbf{C} \cdot \mathbf{N}_2$ that models the coupling between the fibers (in $\mathbf{N}_1$ and $\mathbf{N}_2$ directions) is needed to express the energy function for monoclinic materials as $\mathbf{N}_1$ and $\mathbf{N}_2$ are not perpendicular (see [136, 94, 137]). Therefore

$$\mathbf{S} = \sum_{n=1}^{8} 2W_{I_n} \frac{\partial I_n}{\partial \mathbf{C}^\flat}\,, \qquad W_{I_n} := \frac{\partial W}{\partial I_n}\,, \quad n = 1, \ldots, 8\,. \tag{6.21}$$

Hence[4]

$$\begin{aligned}\mathbf{S} = 2\Big\{ & W_{I_1}\mathbf{G}^\sharp + W_{I_2}\left(I_2\mathbf{C}^{-1} - I_3\mathbf{C}^{-2}\right) + W_{I_3}I_3\mathbf{C}^{-1} + W_{I_4}\left(\mathbf{N}_1\otimes\mathbf{N}_1\right) + W_{I_5}\left(\mathbf{N}_1\otimes\mathbf{C}\cdot\mathbf{N}_1 + \mathbf{N}_1\cdot\mathbf{C}\otimes\mathbf{N}_1\right) \\ & + W_{I_6}\left(\mathbf{N}_2\otimes\mathbf{N}_2\right) + W_{I_7}\left(\mathbf{N}_2\otimes\mathbf{C}\cdot\mathbf{N}_2 + \mathbf{N}_2\cdot\mathbf{C}\otimes\mathbf{N}_2\right) + \frac{W_{I_8}}{2}\left(\mathbf{N}_1\otimes\mathbf{N}_2 + \mathbf{N}_2\otimes\mathbf{N}_1\right)\Big\}\,,\end{aligned} \tag{6.22}$$

and for incompressible solids

$$\begin{aligned}\mathbf{S} = 2\Big\{ & W_{I_1}\mathbf{G}^\sharp + W_{I_2}\left(I_2\mathbf{C}^{-1} - \mathbf{C}^{-2}\right) + W_{I_4}\left(\mathbf{N}_1\otimes\mathbf{N}_1\right) + W_{I_5}\left(\mathbf{N}_1\otimes\mathbf{C}\cdot\mathbf{N}_1 + \mathbf{N}_1\cdot\mathbf{C}\otimes\mathbf{N}_1\right) \\ & + W_{I_6}\left(\mathbf{N}_2\otimes\mathbf{N}_2\right) + W_{I_7}\left(\mathbf{N}_2\otimes\mathbf{C}\cdot\mathbf{N}_2 + \mathbf{N}_2\cdot\mathbf{C}\otimes\mathbf{N}_2\right) + \frac{W_{I_8}}{2}\left(\mathbf{N}_1\otimes\mathbf{N}_2 + \mathbf{N}_2\otimes\mathbf{N}_1\right)\Big\} - p\mathbf{C}^{-1}\,.\end{aligned} \tag{6.23}$$

The Cauchy stress is given in components as

$$\begin{aligned}\sigma^{ab} = 2F^a{}_AF^b{}_B\Big[& (W_{I_1} + I_1W_{I_2})G^{AB} - W_{I_2}C^{AB} + W_{I_4}N_1{}^AN_1{}^B + W_{I_5}\left(N_1{}^QN_1{}^AC^B{}_Q + N_1{}^PN_1{}^BC_P{}^A\right) \\ & + W_{I_6}N_2{}^AN_2{}^B + W_{I_7}\left(N_2{}^SN_2{}^AC^B{}_S + N_2{}^KN_2{}^BC_K{}^A\right) + \frac{W_{I_8}}{2}\left(N_1{}^AN_2{}^B + N_2{}^AN_1{}^B\right)\Big] - pg^{ab}\,.\end{aligned} \tag{6.24}$$

Cartan's Moving Frame. At a point X of a manifold $\mathcal{B}$ consider an orthonormal frame field $\{\mathbf{e}_\alpha\}_{\alpha=1}^N$ forming a basis for $T_X\mathcal{B}$. This frame field is not necessarily a coordinate basis for the

[4] Note that $\frac{\partial I_8}{\partial \mathbf{C}^\flat} = \mathbf{N}_1 \otimes \mathbf{N}_2 + \mathbf{N}_2 \otimes \mathbf{N}_1$.

tangent space. However, given a coordinate basis $\{\frac{\partial}{\partial X^A}\}$, one can obtain an arbitrary frame field $\{\mathbf{e}_\alpha\}$ using an $SO(N,\mathbb{R})$-rotation of the coordinate basis such that $\mathbf{e}_\alpha = \mathsf{F}^A{}_\alpha \frac{\partial}{\partial X^A}$. For a coordinate frame $\left[\frac{\partial}{\partial X^A}, \frac{\partial}{\partial X^B}\right] = 0$,[5] whereas for the non-coordinate frame, $[\mathbf{e}_\alpha, \mathbf{e}_\beta] = -c^\gamma{}_{\alpha\beta}\mathbf{e}_\gamma$, where $c^\gamma{}_{\alpha\beta}$ are the componenets of the object of anhonolomy. One can show that $c^\gamma{}_{\alpha\beta} = \mathsf{F}^A{}_\alpha \mathsf{F}^B{}_\beta \left(\partial_A \mathsf{F}^\gamma{}_B - \partial_B \mathsf{F}^\gamma{}_A\right)$, where $\mathsf{F}^\gamma{}_A$ is the inverse of $\mathsf{F}^A{}_\gamma$. Connection 1-forms are defined by $\nabla \mathbf{e}_\alpha = \mathbf{e}_\gamma \otimes \omega^\gamma{}_\alpha$, and in components, $\nabla_{\mathbf{e}_\beta}\mathbf{e}_\alpha = \langle \omega^\gamma{}_\alpha, \mathbf{e}_\beta \rangle \mathbf{e}_\gamma = \omega^\gamma{}_{\beta\alpha}\mathbf{e}_\gamma$. In terms of the co-frame field $\{\vartheta^\alpha\}_{\alpha=1}^N$ corresponding to $\{\mathbf{e}_\alpha\}$, one has $\omega^\gamma{}_\alpha = \omega^\gamma{}_{\beta\alpha}\vartheta^\beta$. Similarly, one obtains $\nabla\vartheta^\alpha = -\omega^\alpha{}_\gamma \vartheta^\gamma$ and $\nabla_{\mathbf{e}_\beta}\vartheta^\alpha = -\omega^\alpha{}_{\beta\gamma}\vartheta^\gamma$. The metric tensor is represented as $\mathbf{G} = \delta_{\alpha\beta}\vartheta^\alpha \otimes \vartheta^\beta$. Metric compatibility of ∇ gives the following constraints on the connection 1-forms $\delta_{\alpha\gamma}\omega^\gamma{}_\beta + \delta_{\beta\gamma}\omega^\gamma{}_\alpha = 0$. In a non-coordinate basis, the torsion and curvature have the following components $T^\alpha{}_{\beta\gamma} = \omega^\alpha{}_{\beta\gamma} - \omega^\alpha{}_{\gamma\beta} + c^\alpha{}_{\beta\gamma}$ and $\mathcal{R}^\alpha{}_{\beta\lambda\mu} = \partial_\beta \omega^\alpha{}_{\lambda\mu} - \partial_\lambda \omega^\alpha{}_{\beta\mu} + \omega^\alpha{}_{\beta\xi}\omega^\xi{}_{\lambda\mu} - \omega^\alpha{}_{\lambda\xi}\omega^\xi{}_{\beta\mu} + \omega^\alpha{}_{\xi\mu}c^\xi{}_{\beta\lambda}$, respectively. Torsion and curvature 2-forms are, respectively, given by $\mathcal{T}^\alpha = d\vartheta^\alpha + \omega^\alpha{}_\beta \wedge \vartheta^\beta$ and $\mathcal{R}^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta$. These are Cartan's first and second structural equations. The density of Burgers' vector $\mathbf{b}$ at a point X of $\mathcal{B}$ is related to torsion 2-from as follows

$$b^\alpha(X; C_s) = \int_{\Omega_s} \mathsf{P}^\alpha{}_\beta \mathcal{T}^\beta \,, \tag{6.25}$$

where $\Omega_s \in \mathcal{B}$ is a smooth surface with a boundary given by the curve C_s, and $\mathsf{P}(C_s)^t_\tau : T_{C_s(\tau)}\mathcal{B} \to T_{C_s(t)}\mathcal{B}$ parallel transports vectors tangent to the manifold at $C_s(\tau)$ to $C_s(t)$ (see [138, 139] for more details).

6.3 Examples of Anisotropic Bodies with Distributed Defects

In this section, we consider several examples of distributed defects in cylindrical bars made of orthotropic and monoclinic solids as well as distributed defects in spherical balls made of transversely isotropic solids. Particularly, we consider cylindrically-symmetric distributions of parallel screw dislocations and disclinations in an orthotropic medium, a spherically-symmetric distribution of point defects in a transversely isotropic spherical ball, and a cylindrically-symmetric distribution of screw

[5] Note that for any pair of vector fields $\mathbf{U}$ and $\mathbf{V}$ on $\mathcal{B}$, one can define a new vector field —the commutator —given by $[\mathbf{U},\mathbf{V}]_X f := \mathbf{U}_X(\mathbf{V}f) - \mathbf{V}_X(\mathbf{V}f)$, for any smooth function at X on $\mathcal{B}$.

dislocations in a monoclinic medium. We also discuss the effects of the constitutive parameters on the induced stress fields for different types of defects.

6.3.1 A Cylindrically-Symmetric Distribution of Parallel Screw Disclocations in an Orthotropic Medium

Let us consider a cylindrically-symmetric distribution of screw dislocations parallel to the Z-axis with a radially-symmetric Burgers' vector density $b(R)$ (in a cylindrical coordinate system (R, Θ, Z)) in an infinite orthotropic medium. We assume that in the reference configuration the dislocated body is orthotropic. The material preferred directions at a material point X are denoted by $\mathbf{N}_1(\mathrm{X})$, $\mathbf{N}_2(\mathrm{X})$, and $\mathbf{N}_3(\mathrm{X})$ in the reference configuration. In the current configuration, the preferred directions are given by $\mathbf{n}_1(\mathrm{x})$, $\mathbf{n}_2(\mathrm{x})$, and $\mathbf{n}_3(\mathrm{x})$ at the point x corresponding to the material point X. We assume that $\mathbf{N}_1$ and $\mathbf{N}_2$ are in the radial and axial directions, respectively. Note that $\mathbf{N}_3$, which is perpendicular to $\mathbf{N}_1$ and $\mathbf{N}_2$, explicitly depends on the distribution of screw dislocations as will be seen in the following. This is because the geometry of the material manifold has an explicit nontrivial dependence on the dislocations distribution (see (6.26)). In the current configuration, the body will have monoclinic anisotropy as $\mathbf{n}_1$ will be perpendicular to the plane of $\mathbf{n}_2$ and $\mathbf{n}_3$, which will not be orthogonal in the ambient space. It is known that the material manifold for a nonlinear solid with distributed dislocations is a Weitzenböck manifold, i.e., a manifold with torsion having a flat connection and vanishing non-metricity (see [44, 139] for more details). Therefore, the material metric for the dislocated body is written as

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R^2 + f(R)^2 & f(R) \\ 0 & f(R) & 1 \end{pmatrix}, \tag{6.26}$$

where $f(R)$ is related to the Burgers' vector density $b(R)$ such that $f'(R) = \frac{R}{2\pi} b(R)$. Let us endow the ambient space with the Euclidean metric $\mathbf{g} = \mathrm{diag}\{1, r^2, 1\}$. We then assume an embedding of the material manifold into the ambient space of the form $(r, \theta, z) = (r(R), \Theta, \alpha Z)$, where α is a positive constant denoting the longitudinal stretch. Hence, $\mathbf{F} = \mathrm{diag}\{r'(R), 1, \alpha\}$. Assuming incompressibility, i.e., $J = \sqrt{\frac{\det\mathbf{g}}{\det\mathbf{G}}}\det\mathbf{F} = 1$, one obtains $\frac{r(R)}{R}r'(R)\alpha = 1$. Eliminating the rigid body translation by setting $r(0) = 0$, one obtains $r(R) = \frac{1}{\sqrt{\alpha}}R$. Therefore, the right Cauchy-Green deformation tensor is written as[6]

$$\mathbf{C} = \begin{pmatrix} \frac{1}{\alpha} & 0 & 0 \\ 0 & \frac{1}{\alpha} & -\frac{\alpha^2 f(R)}{R^2} \\ 0 & -\frac{f(R)}{\alpha} & \frac{\alpha^2}{R^2}(R^2 + f(R)^2) \end{pmatrix}. \tag{6.27}$$

Note that $\mathbf{N}_1 = \mathbf{E}_R$, $\mathbf{N}_2 = \mathbf{E}_Z$, and $\mathbf{N}_3 = \frac{1}{R}\mathbf{E}_\Theta - \frac{f(R)}{R}\mathbf{E}_Z$. Note also that $\mathbf{N}_3$ is obtained using the orthonormality of the material preferred directions, and $\mathbf{E}_R = \partial/\partial R$, $\mathbf{E}_Z = \partial/\partial Z$, and $\mathbf{E}_\Theta = \partial/\partial\Theta$ form a basis for $T_X\mathcal{B}$. Using (6.16), the invariants of the strain energy function are simplified and are written as

$$\begin{aligned} I_1 &= \operatorname{tr}\mathbf{C} = \frac{2}{\alpha} + \frac{\alpha^2}{R^2}(R^2 + f(R)^2)\,, \quad I_2 = \frac{1}{2}\left[\operatorname{tr}(\mathbf{C}^2) - (\operatorname{tr}\mathbf{C})^2\right] = \frac{1}{\alpha^2} + 2\alpha + \alpha\frac{f(R)^2}{R^2}\,, \\ I_4 &= \frac{1}{\alpha}\,, \qquad I_5 = \frac{1}{\alpha^2}\,, \qquad I_6 = \alpha^2\,, \quad I_7 = \frac{\alpha^4}{R^2}(R^2 + f(R)^2)\,. \end{aligned} \tag{6.28}$$

The non-zero components of the Cauchy stress tensor following (6.20) read

$$\sigma^{rr} = \frac{2}{\alpha^2}\left[W_{I_2}\left(\alpha^3 + \frac{\alpha^3 f(R)^2}{R^2} + 1\right) + \alpha W_{I_4} + 2W_{I_5}\right] + \frac{2W_{I_1}}{\alpha} - p(R)\,, \tag{6.29}$$

$$\sigma^{\theta\theta} = \frac{2\alpha W_{I_1} + 2\left(\alpha^3 + 1\right)W_{I_2} - \alpha^2 p(R)}{\alpha R^2}\,, \tag{6.30}$$

$$\sigma^{zz} = \frac{2\alpha}{R^2}\Big[\left(f(R)^2 + R^2\right)\left(\alpha W_{I_1} + W_{I_2} + 2\alpha^3 W_{I_7}\right) + R^2(W_{I_2} + \alpha W_{I_6})\Big] - p(R)\,, \tag{6.31}$$

$$\sigma^{\theta z} = -\frac{2f(R)}{R^2}\left(\alpha W_{I_1} + W_{I_2} + \alpha^3 W_{I_7}\right)\,. \tag{6.32}$$

[6]The symbolic computations in this chapter were performed using Mathematica [93].

We assume that the stress vanishes when the body is dislocation-free and the longitudinal stretch $\alpha = 1$ (see also [92, 94, 27]). Thus

$$(W_{I_4} + 2W_{I_5})\,|_{I_1=I_2=3, I_4=I_5=I_6=I_7=1} = 0\,, \quad \text{and} \quad (W_{I_6} + 2W_{I_7})\,|_{I_1=I_2=3, I_4=I_5=I_6=I_7=1} = 0\,. \tag{6.33}$$

In the absence of body and inertial forces, the only non-trivial equilibrium equation is $\sigma^{rb}{}_{|b} = 0$, implying[7] that (cf. (6.3)) $\sigma^{rr}{}_{,r} + \frac{\sigma^{rr}}{r} - r\sigma^{\theta\theta} = 0$. Therefore, $p'(R) = h(R)$, where

$$\begin{aligned}
h(R) =& \frac{2}{\alpha R^5}\Big[2R^3 f(R) f'(R)\Big(\alpha^2 W_{I_2} + \alpha^2 W_{I_1 I_1} + \alpha(2+\alpha^3) W_{I_1 I_2} + \alpha^2 W_{I_1 I_4} + 2\alpha W_{I_1 I_5} + \alpha^4 W_{I_1 I_7} \\
&+ (\alpha^3+1) W_{I_2 I_2} + \alpha W_{I_2 I_4} + 2W_{I_2 I_5} + \alpha^3(\alpha^3+1) W_{I_2 I_7} + \alpha^4 W_{I_4 I_7} + 2\alpha^3 W_{I_5 I_7}\Big) \\
&+ 2\alpha^3 R f(R)^3 f'(R)\left(\alpha W_{I_1 I_2} + W_{I_2 I_2} + \alpha^3 W_{I_2 I_7}\right) - R^2 f(R)^2 \Big\{2\alpha W_{I_1 I_2}(2+\alpha^3) \\
&+ 2\alpha^2 W_{I_1 I_4} + 4\alpha W_{I_1 I_5} + 2\alpha^4 W_{I_1 I_7} + \alpha^2 W_{I_2} + 2(\alpha^3+1) W_{I_2 I_2} + 2\alpha W_{I_2 I_4} + 4W_{I_2 I_5} \\
&+ 2\alpha^3(\alpha^3+1) W_{I_2 I_7} + 2\alpha^4 W_{I_4 I_7} + 4\alpha^3 W_{I_5 I_7} + 2\alpha^2 W_{I_1 I_1}\Big\} \\
&- 2\alpha^3 f(R)^4 \left(\alpha W_{I_1 I_2} + W_{I_2 I_2} + \alpha^3 W_{I_2 I_7}\right) + R^4 W_{I_4}\Big] + \frac{4W_{I_5}}{\alpha^2 R}\,.
\end{aligned} \tag{6.34}$$

If one assumes that the medium is a cylinderical bar with a finite radius R_o and the surface $R = R_o$ is traction-free, one obtains

$$p(R) = \int_{R_o}^{R} h(\zeta) d\zeta + \frac{2}{\alpha^2}\left[\Big(\alpha^3 + \frac{\alpha^3 f(R_o)^2}{R_o^2} + 1\Big) W_{I_2}|_{R=R_o} + \alpha W_{I_4}|_{R=R_o} + 2W_{I_5}|_{R=R_o}\right] + \frac{2}{\alpha} W_{I_1}|_{R=R_o}\,. \tag{6.35}$$

Let us employ the so called *standard reinforcing model* for compressible materials, which is defined as [140, 92, 97]

$$W = W\left(I_1, I_2, I_4, I_5, I_6, I_7\right) = W_{\text{iso}}\left(I_1, I_2\right) + W^R_{\text{fib}}\left(I_4, I_5\right) + W^Z_{\text{fib}}\left(I_6, I_7\right)\,, \tag{6.36}$$

[7] Note that $p = p(R)$ is implied from the other equilibrium equations.

where W_{iso} denotes the strain energy function for the isotropic base material, whereas W_{fib}^{R} and W_{fib}^{Z} represent the anisotropic effects due to the fiber reinforcement in the radial and longitudinal directions, respectively. Consider as an example a cylindrical body made of a Mooney-Rivlin solid reinforced with fibers in the radial and longitudinal directions such that

$$\begin{aligned} W(I_1, I_2, I_4, I_5, I_6, I_7) =& \frac{\mu_1}{2}(I_1 - 3) + \frac{\mu_2}{2}(I_2 - 3) + \frac{\gamma_1}{2}(I_4 - 1)^2 \\ &+ \frac{\gamma_2}{2}(I_5 - 1)^2 + \frac{\xi_1}{2}(I_6 - 1)^2 + \frac{\xi_2}{2}(I_7 - 1)^2 \ . \end{aligned} \tag{6.37}$$

Using (6.34), we have

$$h(R) = \alpha\mu_2 \frac{f(R)}{R^3}\left[2Rf'(R) - f(R)\right] + \frac{2}{R}\frac{1}{\alpha^2}\left(\frac{1}{\alpha} - 1\right)\left[\alpha\gamma_1 + 2\gamma_2\left(1 + \frac{1}{\alpha}\right)\right] . \tag{6.38}$$

Thus, from (6.35)

$$\begin{aligned} p(R) =& \alpha\mu_2 \int_{R_o}^{R} \frac{f(\zeta)}{\zeta^3}\left[2\zeta f'(\zeta) - f(\zeta)\right] d\zeta + \frac{2}{\alpha^2}\left(\frac{1}{\alpha} - 1\right)\left[\alpha\gamma_1 + 2\gamma_2\left(1 + \frac{1}{\alpha}\right)\right]\left[1 + \ln\frac{R}{R_o}\right] \\ &+ \frac{\mu_1}{\alpha} + \frac{\mu_2}{\alpha^2}\Big(\alpha^3 + \frac{\alpha^3 f(R_o)^2}{R_o^2} + 1\Big) . \end{aligned} \tag{6.39}$$

The physical components of the Cauchy stress read[8]

$$
\begin{aligned}
\hat{\sigma}^{rr} =& \alpha\mu_2\Big(\frac{f(R)^2}{R^2}-\frac{f(R_o)^2}{R_o^2}\Big)+\alpha\mu_2\int_R^{R_o}\frac{f(\zeta)}{\zeta^3}\left[2\zeta f'(\zeta)-f(\zeta)\right]d\zeta \\
&-\frac{2}{\alpha^2}(\frac{1}{\alpha}-1)\left\{\alpha\gamma_1+2\gamma_2(1+\frac{1}{\alpha})\right\}\ln\frac{R}{R_o}\,,
\end{aligned} \tag{6.40}
$$

$$
\begin{aligned}
\hat{\sigma}^{\theta\theta} =& \alpha\mu_2\int_R^{R_o}\frac{f(\zeta)}{\zeta^3}\left[2\zeta f'(\zeta)-f(\zeta)\right]d\zeta-\alpha\mu_2\frac{f(R_o)^2}{R_o^2} \\
&-\frac{2}{\alpha^2}(\frac{1}{\alpha}-1)\left\{\alpha\gamma_1+2\gamma_2\left(1+\frac{1}{\alpha}\right)\right\}\left[1+\ln\frac{R}{R_o}\right]\,,
\end{aligned} \tag{6.41}
$$

$$
\begin{aligned}
\hat{\sigma}^{zz} =& 2\alpha^2(\alpha^2-1)\xi_1+\frac{4\alpha^4\xi_2}{R^2}\left(f(R)^2+R^2\right)\left[\frac{\alpha^4}{R^2}\left(f(R)^2+R^2\right)-1\right] \\
&+\alpha\mu_2\Big(\frac{f(R)^2}{R^2}-\frac{f(R_o)^2}{R_o^2}\Big)-\frac{2}{\alpha^2}\left(\frac{1}{\alpha}-1\right)\left\{\alpha\gamma_1+2\gamma_2\left(1+\frac{1}{\alpha}\right)\right\}\left[1+\ln\frac{R}{R_o}\right] \\
&+\alpha\mu_2\int_R^{R_o}\frac{f(\zeta)}{\zeta^3}\left[2\zeta f'(\zeta)-f(\zeta)\right]d\zeta+(\alpha\mu_1+\mu_2)(\alpha-\frac{1}{\alpha^2})+\alpha^2\mu_1\frac{f(R)^2}{R^2}\,,
\end{aligned} \tag{6.42}
$$

$$
\hat{\sigma}^{\theta z} = -\frac{f(R)}{\alpha^{\frac{1}{2}}R}\left(\alpha\mu_1+\mu_2+2\alpha^3\xi_2\left[\frac{\alpha^4}{R^2}\left(R^2+f(R)^2\right)-1\right]\right)\,. \tag{6.43}
$$

Remark 6.3.1. From (6.39), for an arbitrary cylindrically-symmetric distribution of parallel screw dislocations, the pressure $p(R)$, and hence, $\hat{\sigma}^{rr}$, $\hat{\sigma}^{\theta\theta}$, and $\hat{\sigma}^{zz}$ exhibit a logarithmic singularity on the dislocation axis ($R=0$) unless $\alpha=1$. Note that this singularity is inherently due to the anisotropy effects, i.e., the presence of the reinforcement in the radial direction. In particular, the singularity does not occur when $\gamma_1=\gamma_2=0$, e.g., when the material is isotropic. Note also that as $R\to 0$, $f(R)=\frac{b(0)}{4\pi}R^2+\mathcal{O}(R^3)$, and thus, $\frac{f(R)}{R}$ is finite at $R=0$. This implies that unlike the other stress components, $\hat{\sigma}^{\theta z}$ is nonsingular.

Remark 6.3.2. Note that in the case of fiber-reinforced neo-Hookean materials ($\mu_2=0$) and a given arbitrary cylindrically-symmetric distribution of screw dislocations supported on a cylinder of radius R_i, the stress field for $R>R_i$ is independent of $b(R)$ and is identical to that of a single screw dislocation with Burgers vector $b_0=\int_0^{R_i}\eta b(\eta)d\eta$. Acharya [126] and Yavari and Goriely [44] had observed this for isotropic neo-Hookean solids.

[8] The physical components of the Cauchy stress tensor, i.e., $\hat{\sigma}^{ab}=\sigma^{ab}\sqrt{g_{aa}g_{bb}}$ (no summation) [67] are given as $\hat{\sigma}^{rr}=\sigma^{rr}$, $\hat{\sigma}^{\theta\theta}=r^2(R)\sigma^{\theta\theta}$, $\hat{\sigma}^{zz}=\sigma^{zz}$, and $\hat{\sigma}^{\theta z}=r(R)\sigma^{\theta z}$.

As an example, let us assume the following Burgers' vector density distribution:

$$b(R) = \begin{cases} b_0 & 0 < R \leq R_i \,, \\ 0 & R_i < R \leq R_o \,, \end{cases} \tag{6.44}$$

where $R_i \leq R_o$. Thus

$$f(R) = \frac{1}{2\pi} \int_0^R \eta b(\eta) d\eta = \frac{b_0}{4\pi} \begin{cases} R^2 & 0 < R \leq R_i \,, \\ R_i^2 & R_i < R \leq R_o \,. \end{cases} \tag{6.45}$$

Fig. 6.1 depicts the variation of the different components of the Cauchy stress for the Burgers' vector density distribution (6.44) such that $R_i/R_o = 0.5$ and $b_0 R_o = 20$. Notice that the $\hat{\sigma}^{rr}$ and $\hat{\sigma}^{\theta\theta}$ vanish for a neo-Hookean solid.

Remark 6.3.3. As noted by Zubov [124], the energy per unit length (along the dislocation line) of a single screw dislocation in a Mooney-Rivlin solid is unbounded.[9] This is also the case for a fiber-reinforced Mooney-Rivlin material due to the standard reinforcing model considered here (cf. (6.36)). Let us consider incompressible isotropic base materials, for which the energy per unit length of a single screw dislocation remains bounded, i.e., $2\pi \int_0^{R_o} W_{\text{iso}}(I_1(\xi), I_2(\xi))\xi d\xi < \infty$, for finite R_o (examples include Varga [124], incompressible power-law [141, 125], generalized incompressible neo-Hookean materials [44], and Hencky material [22]). Exploiting the standard reinforcing model, the energy function for the fiber-reinforced material with the isotropic base with the energy function $W_{\text{iso}}(I_1, I_2)$ is assumed to be given as

$$W = W_{\text{iso}}(I_1, I_2) + \frac{\gamma_1}{2}(I_4 - 1)^2 + \frac{\gamma_2}{2}(I_5 - 1)^2 + \frac{\xi_1}{2}(I_6 - 1)^2 + \frac{\xi_2}{2}(I_7 - 1)^\beta \,. \tag{6.46}$$

Then the energy per unit length along a single screw dislocation line is finite if $\beta < 1$. To see this, we

[9] Note, however, that the energy of distributed screw dislocations is not necessarily unbounded (see also [23]). In particular, a Mooney-Rivlin reinforced material with the energy function (6.37) and the Burgers' vector distribution (6.44) has a finite energy per unit length.

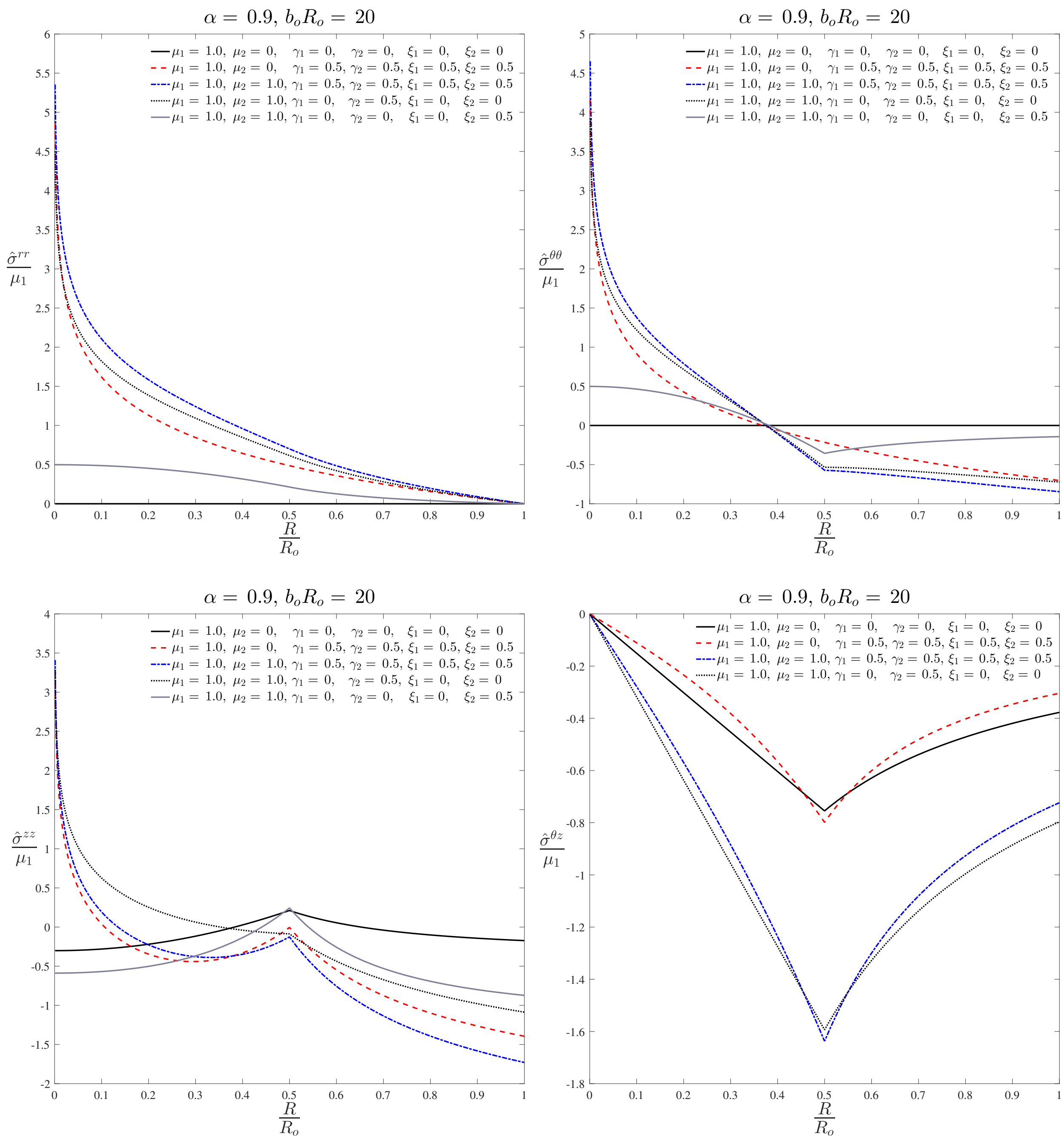


Figure 6.1: *Stress distribution in a medium with the constitutive equation* (6.37) *and the dislocation distribution* (6.44) *such that* $R_i/R_o = 0.5$*,* $b_0 R_o = 20$*, and* $\alpha = 0.9$ *for different values of the constitutive parameters.*

need to show that $2\pi \int_0^{R_o} \frac{\xi_2}{2} \left(I_7(\zeta)-1\right)^\beta \zeta d\zeta < \infty$ as the finiteness of the contribution of the other terms in the energy per unit length is trivial (cf. (6.28)). Noting that for a single screw dislocation with Burgers vector b_i, $b(R) = 2\pi b_i\, \delta^2(R)$, and hence, $f(R) = \frac{b_i}{2\pi} H(R)$, we have

$$\pi\xi_2 \int_0^{R_o} \left(I_7(\zeta)-1\right)^\beta \zeta d\zeta = \pi\xi_2 \int_0^{R_o} \left[\frac{\alpha^4}{\zeta^2}(\zeta^2+f(\zeta)^2)-1\right]^\beta \zeta d\zeta = \pi\xi_2 \int_0^{R_o} \left[\alpha^4-1+\frac{\alpha^4 b_i^2}{4\pi^2\zeta^2}\right]^\beta \zeta d\zeta < \infty\,, \tag{6.47}$$

provided that $\beta < 1$. Similarly, one can show that if the resultant longitudinal force, i.e., $F_Z = 2\pi \int_0^{R_o} \hat{\sigma}^{zz}(\zeta)\zeta d\zeta$, induced by a single screw dislocation is finite for the isotropic base material with the energy function $W_{\text{fib}}(I_1, I_2)$, so is the axial force for the fiber-reinforced material with the energy function (6.46) when $\beta < 1$.

As the underlying geometry of the material manifold explicitly depends on the distribution of defects, so are the material preferred directions (and thus, the material symmetry group). One of the consequences of this is that the class of anisotropy of the defective body is, in general, different in the reference and current configurations given that the reference configuration has a nontrivial geometry, whereas the geometry of the current configuration is trivial. The next section is aimed at illustrating that depending on whether the dislocated body is orthotropic in its reference configuration or in its current configuration, the induced residual stresses are different.

6.3.2 A Cylindrically-Symmetric Distribution of Parallel Screw Disclocations in a Monoclinic Medium

In the previous section, we assumed that the dislocated body is orthotropic in the reference configuration. Instead, let us assume that the medium with the cylindrically-symmetric distribution of parallel screw dislocations is orthotropic in its current configuration such that the orthotropic axes are in the radial, circumferential, and axial directions in the ambient space. In the reference configuration, the material will be monoclinic such that $\mathbf{N}_3 = \hat{\mathbf{R}}$ is perpendicular to the plane of $\mathbf{N}_1 = \hat{\mathbf{\Theta}}$ and $\mathbf{N}_2 = \hat{\mathbf{Z}}$.[10] We assume the same class of deformations as was assumed in the previous section, and thus, $r(R) = \frac{1}{\sqrt{\alpha}} R$. Hence, the right Cauchy-Green deformation tensor is given by (6.27). The

[10]Note that $\mathbf{N}_1 = \mathbf{E}_\Theta/(R^2+f(R)^2)^{1/2}$ and $\mathbf{N}_2 = \mathbf{E}_Z$ are not orthogonal in the nontrivial geometry of the reference configuration (cf. (6.26)).

invariants of the strain energy function for the monoclinic material are given as

$$
\begin{aligned}
I_1 =& \operatorname{tr}(\mathbf{C}) = \frac{2}{\alpha} + \frac{\alpha^2}{R^2}(R^2 + f(R)^2)\,, \quad I_2 = \frac{1}{2}\left[\operatorname{tr}(\mathbf{C}^2) - (\operatorname{tr}\mathbf{C})^2\right] = \frac{1}{\alpha^2} + 2\alpha + \alpha\frac{f(R)^2}{R^2}\,, \\
I_4 =& \frac{R^2}{\alpha(R^2 + f(R)^2)}\,,\ I_5 = \frac{R^2}{\alpha^2(R^2 + f(R)^2)}\,,\ I_6 = \alpha^2\,,\ I_7 = \frac{\alpha^4}{R^2}(R^2 + f(R)^2)\,,\ I_8 = 0\,.
\end{aligned}
\tag{6.48}
$$

From (6.24), the non-zero components of the Cauchy stress tensor read

$$
\sigma^{rr} = \frac{2W_{I_2}}{\alpha^2}\Big(\alpha^3 + \frac{\alpha^3 f(R)^2}{R^2} + 1\Big) + \frac{2W_{I_1}}{\alpha} - p(R)\,, \tag{6.49}
$$

$$
\sigma^{\theta\theta} = \frac{2\alpha W_{I_1} + 2\left(\alpha^3+1\right)W_{I_2} - \alpha^2 p(R)}{\alpha R^2} + \frac{2(\alpha W_{I_4} + 2W_{I_5})}{\alpha(R^2 + f(R)^2)}\,, \tag{6.50}
$$

$$
\sigma^{zz} = \frac{2\alpha}{R^2}\Big[\left(f(R)^2 + R^2\right)\left(\alpha W_{I_1} + W_{I_2} + 2\alpha^3 W_{I_7}\right) + R^2(W_{I_2} + \alpha W_{I_6})\Big] - p(R)\,, \tag{6.51}
$$

$$
\sigma^{\theta z} = -\frac{2f(R)}{R^2}\left(\alpha W_{I_1} + W_{I_2} + \alpha^3 W_{I_7}\right) - \frac{2f(R)}{R^2 + f(R)^2}W_{I_5} + \frac{\alpha}{(R^2 + f(R)^2)^{\frac{1}{2}}}W_{I_8}\,. \tag{6.52}
$$

Note that for the stress to vanish when $\alpha = 1$ and the body is dislocation-free, i.e., $f(R) = 0$ (identically), one needs to have $(W_{I_4} + 2W_{I_5}) = (W_{I_6} + 2W_{I_7}) = W_{I_8} = 0$, evaluated at $I_1 = I_2 = 3$, $I_4 = I_5 = I_6 = I_7 = 1$, $I_8 = 0$. The equilibrium equation implies that $p'(R) = S(R)$, where

$$
\begin{aligned}
S(R) =& -\frac{1}{\alpha^4 R^3}\Bigg[-4\alpha f(R)\left(Rf'(R) - f(R)\right)\left(\alpha^4 W_{I_1I_1} + \alpha^3 W_{I_1I_2} + \alpha^6 W_{I_1I_7} - \frac{R^4(\alpha W_{I_1I_4} + W_{I_1I_5})}{\left(f(R)^2 + R^2\right)^2}\right) \\
&- 4f(R)\Big(\alpha^3 + \frac{\alpha^3 f(R)^2}{R^2} + 1\Big)\left(Rf'(R) - f(R)\right)\Big\{\alpha^4 W_{I_1I_2} + \alpha^3 W_{I_2I_2} + \alpha^6 W_{I_2I_7} \\
&- \frac{R^4(\alpha W_{I_2I_4} + W_{I_2I_5})}{\left(f(R)^2 + R^2\right)^2}\Big\} + 4\alpha^5 f(R) W_{I_2}\left(f(R) - Rf'(R)\right) + \frac{2\alpha^2 R^4(\alpha W_{I_4} + 2W_{I_5})}{f(R)^2 + R^2} \\
&- 2\alpha^5 f(R)^2 W_{I_2}\Bigg]\,.
\end{aligned}
\tag{6.53}
$$

Assuming that the surface $R = R_o$ is traction-free the pressure field is obtained as

$$
p(R) = \int_{R_o}^{R} S(\zeta)d\zeta + \frac{2}{\alpha^2}\Big(\alpha^3 + \frac{\alpha^3 f(R_o)^2}{R_o^2} + 1\Big)W_{I_2}|_{R=R_o} + \frac{2}{\alpha}W_{I_1}|_{R=R_o}\,. \tag{6.54}
$$

Let us consider the following model for the strain energy function

$$W = W\left(I_1, I_2, I_4, I_5, I_6, I_7, I_8\right) = W_{\text{iso}}\left(I_1, I_2\right) + W_{\text{fib}}^{\Theta}\left(I_4, I_5\right) + W_{\text{fib}}^{Z}\left(I_6, I_7\right) + W_{\text{fib}}^{Z\Theta}(I_8)\,, \quad (6.55)$$

where W_{iso} describes that part of the energy function pertaining to the isotropic base material, while W_{fib}^{Θ} and W_{fib}^{Z} represent the reinforcement effects in the circumferential and axial directions. $W_{\text{fib}}^{Z\Theta}(I_8)$ models the coupling between the axial and circumferential fibers. Note, however, that $I_8 = 0$, and for the stress to vanish for the dislocation-free body, one needs $W_{I_8} = 0$ at $I_8 = 0$, which implies that $W_{\text{fib}}^{Z\Theta}(I_8) = 0$, i.e., the coupling term must vanish. A way out would be to require that the coupling term depend on some other invariants as well, e.g., one can define $W_{\text{fib}}^{Z\Theta}(I_1, I_8) = \eta I_8(I_1 - 3)$ for some positive constant η. For the sake of simplicity, as an example, we consider a fiber-reinforced Mooney-Rivlin material with the following energy function

$$\begin{aligned} W(I_1, I_2, I_4, I_5, I_6, I_7) =& \frac{\mu_1}{2}\left(I_1 - 3\right) + \frac{\mu_2}{2}\left(I_2 - 3\right) + \frac{\lambda_1}{2}\left(I_4 - 1\right)^2 \\ &+ \frac{\lambda_2}{2}\left(I_5 - 1\right)^2 + \frac{\xi_1}{2}\left(I_6 - 1\right)^2 + \frac{\xi_2}{2}\left(I_7 - 1\right)^2 \,. \end{aligned} \quad (6.56)$$

Therefore, one obtains

$$\begin{aligned} S(R) =& \frac{1}{\alpha^2 R^3\left(f(R)^2 + R^2\right)}\Big[\alpha^3\mu_2 f(R)(R^2 + f(R)^2)(2Rf'(R) - f(R)) \\ &+ 2R^4\Big\{\alpha\lambda_1 - \frac{R^2}{f(R)^2 + R^2}(\frac{2\lambda_2}{\alpha^2} + \lambda_1) + 2\lambda_2\Big\}\Big]\,. \end{aligned} \quad (6.57)$$

Thus, the physical components of the stress are given as

$$\hat{\sigma}^{rr} = \alpha\mu_2 \left(\frac{f(R)^2}{R^2} - \frac{f(R_o)^2}{R_o^2}\right) + \int_R^{R_o} S(\zeta)d\zeta\,, \tag{6.58}$$

$$\begin{aligned}\hat{\sigma}^{\theta\theta} &= \int_R^{R_o} S(\zeta)d\zeta - \alpha\mu_2\frac{f(R_o)^2}{R_o^2} + \frac{2R^2}{\alpha^2(R^2+f(R)^2)}\Big[\alpha\lambda_1\Big(\frac{R^2}{\alpha(R^2+f(R)^2)} - 1\Big) \\ &\quad + 2\lambda_2\Big(\frac{R^2}{\alpha^2(R^2+f(R)^2)} - 1\Big)\Big]\,,\end{aligned} \tag{6.59}$$

$$\begin{aligned}\hat{\sigma}^{zz} &= 2\alpha^2\xi_1(\alpha^2-1) + 4\alpha^4\xi_2\Big(1+\frac{f(R)^2}{R^2}\Big)\Big[\alpha^4\Big(1+\frac{f(R)^2}{R^2}\Big) - 1\Big] + \int_R^{R_o} S(\zeta)d\zeta \\ &\quad + \alpha\mu_2\left(\frac{f(R)^2}{R^2} - \frac{f(R_o)^2}{R_o^2}\right) + (\alpha\mu_1+\mu_2)(\alpha - \frac{1}{\alpha^2}) + \alpha^2\mu_1\frac{f(R)^2}{R^2}\,,\end{aligned} \tag{6.60}$$

$$\begin{aligned}\hat{\sigma}^{\theta z} &= -\frac{1}{\sqrt{\alpha}}\frac{f(R)}{R}(\alpha\mu_1+\mu_2) - 2\xi_2\alpha^{\frac{5}{2}}\frac{f(R)}{R}\Big[\alpha^4\Big(1+\frac{f(R)^2}{R^2}\Big) - 1\Big] \\ &\quad - \frac{2\lambda_2}{\sqrt{\alpha}}\frac{Rf(R)}{R^2+f(R)^2}\Big(\frac{R^2}{\alpha^2(R^2+f(R)^2)} - 1\Big)\,.\end{aligned} \tag{6.61}$$

Remark 6.3.4. For an arbitrary cylindrically-symmetric distribution of parallel screw dislocations with a smooth Burgers' vector density $b(R)$ in a monoclinic material, the pressure, and hence, $\hat{\sigma}^{rr}$, $\hat{\sigma}^{\theta\theta}$, and $\hat{\sigma}^{zz}$ have a logarithmic singularity on the dislocation axis unless $\alpha = 1$. Nevertheless, the shear component $\hat{\sigma}^{\theta z}$ is finite and vanishes at $R = 0$. This is because as $R \to 0$, $f(R) = \frac{b(0)}{4\pi}R^2 + \mathcal{O}(R^3)$, and thus, from (6.57)

$$S(R) = \frac{2\,(\alpha-1)}{\alpha^2}\left[\lambda_1 + \frac{2\lambda_2}{\alpha}\left(1+\frac{1}{\alpha}\right)\right]\frac{1}{R} + \mathcal{O}(R)\,. \tag{6.62}$$

Therefore, $p(R) = C - \frac{2(\alpha-1)}{\alpha^2}\left[\lambda_1 + \frac{2\lambda_2}{\alpha}\left(1+\frac{1}{\alpha}\right)\right]\ln\frac{R}{R_o} + \mathcal{O}(R^2)$ as $R \to 0$, where C is a constant. It is straightforward to see that when $\alpha = 1$, the stress is finite and $\hat{\sigma}^{rr} = \hat{\sigma}^{\theta\theta} = \hat{\sigma}^{zz}$ at $R = 0$.

In Fig. 6.2 the stress field is shown for the dislocation distribution (6.44), where $R_i/R_o = 0.5$ and $b_0 R_o = 20$ for different values of the constitutive parameters given by (6.56).

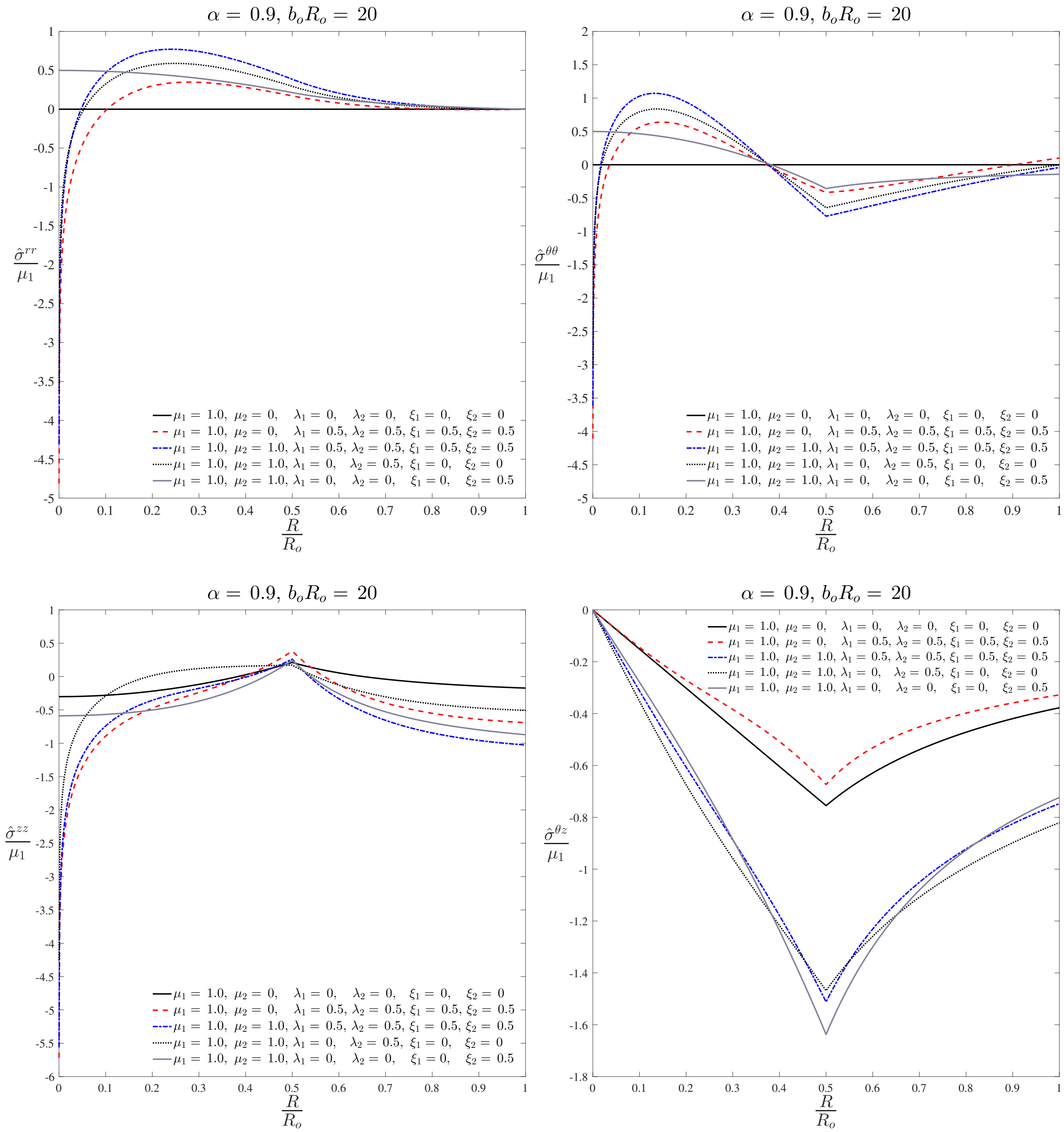


Figure 6.2: *Stress distribution in a medium with the constitutive equation* (6.56) *and the dislocation distribution* (6.44) *such that* $R_i/R_o = 0.5$, $b_oR_o = 20$, *and* $\alpha = 0.9$ *for different values of the constitutive parameters.*

6.3.3 A Parallel Cylindrically Symmetric Distribution of Wedge Disclinations in an Orthotropic Medium

Let us consider a parallel cylindrically-symmetric distribution of wedge disclinations in an infinite orthotropic medium in the reference configuration. In the cylindrical coordinates (R, Θ, Z), assume that the material orthotropic axes are in the R, Θ, and Z directions. The radial density of the wedge disclinations is denoted by $w(R)$. The material manifold for a body having a distribution of wedge disclinations is a Riemannian manifold with a non-vanishing curvature. The material metric for the disclinated body is given by [128]

$$\mathbf{G} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & f(R)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{6.63}$$

where $f''(R) = -\frac{R}{2\pi} w(R)$. The ambient space is endowed with the Euclidean metric $\mathbf{g} = \mathrm{diag}\{1, r^2, 1\}$. We embed the material manifold into the ambient space by looking for mappings[11] of the form $(r, \theta, z) = (r(R), \Theta, \alpha Z)$, where α is a constant representing the axial stretch of the bar that depends on the axial boundary conditions. Therefore, the deformation gradient reads $\mathbf{F} = \mathrm{diag}\,(r'(R), 1, \alpha)$. Incompressibility constraint dictates that $J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}} \det \mathbf{F} = \alpha \frac{r(R)}{f(R)} r'(R) = 1$. Thus, imposing $r(0) = 0$, we have $r(R) = \left(\frac{2}{\alpha} \int_0^R f(\xi) d\xi \right)^{\frac{1}{2}}$. The right Cauchy-Green deformation tensor reads $\mathbf{C} = \mathrm{diag} \left\{ \frac{1}{\alpha^2} \frac{f(R)^2}{r(R)^2}, \frac{r(R)^2}{f(R)^2}, \alpha^2 \right\}$. From (6.16), the invariants of the strain energy function are simplified to read

$$\begin{aligned} I_1 &= \mathrm{tr}(\mathbf{C}) = \alpha^2 + \frac{1}{\alpha^2} \frac{f(R)^2}{r(R)^2} + \frac{r(R)^2}{f(R)^2}, \quad I_2 = \frac{1}{2}(\mathrm{tr}(\mathbf{C}^2) - \mathrm{tr}(\mathbf{C})^2) = \frac{1}{\alpha^2} + \alpha^2 \frac{r(R)^2}{f(R)^2} + \frac{f(R)^2}{r(R)^2}, \\ I_4 &= \frac{1}{\alpha^2} \frac{f(R)^2}{r(R)^2}, \qquad I_5 = \frac{1}{\alpha^4} \frac{f(R)^4}{r(R)^4}, \qquad I_6 = \alpha^2, \quad I_7 = \alpha^4. \end{aligned} \tag{6.64}$$

[11]Note that for the class of deformations that is considered, the material will be orthotropic in its current configuration as well. The orthotropic axes in the current configuration will be in the radial, circumferential, and axial directions (similar to those in the reference configuration).

The non-zero physical components of the Cauchy stress are as follows[12]

$$
\begin{aligned}
\hat{\sigma}^{rr} =& \frac{2}{\alpha^2}\frac{f(R)^2}{r(R)^2}\left(W_{I_1}+\alpha^2 W_{I_2}+W_{I_4}\right)+\frac{4}{\alpha^4}\frac{f(R)^4}{r(R)^4}W_{I_5}+\frac{2}{\alpha^2}W_{I_2}-p(R)\,, && (6.65)\\
\hat{\sigma}^{\theta\theta} =& 2\frac{r(R)^2}{f(R)^2}\left(W_{I_1}+\alpha^2 W_{I_2}\right)+\frac{2}{\alpha^2}W_{I_2}-p(R)\,, && (6.66)\\
\hat{\sigma}^{zz} =& 2\alpha^2\left(W_{I_1}+W_{I_6}+2\alpha^2 W_{I_7}\right)+2W_{I_2}\left(\frac{f(R)^2}{r(R)^2}+\alpha^2\frac{r(R)^2}{f(R)^2}\right)-p(R)\,. && (6.67)
\end{aligned}
$$

The equilibrium equation implies that $p'(R)=k(R)$, where

$$
\begin{aligned}
k(R)=\frac{2}{\alpha^8 f^3 r^9}\Big[& 2\alpha^4 f^4 r^7 f'\left(\alpha^2 W_{I_1}+W_{I_1I_2}-2W_{I_1I_5}+\alpha^4 W_{I_2}+\alpha^2 W_{I_2I_2}+W_{I_2I_4}-2\alpha^2 W_{I_2I_5}+\alpha^2 W_{I_4}\right)\\
&+2\alpha^2 f^6 r^5 f'\left(\alpha^2 W_{I_1I_1}+2\alpha^4 W_{I_1I_2}+2\alpha^2 W_{I_1I_4}+\alpha^6 W_{I_2I_2}+2\alpha^4 W_{I_2I_4}+2W_{I_2I_5}+\alpha^2 W_{I_4I_4}+4\alpha^2 W_{I_5}\right)\\
&-2\alpha^6 f^2 r^9 f'\left(W_{I_1I_1}+2\alpha^2 W_{I_1I_2}+W_{I_1I_4}+\alpha^4 W_{I_2I_2}+\alpha^2 W_{I_2I_4}\right)\\
&+8\alpha^2 f^8 r^3 f'\left(W_{I_1I_5}+\alpha^2 W_{I_2I_5}+W_{I_4I_5}\right)+8f^{10}W_{I_5I_5}rf'-2\alpha^6 r^{11}f'\left(W_{I_1I_2}+\alpha^2 W_{I_2I_2}\right)\\
&-\alpha^3 f^6 r^5\left(\alpha^2 W_{I_1}+2W_{I_1I_2}-4W_{I_1I_5}+\alpha^4 W_{I_2}+2\alpha^2 W_{I_2I_2}+2W_{I_2I_4}-4\alpha^2 W_{I_2I_5}+\alpha^2 W_{I_4}\right)\\
&-\alpha^5 f^2 r^9\left(\alpha^2 W_{I_1}-2W_{I_1I_2}+\alpha^4 W_{I_2}-2\alpha^2 W_{I_2I_2}\right)-8\alpha f^{10} r\left(W_{I_1I_5}+\alpha^2 W_{I_2I_5}+W_{I_4I_5}\right)\\
&-2\alpha f^8 r^3\left(\alpha^2 W_{I_1I_1}+2\alpha^4 W_{I_1I_2}+2\alpha^2 W_{I_1I_4}+\alpha^6 W_{I_2I_2}+2\alpha^4 W_{I_2I_4}+2W_{I_2I_5}+\alpha^2 W_{I_4I_4}+3\alpha^2 W_{I_5}\right)\\
&+2\alpha^5 f^4 r^7\left(W_{I_1I_1}+2\alpha^2 W_{I_1I_2}+W_{I_1I_4}+\alpha^4 W_{I_2I_2}+\alpha^2 W_{I_2I_4}\right)-\frac{8f^{12}W_{I_5I_5}}{\alpha r}\Big]\,.
\end{aligned}
\tag{6.68}
$$

Assuming (6.37) for the energy function, one obtains

$$
\begin{aligned}
p'(R)=-\frac{1}{\alpha^9 f r^{10}}\Big[& -2\alpha^7 f^2 r^8 f'\left(\alpha^2\mu_2-2\gamma_1+\mu_1\right)+\alpha^5 f^4 r^6\left\{\alpha\left(\alpha^2\mu_2-2\gamma_1+\mu_1\right)-8(\gamma_1-2\gamma_2)f'\right\}\\
&-32\alpha\gamma_2 f^8 r^2 f'+6\alpha^4(\gamma_1-2\gamma_2)f^6 r^4+28\gamma_2 f^{10}+\alpha^8 r^{10}\left(\alpha^2\mu_2+\mu_1\right)\Big]\,.
\end{aligned}
\tag{6.69}
$$

Knowing that the traction vanishes on the outer boundary $R=R_o$, one finds

$$
p(R_o)=\frac{1}{\alpha^2}\frac{f(R_o)^2}{r(R_o)^2}\left\{\mu_1+\alpha^2\mu_2+2\gamma_1\left[\frac{1}{\alpha^2}\frac{f(R_o)^2}{r(R_o)^2}-1\right]\right\}+\frac{4\gamma_2}{\alpha^4}\frac{f(R_o)^4}{r(R_o)^4}\left[\frac{1}{\alpha^4}\frac{f(R_o)^4}{r(R_o)^4}-1\right]+\frac{\mu_2}{\alpha^2}\,.
\tag{6.70}
$$

[12]When the body is disclination-free $f(R)=R$, and the stress vanishes if the energy function satisfies (6.33).

Therefore, $p(R) = \int_{R_o}^{R} p'(\xi)d\xi + p(R_o)$. The stress components are simplified and read

$$\hat{\sigma}^{rr} = \frac{1}{\alpha^2}\frac{f(R)^2}{r(R)^2}\left\{\mu_1 + \alpha^2\mu_2 + 2\gamma_1\left[\frac{1}{\alpha^2}\frac{f(R)^2}{r(R)^2} - 1\right]\right\} + \frac{4\gamma_2}{\alpha^4}\frac{f(R)^4}{r(R)^4}\left[\frac{1}{\alpha^4}\frac{f(R)^4}{r(R)^4} - 1\right] + \frac{\mu_2}{\alpha^2} - p(R)\,, \tag{6.71}$$

$$\hat{\sigma}^{\theta\theta} = \frac{r(R)^2}{f(R)^2}\left(\mu_1 + \alpha^2\mu_2\right) + \frac{\mu_2}{\alpha^2} - p(R)\,, \tag{6.72}$$

$$\hat{\sigma}^{zz} = \alpha^2\left[\mu_1 + 2\xi_1(\alpha^2 - 1) + 4\alpha^2\xi_2(\alpha^4 - 1)\right] + \mu_2\left[\frac{f(R)^2}{r(R)^2} + \alpha^2\frac{r(R)^2}{f(R)^2}\right] - p(R)\,. \tag{6.73}$$

Example 6.3.5. For a uniform disclination distribution $w(R) = w_o$, one has $f''(R) = -\frac{R}{2\pi}w_o$, and thus, $f(R) = R - \frac{w_o}{12\pi}R^3$. Therefore

$$r(R) = \frac{R}{\alpha^{\frac{1}{2}}}\left(1 - \frac{w_o}{24\pi}R^2\right)^{\frac{1}{2}}, \tag{6.74}$$

provided that $w_o < 24\pi/R_o^2$.

Remark 6.3.6. For the uniform disclination distribution, the stress field exhibits a logarithmic singularity on the disclinations axis unless the axial stretch $\alpha = 1$. Moreover, when $\alpha = 1$, the stress is finite and hydrostatic at $R = 0$. To see this, as $R \to 0$, $r(R) = \frac{R}{\alpha^{\frac{1}{2}}} + \mathcal{O}(R^3)$, and $f(R) = R + \mathcal{O}(R^3)$. From (6.69), therefore

$$p(R) = C + \frac{2(1-\alpha)}{\alpha^4}\left[\alpha^2\gamma_1 + 2\gamma_2(1+\alpha)\right]\ln R + \mathcal{O}(R^2)\,, \tag{6.75}$$

where C is a constant. Hence, the stress is logarithmically unbounded at $R = 0$ unless $\alpha = 1$. Similar to the case of parallel screw dislocations in an orthotropic medium (cf. Remark. 6.3.1), the singularity arises as a result of radial reinforcement effects, and does not, in particular, occur in isotropic materials. Note that for $\alpha = 1$, at $R = 0$, one has $\hat{\sigma}^{rr} = \hat{\sigma}^{\theta\theta} = \hat{\sigma}^{zz} = \mu_1 + 2\mu_2 + \int_0^{R_o} p'(\xi)d\xi - p(R_o)$.

In Fig. 6.3, we show the variation of the stress components for the uniform disclination distribution

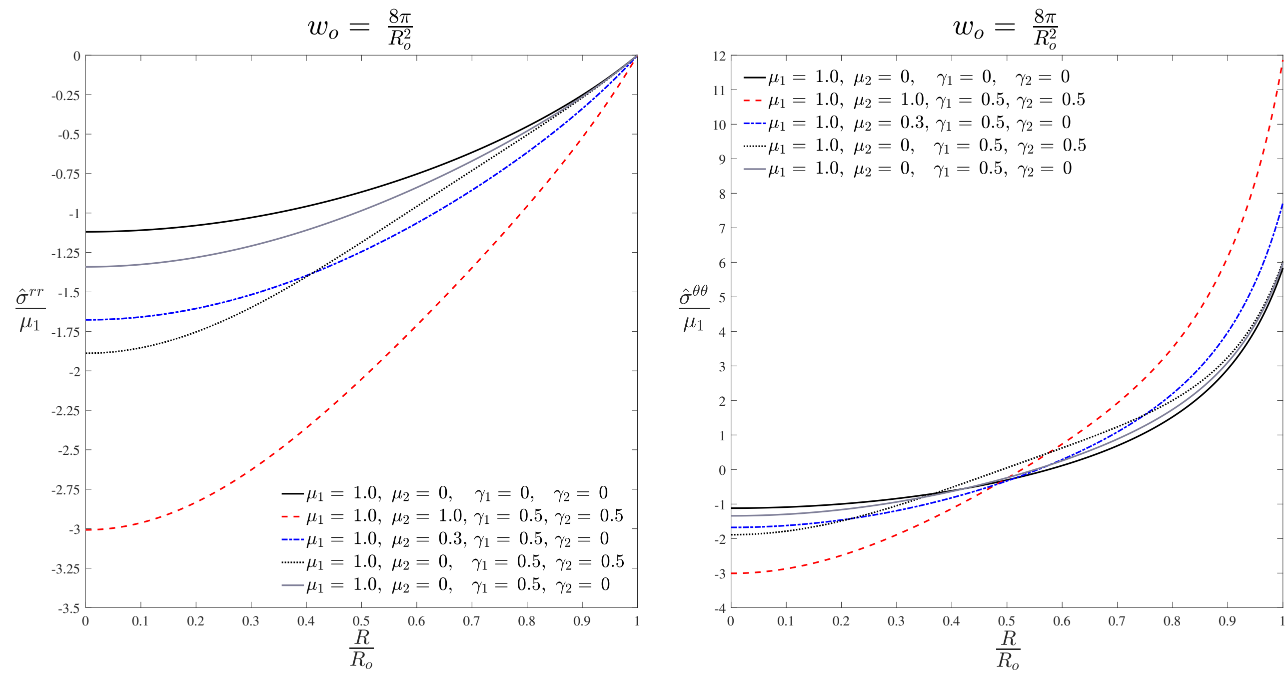


Figure 6.3: $\hat{\sigma}^{rr}$ *and* $\hat{\sigma}^{\theta\theta}$ *distributions for different values of the constitutive parameters for a uniform disclination distribution with* $\omega_o = 8\pi/R_o^2$ *such that* $\alpha = 1$.

with $w_o = 8\pi/R_o^2$ and for some different values of the constitutive parameters.[13]

Example 6.3.7. For a single wedge disclination $\omega(R) = 2\pi\Theta_o\delta^2(R)$, where Θ_o is the angle of the wedge shape region that is removed in Volterra's cut-and-weld operation (see [128] for more details). Therefore, $f''(R) = -\frac{\Theta_o}{2\pi}\delta(R)$, which implies that $f(R) = R(1-\frac{\Theta_o}{2\pi})$, and thus, $r(R) = \frac{R}{\alpha^{\frac{1}{2}}}(1-\frac{\Theta_o}{2\pi})^{\frac{1}{2}}$. Fig. 6.4 illustrates the stress distribution for different values of the reinforcement and the base material parameters in the case of a single wedge disclination of positive sign with $\Theta_o = \frac{\pi}{2}$.

6.3.4 Distributed Edge Dislocations in an Orthotropic Medium

Next, we consider a distribution of edge dislocations in an orthotropic medium such that the material preferred directions are parallel to the Cartesian axes in the Cartesian coordinates (X, Y, Z). Let us

[13]Note that the numerical values shown in [128]'s Fig. 4 are not correct. This was caused by a typo in the sign of the integral term in the numerical evaluation of the pressure function from Eq. (4.23). In other words, the numerical values in that figure correspond to the following (incorrect) relation for the pressure with a positive sign for the integral term

$$p(R) = \mu\frac{f^2(R_o)}{r^2(R_o)} + \mu\int_R^{R_o}\left[\frac{f(\eta)f'(\eta)}{\int_0^\eta f(\xi)d\xi} - \frac{f^3(\eta)}{4(\int_0^\eta f(\xi)d\xi)^2} - \frac{1}{f(\eta)}\right]d\eta\,.$$

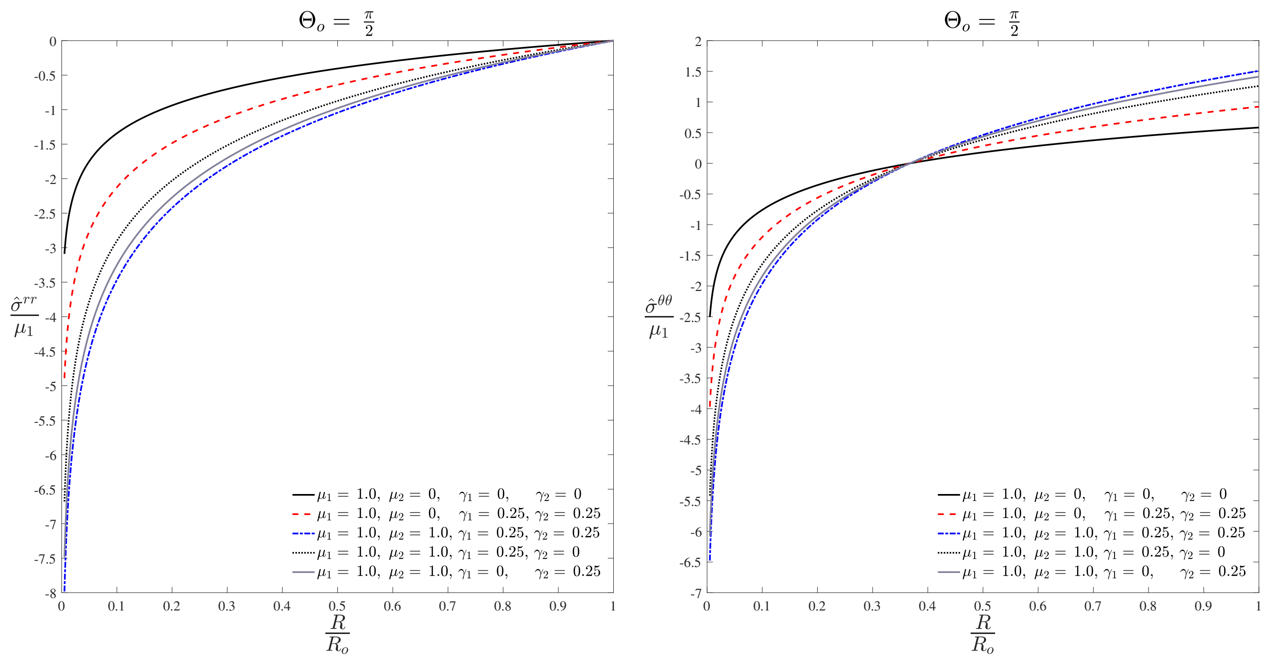


Figure 6.4: *$\hat{\sigma}^{rr}$ and $\hat{\sigma}^{\theta\theta}$ distributions for different values of the constitutive parameters for a single positive wedge disclination with $\Theta_o = \pi/2$ such that $\alpha = 1$.*

consider the orthonormal frame field $\{\mathbf{e}_\alpha(X,Y,Z)\}_{\alpha=1}^{3}$, where $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ are in the X, Y, and Z-directions, respectively. We assume that the edge dislocation distribution consists of dislocations with i) the dislocation line parallel to the Z-axis such that the Burgers' vector density is given by $b_1(Z)\mathbf{e}_1 + c_1(Z)\mathbf{e}_2$, ii) X-oriented Burgers' vector density $b_2(X,Y,Z)\mathbf{e}_1$ such that the dislocation line is parallel to the Y-axis, iii) Y-oriented Burgers' vector $c_2(X,Y,Z)\mathbf{e}_2$ with the dislocation line parallel to the X-axis. Let us consider the following co-frame field (see also [44])

$$\vartheta^1 = e^{\xi(Z)+\gamma(Y)}dX\,, \quad \vartheta^2 = e^{\eta(Z)+\lambda(X)}dY\,, \quad \vartheta^3 = e^{\psi(Z)}dZ\,, \tag{6.76}$$

where $\xi(Z)$, $\gamma(Y)$, $\eta(Z)$, $\lambda(X)$, and $\psi(Z)$ are scalar functions to be determined. The corresponding frame field reads

$$\mathbf{e}_1 = e^{-\xi(Z)-\gamma(Y)}\partial_X\,, \quad \mathbf{e}_2 = e^{-\eta(Z)-\lambda(X)}\partial_Y\,, \quad \mathbf{e}_3 = e^{-\psi(Z)}\partial_Z\,. \tag{6.77}$$

Note that $\mathbf{G} = \delta_{\alpha\beta}\vartheta^\alpha \otimes \vartheta^\beta$, and thus

$$\mathbf{G} = \operatorname{diag}\left\{ e^{2\left(\xi(Z)+\gamma(Y)\right)}, e^{2\left(\eta(Z)+\lambda(X)\right)}, e^{2\psi(Z)} \right\}. \tag{6.78}$$

The above dislocation distribution corresponds to the following torsion 2-forms (cf. (6.25))

$$\mathcal{T}^1 = b_1(Z)\vartheta^3 \wedge \vartheta^1 + b_2(X,Y,Z)\vartheta^1 \wedge \vartheta^2 , \quad \mathcal{T}^2 = c_1(Z)\vartheta^2 \wedge \vartheta^3 + c_2(X,Y,Z)\vartheta^1 \wedge \vartheta^2 , \quad \mathcal{T}^3 = 0 . \tag{6.79}$$

This represents a distribution of edge dislocations with the following total Burgers' vector density

$$\begin{aligned} \mathbf{b}(X,Y,Z) &= \left(b_1(Z) + b_2(X,Y,Z)\right)\mathbf{e}_1 + \left(c_1(Z) + c_2(X,Y,Z)\right)\mathbf{e}_2 \\ &= e^{-\xi(Z)-\gamma(Y)}\left[b_1(Z) + b_2(X,Y,Z)\right]\partial_X + e^{-\eta(Z)-\lambda(X)}\left[c_1(Z) + c_2(X,Y,Z)\right]\partial_Y . \end{aligned} \tag{6.80}$$

From (6.76), one obtains

$$\begin{aligned} d\vartheta^1 &= e^{-\psi(Z)}\xi'(Z)\vartheta^3 \wedge \vartheta^1 + e^{-\eta(Z)-\lambda(X)}\gamma'(Y)\vartheta^2 \wedge \vartheta^1 , \\ d\vartheta^2 &= e^{-\psi(Z)}\eta'(Z)\vartheta^3 \wedge \vartheta^2 + e^{-\xi(Z)-\gamma(Y)}\lambda'(X)\vartheta^1 \wedge \vartheta^2 , \qquad d\vartheta^3 = 0 . \end{aligned} \tag{6.81}$$

Metric compatibility implies the following connection 1-forms matrix

$$\boldsymbol{\omega} = [\omega^\alpha{}_\beta] = \begin{pmatrix} 0 & \omega^1{}_2 & -\omega^3{}_1 \\ -\omega^1{}_2 & 0 & \omega^2{}_3 \\ \omega^3{}_1 & -\omega^2{}_3 & 0 \end{pmatrix} . \tag{6.82}$$

Cartan's first structural equation gives the following connection 1-forms

$$\begin{aligned} \omega^1{}_2 &= \left(b_2(X,Y,Z) + \gamma'(Y)e^{-\eta(Z)-\lambda(X)}\right)\vartheta^1 + \left(c_2(X,Y,Z) - \lambda'(X)e^{-\xi(Z)-\gamma(Y)}\right)\vartheta^2 , \\ \omega^2{}_3 &= \left(c_1(Z) + \eta'(Z)e^{-\psi(Z)}\right)\vartheta^2 , \qquad \omega^3{}_1 = \left(b_1(Z) - \xi'(Z)e^{-\psi(Z)}\right)\vartheta^1 . \end{aligned} \tag{6.83}$$

The second structural equation, i.e., $\mathcal{R}^{\alpha}{}_{\beta} = 0$ is trivially satisfied if one assumes that

$$\xi'(Z) = b_1(Z)e^{\psi(Z)}\,, \quad \gamma'(Y) = -b_2(X,Y,Z)e^{\eta(Z)+\lambda(X)}\,, \quad \eta'(Z) = -c_1(Z)e^{\psi(Z)}\,, \\ \lambda'(X) = c_2(X,Y,Z)e^{\xi(Z)+\gamma(Y)}\,. \tag{6.84}$$

Thus

$$\eta'(Z) = -\frac{1}{b_2}\frac{\partial b_2}{\partial Z},\ \lambda'(X) = -\frac{1}{b_2}\frac{\partial b_2}{\partial X},\ \gamma'(Y) = -\frac{1}{c_2}\frac{\partial c_2}{\partial Y},\ \xi'(Z) = -\frac{1}{c_2}\frac{\partial c_2}{\partial Z},\ \psi(Z) = \ln\left[\frac{-1}{b_1 c_2}\frac{\partial c_2}{\partial Z}\right]\,, \tag{6.85}$$

where one needs to have $\frac{\partial b_2}{\partial Z} = -\frac{c_1 b_2}{b_1 c_2}\frac{\partial c_2}{\partial Z}$ and $-\frac{1}{b_1 c_2}\frac{\partial c_2}{\partial Z} > 0$. If we assume that b_2 and c_2 are separable in X, Y, and Z, i.e., $b_2 = b_{2X}(X)b_{2Y}(Y)b_{2Z}(Z)$ and $c_2 = c_{2X}(X)c_{2Y}(Y)c_{2Z}(Z)$, then

$$e^{\eta(Z)} = \frac{C_1}{b_{2Z}(Z)}\,, \quad e^{\lambda(X)} = \frac{C_2}{b_{2X}(X)}\,, \quad e^{\gamma(Y)} = \frac{C_3}{c_{2Y}(Y)}\,, \quad e^{\xi(Z)} = \frac{C_4}{c_{2Z}(Z)}\,, \quad e^{\psi(Z)} = -\frac{c_{2Z}(Z)'}{b_1(Z)c_{2Z}(Z)}\,, \tag{6.86}$$

where C_i, $i = 1, \ldots, 4$ are constants of integration. The compatibility conditions are written as

$$C_1C_2\, b_{2Y}(Y) = \frac{c'_{2Y}(Y)}{c_{2Y}(Y)}\,, \quad C_3C_4\, c_{2X}(X) = -\frac{b'_{2X}(X)}{b_{2X}(X)}\,, \quad \frac{b'_{2Z}(Z)}{b_{2Z}(Z)} = -\frac{c_1(Z)}{b_1(Z)}\frac{c'_{2Z}(Z)}{c_{2Z}(Z)}\,. \tag{6.87}$$

Therefore, we have the material manifold (6.78) for the edge dislocation distributions with the Burgers' vector density (6.80). For the sake of simplicity of calculations, in the remaining of this section we consider two simplified cases of the distribution (6.80): (i) $b_2(X,Y,Z) = c_2(X,Y,Z) = 0$, $\gamma(Y) = 0$, $\lambda(X) = 0$, $\psi(Z) = 0$, and (ii) $b_2(X,Y,Z) = c_2(X,Y,Z) = 0$, $\gamma(Y) = 0$, $\lambda(X) = 0$, $c_1(Z) = 0$, $\eta(Z) = 0$.

Case (i). From (6.80), the Burgers' vector density reads $\mathbf{b} = \mathbf{b}(Z) = b_1(Z)\mathbf{e}_1 + c_1(Z)\mathbf{e}_2 = b_1(Z)e^{-\xi(Z)}\partial_X + e^{-\eta(Z)}c_1(Z)\partial_Y$, where, using (6.84), $\xi'(Z) = b_1(Z)$ and $\eta'(Z) = -c_1(Z)$. The material metric (6.78) is simplified as $\mathbf{G} = \operatorname{diag}\left\{e^{2\xi(Z)}, e^{2\eta(Z)}, 1\right\}$. Looking for solutions of the form $(x,y,z) = (X,Y,\alpha Z)$, the incompressibility constraint implies that $J = \frac{\alpha}{e^{\xi(Z)+\eta(Z)}} = 1$, and thus, $\xi(Z) + \eta(Z) = \ln\alpha$. This means that $\xi'(Z) + \eta'(Z) = 0$, and hence, $c_1(Z) = b_1(Z)$. Choosing orthonormal vectors $\mathbf{N}_1 = e^{-\xi(Z)}\partial_X$, $\mathbf{N}_2 = e^{-\eta(Z)}\partial_Y$, and $\mathbf{N}_3 = \partial_Z$ as the orthotropic axes, the invariants of the energy function are obtained from (6.16) as follows

$$I_1 = \alpha^2 + e^{-2\xi(Z)} + \frac{1}{\alpha^2}e^{2\xi(Z)}\,, \quad I_2 = \frac{1}{\alpha^2} + e^{2\xi(Z)} + \alpha^2 e^{-2\xi(Z)}\,, \quad I_5 = I_4^2 = e^{-4\xi(Z)}\,, \quad I_7 = I_6^2 = \frac{e^{4\xi(Z)}}{\alpha^4}\,. \tag{6.88}$$

Therefore, the non-zero components of the Cauchy stress tensor read

$$\hat{\sigma}^{xx} = 2e^{-2\xi(Z)}\left[W_{I_1} + \Big(\frac{e^{2\xi(Z)}}{\alpha^2} + \alpha^2\Big)W_{I_2} + W_{I_4}\right] + 4e^{-4\xi(Z)}W_{I_5} - p(Z)\,, \tag{6.89}$$

$$\hat{\sigma}^{yy} = \frac{2}{\alpha^2}e^{2\xi(Z)}\left[W_{I_1} + \Big(e^{-2\xi(Z)} + \alpha^2\Big)W_{I_2} + W_{I_6}\right] + \frac{4}{\alpha^4}e^{4\xi(Z)}W_{I_7} - p(Z)\,, \tag{6.90}$$

$$\hat{\sigma}^{zz} = 2\alpha^2\left[W_{I_1} + \Big(e^{-2\xi(Z)} + \frac{e^{2\xi(Z)}}{\alpha^2}\Big)W_{I_2}\right] - p(Z)\,. \tag{6.91}$$

Equilibrium equations imply that $\hat{\sigma}^{zz} = C$, where C is a constant. Vanishing of the traction vector on surfaces parallel to the $\mathrm{X}-\mathrm{Y}$ plane gives the pressure as

$$P(Z) = 2\alpha^2\left[W_{I_1} + \Big(e^{-2\xi(Z)} + \frac{e^{2\xi(Z)}}{\alpha^2}\Big)W_{I_2}\right]\,. \tag{6.92}$$

Case (ii). The Burgers' vector density is given by $\mathbf{b} = \mathbf{b}(Z) = b_1(Z)\mathbf{e}_1 = b_1(Z)e^{-\xi(Z)}\partial_X$, where $\xi'(Z) = b_1(Z)$. From (6.78), the material metric reads $\mathbf{G} = \mathrm{diag}\left\{e^{2\xi(Z)}, 1, e^{2\psi(Z)}\right\}$. We then look for solutions of the form $(x,y,z) = (X, Y, \alpha Z)$. Incompressibility implies that $J = \frac{\alpha}{e^{\xi(Z)+\psi(Z)}} = 1$, and hence, $\xi(Z)+\psi(Z) = \ln\alpha$. The orthotropic axes are $\mathbf{N}_1 = e^{-\xi(Z)}\partial_X$, $\mathbf{N}_2 = \partial_Y$, and $\mathbf{N}_3 = e^{-\psi(Z)}\partial_Z$. The invariants of the strain energy function read

$$I_1 = 1 + e^{-2\xi(Z)} + e^{2\xi(Z)}\,, \quad I_2 = 1 + e^{2\xi(Z)} + e^{-2\xi(Z)}\,, \quad I_5 = I_4^2 = e^{-4\xi(Z)}\,, \quad I_6 = I_7 = 1\,. \tag{6.93}$$

The non-zero components of the Cauchy stress are given as

$$\hat{\sigma}^{xx} = 2W_{I_2} + 2e^{-2\xi(Z)}\left(W_{I_1} + W_{I_2} + W_{I_4}\right) + 4e^{-4\xi(Z)}W_{I_5} - p(Z)\,, \tag{6.94}$$

$$\hat{\sigma}^{yy} = 2W_{I_1} + 2W_{I_2}\left(e^{-2\xi(Z)} + e^{2\xi(Z)}\right) + 2W_{I_6} + 4W_{I_7} - p(Z)\,, \tag{6.95}$$

$$\hat{\sigma}^{zz} = 2\left(W_{I_2} + e^{2\xi(Z)}(W_{I_1} + W_{I_2})\right) - p(Z)\,. \tag{6.96}$$

The equilibrium equation and the vanishing of traction vector on surfaces parallel to $\mathrm{X}-\mathrm{Y}$ plane give

$$p(Z) = 2\left(W_{I_2} + e^{2\xi(Z)}(W_{I_1} + W_{I_2})\right)\,. \tag{6.97}$$

6.3.5 A Spherically-Symmetric Distribution of Point Defects in a Transversely Isotropic Ball

In this section, we calculate the stress field of a spherically-symmetric distribution of point defects with the volume density $\mathsf{n}(R)$[14] in a transversely isotropic ball of radius R_o. The material manifold of a medium with distributed point defects is a flat Weyl manifold [63]. Let us assume that the material preferred direction is radial, i.e., $\mathbf{N} = \hat{\mathbf{R}}$,[15] where $\hat{\mathbf{R}}$ is a unit vector in the radial direction. The material metric for the body with a radial distribution of point defects in the spherical coordinates (R, Θ, Φ) reads $\mathbf{G} = \mathrm{diag}\left\{f^2(R), R^2, R^2\sin^2\Theta\right\}$, where

$$f(R) = \frac{1-\mathsf{n}(R)}{1-\frac{1}{R^3}\int_0^R 3y^2\mathsf{n}(y)dy}\,. \tag{6.98}$$

We endow the ambient space with the flat Euclidean metric $\mathbf{g} = \mathrm{diag}\left\{1, r^2, r^2\sin^2\theta\right\}$ in the spherical coordinates (r, θ, ϕ). Given an embedding of the form $(r, \theta, \phi) = (r(R), \Theta, \Phi)$, the deformation gradient is written as $\mathbf{F} = \mathrm{diag}\{r'(R), 1, 1\}$. The right Cauchy-Green deformation tensor reads

[14]Note that $\mathsf{n}(R) < 0$ for a distribution of vacancies, and $\mathsf{n}(R) > 0$ for a distribution of interstitials.

[15]Note that $\hat{\mathbf{R}} = \frac{1}{f(R)}\mathbf{E}_R$ is the unit vector identifying the material preferred direction, where $\mathbf{E}_R = \frac{\partial}{\partial R}$ such that $\langle\!\langle \mathbf{E}_R, \mathbf{E}_R\rangle\!\rangle_{\mathbf{G}} = G_{RR}$.

$\mathbf{C} = \mathrm{diag}\left\{\frac{r'^2(R)}{f^2(R)}, \frac{r^2(R)}{R^2}, \frac{r^2(R)}{R^2}\right\}$. Assuming incompressibility, the Jacobean is expressed as

$$J = \sqrt{\frac{\mathrm{det}\mathbf{g}}{\mathrm{det}\mathbf{G}}}\mathrm{det}\mathbf{F} = \frac{r^2(R)r'(R)}{R^2 f(R)} = 1\,. \tag{6.99}$$

This gives $r(R) = \left(\int_0^R 3\xi^2 f(\xi)d\xi\right)^{\frac{1}{3}}$. Using (6.8), the invariants are written as

$$I_1 = \mathrm{tr}(\mathbf{C}) = \frac{R^4}{r^4(R)} + 2\frac{r^2(R)}{R^2}\,, \quad I_2 = \frac{1}{2}(\mathrm{tr}(\mathbf{C}^2) - \mathrm{tr}(\mathbf{C})^2) = \frac{r^4(R)}{R^4} + \frac{2R^2}{r(R)^2}\,, \quad I_5 = I_4^2 = \frac{R^8}{r^8(R)}\,. \tag{6.100}$$

Using (6.99), the non-zero stress components read[16]

$$\begin{aligned}\hat{\sigma}^{rr} &= 2\frac{R^4}{r^4(R)}\left(W_{I_1} + W_{I_4}\right) + 4\frac{R^2}{r^2(R)}W_{I_2} + 4\frac{R^8}{r^8(R)}W_{I_5} - p(R)\,,\\ \hat{\sigma}^{\theta\theta} &= \hat{\sigma}^{\phi\phi} = 2\frac{r^2(R)}{R^2}W_{I_1} + 2\frac{R^2}{r^2(R)}W_{I_2} + 2\frac{r^4(R)}{R^4}W_{I_2} - p(R)\,.\end{aligned} \tag{6.102}$$

The non-trivial equilibrium equation is simplified to read $\frac{1}{r'(R)}\sigma^{rr}{}_{,R} + \frac{2}{r}\sigma^{rr} - 2r\sigma^{\theta\theta} = 0$. This gives $p'(R) = q(R)$, where

$$\begin{aligned}q(R) = &-\frac{4}{R^3 r^{19}}\Big[f\Big\{R^9 r^{12}\left(W_{I_1} + 4W_{I_2I_2} - 4W_{I_2I_5} + W_{I_4}\right) + R^3 r^{18}\left(W_{I_1} - 4W_{I_2I_2}\right)\\ &+ R^7 r^{14}\left[W_{I_2} - 2\left(W_{I_1I_1} + W_{I_1I_4}\right)\right] + 2R^{13}r^8\left(W_{I_1I_1} + 2W_{I_1I_4} + W_{I_4I_4} + 3W_{I_5}\right)\\ &+ 2R^{11}r^{10}\left(3W_{I_1I_2} - 2W_{I_1I_5} + 3W_{I_2I_4}\right) - 2R^5 r^{16}\left(3W_{I_1I_2} + W_{I_2I_4}\right) + 8R^{17}r^4\left(W_{I_1I_5} + W_{I_4I_5}\right)\\ &+ RW_{I_2}r^{20} + 12R^{15}W_{I_2I_5}r^6 + 8R^{21}W_{I_5I_5}\Big\} - 2r^3\Big\{R^6 r^{12}\left(W_{I_1} + 2W_{I_2I_2} - 2W_{I_2I_5} + W_{I_4}\right)\\ &+ R^4 r^{14}\left(W_{I_2} - W_{I_1I_1} - W_{I_1I_4}\right) + R^{10}r^8\left(W_{I_1I_1} + 2W_{I_1I_4} + W_{I_4I_4} + 4W_{I_5}\right)\\ &+ R^8 r^{10}\left(3W_{I_1I_2} - 2W_{I_1I_5} + 3W_{I_2I_4}\right) - R^2 r^{16}\left(3W_{I_1I_2} + W_{I_2I_4}\right) + 4R^{14}r^4\left(W_{I_1I_5} + W_{I_4I_5}\right)\\ &- 2W_{I_2I_2}r^{18} + 6R^{12}W_{I_2I_5}r^6 + 4R^{18}W_{I_5I_5}\Big\}\Big]\,.\end{aligned} \tag{6.103}$$

[16]When the body is defect-free, $f(R) = 1$, and thus, $I_1 = I_2 = 3$ and $I_4 = I_5 = 1$. If one assumes that the stress vanishes in this case, one has (see [92, 94] for similar conditions)

$$\left(2W_{I_5} + W_{I_4}\right)|_{I_1=I_2=3, I_4=I_5=1} = 0\,. \tag{6.101}$$

Next, we assume an energy function corresponding to a radially reinforced Mooney-Rivlin spherical ball of the following form

$$W(I_1, I_2, I_4, I_5) = \frac{\mu_1}{2}(I_1 - 3) + \frac{\mu_2}{2}(I_2 - 3) + \frac{\gamma_1}{2}(I_4 - 1)^2 + \frac{\gamma_2}{2}(I_5 - 1)^2 \,, \tag{6.104}$$

where μ_1 and μ_2 are constants of the Mooney-Rivlin base material, while γ_1 and γ_2 are non-negative material constants pertaining to the reinforcement strength in the radial direction. Thus, (6.103) is simplified to read

$$\begin{aligned} p'(R) = -&\frac{2}{R^2 r^{19}}\Big[f\left\{6R^{12}(\gamma_1 - 2\gamma_2)r^8 + R^8(\mu_1 - 2\gamma_1)r^{12} + \mu_2 R^6 r^{14} + \mu_1 R^2 r^{18} + \mu_2 r^{20} + 28\gamma_2 R^{20}\right\} \\ &- 2R^3 r^3\left(4R^6(\gamma_1 - 2\gamma_2)r^8 + R^2(\mu_1 - 2\gamma_1)r^{12} + \mu_2 r^{14} + 16\gamma_2 R^{14}\right)\Big]\,. \end{aligned} \tag{6.105}$$

The stress components are also simplified and read

$$\begin{aligned} \hat{\sigma}^{rr} &= \frac{R^4}{r^4}\Big[\mu_1 + 2\gamma_1\Big(\frac{R^4}{r^4} - 1\Big)\Big] + 2\mu_2\frac{R^2}{r^2} + 4\gamma_2\frac{R^8}{r^8}\Big(\frac{R^8}{r^8} - 1\Big) - p\,, \\ \hat{\sigma}^{\theta\theta} &= \hat{\sigma}^{\phi\phi} = \mu_1\frac{r^2}{R^2} + \mu_2\frac{R^2}{r^2} + \mu_2\frac{r^4}{R^4} - p\,. \end{aligned} \tag{6.106}$$

Assuming that the boundary of the ball is traction-free, one obtains

$$p(R_o) = \frac{R_o^4}{r^4(R_o)}\left\{\mu_1 + 2\gamma_1\left[\frac{R_o^4}{r^4(R_o)} - 1\right]\right\} + 2\mu_2\frac{R_o^2}{r^2(R_o)} + 4\gamma_2\frac{R_o^8}{r^8(R_o)}\left[\frac{R_o^8}{r^8(R_o)} - 1\right]\,. \tag{6.107}$$

Thus

$$\begin{aligned} p(R) =& 2\int_R^{R_o}\frac{1}{\xi^2 r^{19}(\xi)}\Bigg\{f(\xi)\Big[6\xi^{12}(\gamma_1 - 2\gamma_2)r^8(\xi) + \xi^8(\mu_1 - 2\gamma_1)r^{12}(\xi) + \mu_2\xi^6 r^{14}(\xi) + \mu_1\xi^2 r^{18}(\xi) \\ &+ \mu_2 r^{20} + 28\gamma_2\xi^{20}\Big] - 2\xi^3 r^3(\xi)\Big[4\xi^6(\gamma_1 - 2\gamma_2)r^8(\xi) + \xi^2(\mu_1 - 2\gamma_1)r^{12} + \mu_2 r^{14}(\xi) + 16\gamma_2\xi^{14}\Big]\Bigg\}d\xi \\ &+ p(R_o)\,. \end{aligned} \tag{6.108}$$

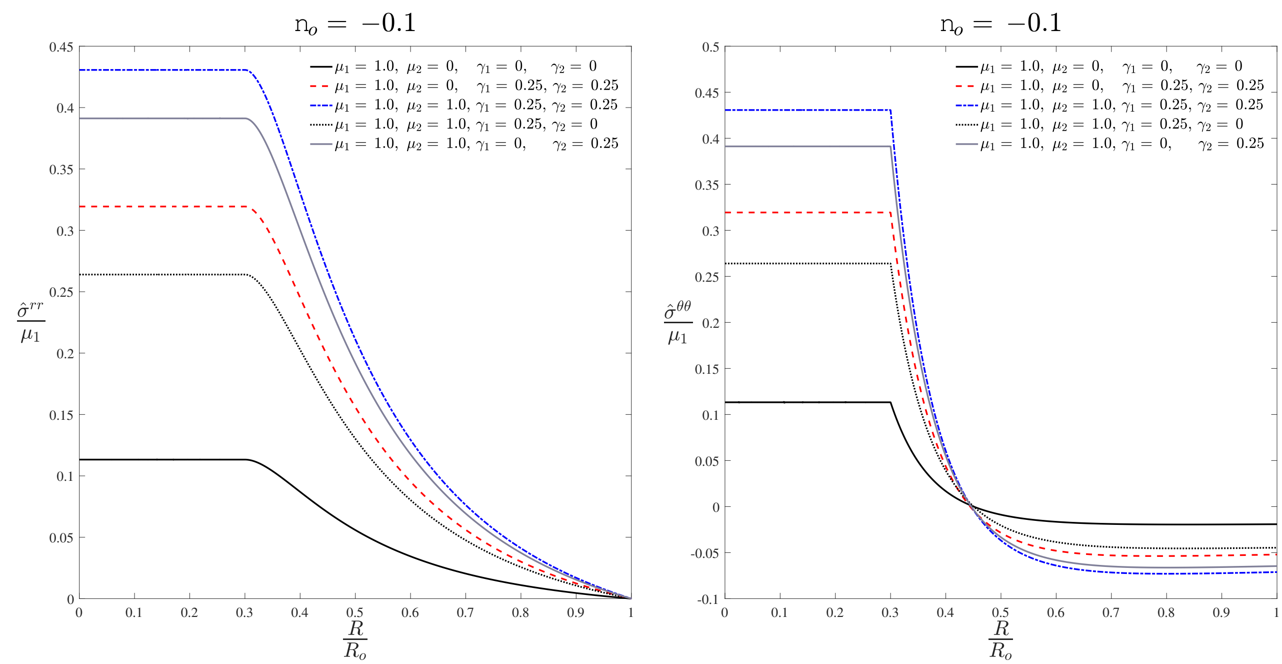


Figure 6.5: $\hat{\sigma}^{rr}$ *and* $\hat{\sigma}^{\theta\theta}$ *distributions for different values of the constitutive parameters for the point defect distribution* (6.109) *with* $R_i/R_o = 0.3$ *and* $\mathsf{n}_o = -0.1$.

Let us consider the following distribution of point defects in the ball

$$\mathsf{n}(R) = \begin{cases} \mathsf{n}_o & 0 \leq R \leq R_i\,, \\ 0 & R_i < R \leq R_o\,. \end{cases} \tag{6.109}$$

Therefore, from (6.98)

$$f(R) = \begin{cases} 1\,, & 0 \leq R \leq R_i\,, \\ (1 - \mathsf{n}_o(R_i/R)^3)^{-1}\,, & R_i < R \leq R_o\,, \end{cases} \tag{6.110}$$

and hence

$$r(R) = \begin{cases} R\,, & 0 \leq R \leq R_i\,, \\ \left[R^3 + \mathsf{n}_o R_i^3 \ln \frac{(R/R_i)^3 - \mathsf{n}_o}{1-\mathsf{n}_o}\right]^{1/3}\,, & R_i < R \leq R_o\,. \end{cases} \tag{6.111}$$

Fig. 6.5 shows the stress field variation for the point defect distribution (6.109), when $R_i/R_o = 0.3$ and $\mathsf{n}_o = -0.1$ for different values of the reinforcement and base material constants in (6.104).

Remark 6.3.8. Consider an arbitrary nonlinear incompressible transversely isotropic spherical ball of radius R_o such that the material preferred direction is radial. Suppose that the ball is subject to a uniform pressure on its boundary and has the point defect distribution (6.109). Then, in the ball $R \leq R_i$, the stress is uniform and hydrostatic. Interestingly, the value of the hydrostatic stress inside the ball $R \leq R_i$ has an explicit dependence on the reinforcement parameters (see Fig. 6.5). To show this, for $R \leq R_i$, $f(R) = 1$ and $r(R) = R$, following (6.110) and (6.111), respectively. Therefore, after some simplification, (6.103) implies that $p'(R) = q(R) = \frac{4}{R}\left(W_{I_4} + 2W_{I_5}\right)|_{I_1=I_2=3,I_4=I_5=1} = 0$ using the relation (6.101). Hence, for $R \leq R_i$, $p(R) = C$, where C is a constant depending on the reinforcement and the base material parameters. From (6.102), $\hat{\sigma}^{rr} = \hat{\sigma}^{\theta\theta} = \hat{\sigma}^{\phi\phi} = 2\left(W_{I_1} + 2W_{I_2}\right)|_{I_1=I_2=3,I_4=I_5=1} - C$ for $R \leq R_i$.

CHAPTER 7

ELASTODYNAMIC TRANSFORMATION CLOAKING

7.1 Introduction

Invisibility has been a dream for centuries. Making objects invisible to electromagnetic waves has been a subject of intense research in recent years. Pendry *et al.* [142] and Leonhardt [143] independently showed the possibility of electromagnetic cloaking. This was later experimentally verified by Schurig *et al.* [144] and Liu *et al.* [145], and Ergin *et al.* [146].[1] In many references, including [142], it is argued that the main idea of cloaking in electromagnetism is the invariance of Maxwell's equations under coordinate transformations (covariance). The covariance of Maxwell's equations has been known for a long time [147]. However, one should note that covariance of Maxwell's equations is not the direct underlying principle of transformation cloaking. In electromagnetic cloaking one maps one problem to another problem with some desirable response. For example, a domain with a hole surrounded by a cloak with unknown physical properties is mapped to a domain without a hole (or with a very small one) made of an isotropic and homogeneous material. Then one tries to find the transformed fields such that both problems satisfy Maxwell's equations [148]. This will then determine the physical properties of the cloak. In particular, the transformed quantities are not necessarily what one would expect under a coordinate transformation, i.e., the two problems are not related by push-forward or pull-back using the cloaking map.

Cloaking in the context of conductivity [149, 150], electrical impedance tomography, and electromagnetism has been studied rigorously and is well understood [151, 152, 153, 154]. Interestingly, the idea of cloaking has been explored in many other fields of science and engineering, e.g., acoustics [155, 156, 157, 158, 159, 160, 161, 162], optics [163], thermodynamics (design of thermal cloaks) [164], diffusion [165], quantum mechanics [166], and elastodynamics [167]. The recent reviews [168,

[1] One should note the frequency limitation in the existing electromagnetic cloaking works; the existing works have been limited to microwaves frequencies.

169] discuss these applications in some detail. The least understood among these applications is elastodynamics. In our opinion, the main problem is that none of the existing works in the literature has formulated the problem of elastodynamic cloaking properly. In particular, boundary and continuity conditions and the restrictions they impose on cloaking maps have not been discussed. In this chapter, we formulate both the nonlinear and linearized cloaking problems in a mathematically precise form. One should note that transformation cloaking is an inherently geometric problem. This is explained in the case of optical cloaking and invisibility in [163]. We will see that this is the case for elastodynamic cloaking as well; geometry plays a critical role in a proper formulation of elastodynamic transformation cloaking.

The first ideas related to cloaking in elasticity go back to the 1930s and 1940s in the works of Gurney [170] and Reissner and Morduchow [171] on reinforced holes in elastic sheets in the framework of linear elasticity. The first systematic study of cloaking in linear elasticity is due to Mansfield [172] who introduced the concept of neutral holes. Mansfield considered a sheet (a plane problem) under a given (far-field) load. For the same far-field load applied to an uncut sheet one knows (or can calculate) the corresponding Airy stress function $\phi = \phi(x, y)$. He then put a hole(s) in the sheet and asked if the hole(s) can be reinforced such that the stress field outside the hole(s) is identical to that of the uncut sheet. In other words, the reinforcement hides the hole(s) from the stress field. Mansfield [172] showed that the boundary of a neutral hole is given by the equation $\phi(x, y) + ax + by + c = 0$, where a, b, c are constants. The reinforcement is a pre-stressed axially-loaded member (no bending stiffness) that may have a non-uniform cross sectional area. Design of such neutral holes depend on the external loading; unlike the cloaking problem in which a hole is to be hidden from arbitrary elastic disturbances, the shape of a neutral hole and the characteristics of its reinforcement explicitly depend on the stress field of the uncut sheet. In other words, a reinforced hole neutral under one far-field load may not be neutral under another one. In this chapter we will present a nonlinear analogue of Mansfield's neutral holes for radial deformations in §7.4.2. Another problem related to cloaking is the idea of neutral inhomogeneities. A neutral inhomogeneity when inserted in an elastic medium that is under some far field loading would not perturb the stress field of the outside medium. This has been extensively studied for spherical [173, 174, 175], cylindrical [176], and ellipsoidal [177] inhomogeneities. It should be noted that the properties of the neutral inhomogeneities depend on the imposed loading.

The main difference between electromagnetic (and optical) cloaking and elastodynamic cloaking is that the governing equations of elasticity are written with respect to two frames and that they are tensor-valued. In elasticity, one writes the governing equations with respect to a reference and a current configuration; this leads to two-point tensors in the governing equations. If formulated properly, both nonlinear and linearized elasticity are spatially covariant, i.e., their governing equations are invariant under arbitrary time-dependent changes of frame (or coordinate transformations if viewed passively) [178, 82]. Invariance under referential changes of frame is more subtle as can be seen in the work of Yavari *et al.* [46], who showed that the balance of energy is not invariant under an arbitrary time-dependent referential diffeomorphism. They obtained the transformed balance of energy that has some new terms corresponding to the velocity of the referential change of frame. Mazzucato and Rachele [132] showed that the balance of linear momentum is invariant under any time-independent change in the reference configuration. In this chapter, we will show that the results of Yavari *et al.* [46] and Mazzucato and Rachele [132] are consistent and the balance of energy and all the governing equations of nonlinear elasticity are invariant under arbitrary time-independent referential coordinate transformations. More recently, motivated by applications in seismology, Al-Attar and Crawford [179] showed the invariance of the governing equations of elastodynamics under referential changes of frame that they called particle relabelling transformations, a term borrowed from fluid mechanics. They also correctly pointed out that the elastic constants with respect to any equilibrium configuration transform tensorially under a particle relabelling transformation.

The goal in elastodynamic cloaking is to make a hole (cavity) invisible to elastic waves. One idea is to cover the boundary of the cavity by a cloak that has inhomogeneous and anisotropic elastic properties, in general. The cloak will deflect the elastic waves resulting in elastic measurements away from the cavity (more specifically, outside the cloak) identical to those when the cavity is absent. Pulling back the homogeneous material properties using a cloaking transformation (that will be defined later), one may obtain the desired inhomogeneous and anisotropic mechanical properties of the cloak. To achieve design of an elastic cloak one would need to answer the following questions: i. Are the governing equations of nonlinear (and linear) elasticity invariant under coordinate transformations? First, one must define what a coordinate transformation means in nonlinear elasticity. There are two types of transformations that have very different physical meanings. ii. Can cloaking be achieved using a spatial or referential coordinate transformation? Or is a cloaking transformation more than a change of coordinates? Related to question i, note that any properly formulated physical field theory has to be covariant. This was Einstein's idea in the theory of general relativity. No coordinate system can be a distinguished one; nature does not discriminate between different observers and they all see the same physical laws. In nonlinear elasticity covariance is understood as the invariance of the governing equations under arbitrary time-dependent coordinate transformations in the ambient space [180, 43, 46]. Regarding question ii we will show that a cloaking transformation is neither a spatial nor a referential change of frame (coordinates); a cloaking transformation maps the boundary-value problem of an isotropic and homogeneous elastic body (virtual problem) to that of an anisotropic and inhomogeneous elastic body with a hole surrounded by a cloak that is to be designed (physical problem).

Traditionally, many workers in solid mechanics start from linear elasticity. This is appropriate for many practical applications, and linear elasticity has been quite successful in numerous engineering applications. The governing equations of linear elasticity are linear partial differential equations, and hence, superposition is applicable, one can use Green's functions, etc. However, there are many problems for which linearized elasticity is not appropriate. The first practical application of nonlinear elasticity was in the rubber industry in the 1940s and 1950s, which motivated Rivlin's seminal contributions [181, 182, 183, 184, 185, 186, 187]. In recent years, nonlinear elasticity has been revived motivated by the biomechanics applications in which biological tissues undergo large strains [188]. Unlike electromagnetism with only one configuration (ambient space), in nonlinear elasticity, there are two inherently different configurations: reference and current. Linear elasticity does not distinguish between these two configurations, and this has been a source of confusion in the recent literature of elastodynamic transformation cloaking. In the reference configuration the body is stress free[2] and any measure of strain is defined with respect to this configuration. Consequently, the stored energy of an elastic body explicitly depends on the reference configuration as well. In the classical formulation of nonlinear elasticity, it is well understood that coordinate transformations in the reference and current configurations are very different. Local referential transformations are related to material symmetries, while the global transformations of the ambient space (current configuration) are related to objectivity (or material frame indifference). This implies that any cloaking study, even when strains are small, should be formulated in the framework of nonlinear elasticity.

A classic example of an improper use of the governing equations of linear elasticity can be seen in almost all the existing discussions on the objectivity of linear elasticity. It has long been argued that linearized elasticity is not objective, i.e., its governing equations are not invariant under rigid body translations and rotations of the ambient space (see [178] for references). This is unnatural and accepting it, one, at least implicitly, is assuming that linear elasticity is a "special" field theory. This cannot be true and linear elasticity, like any other field theory, has to be objective (and, more generally, spatially covariant) if it is properly formulated. This problem was revisited independently by Steigmann [178] and Yavari and Ozakin [82]. These authors showed that if formulated and interpreted properly, linear elasticity is objective (covariant) as expected. In short, Navier's equations are written with respect to one coordinate system and do not have the proper geometric structure to be used in studying the transformation properties of the balance of linear momentum in linear elasticity.

The work of Lodge [35, 36]. Arthur S. Lodge showed that the static equilibrium solutions of certain anisotropic homogeneous linear elastic bodies can be mapped to those of isotropic homogenous linear elastic bodies using affine transformations of position and displacement vectors. In other words, knowing an equilibrium solution for a homogeneous isotropic linear elastic body, new equilibrium solutions can be generated for certain anisotropic bodies. We should point out that in [35, 36] and [191] the matrices of the coordinate and displacement transformations are inverses of each other. In

[2] We should mention that there are recent geometric developments using non-Euclidean reference configurations that allow for sources of residual stress [9, 20, 44, 63, 128, 64, 10, 189, 25, 26, 27, 66, 21, 190, 40].

particular, Lodge [35] in his first paper considered the following transformations for coordinates and displacements: $(x, y, z) \to (x', y', z') = (x, y, \nu^{-\frac{1}{2}} z)$ and $(u, v, w) \to (u', v', w') = (u, v, \nu^{\frac{1}{2}} z)$, for some $\nu > 0$. He showed that the governing equations of linear elasticity are invariant under these transformations. Note that these transformations map the equilibrium solution of an isotropic elastic body to that of another elastic body that is anisotropic. This should not be confused with the transformation of the governing equations of one given body under referential or spatial coordinate transformations. In other words, Lodge [36] (see also Lang [192]) finds an equilibrium solution for an anisotropic body using that of another elastic body, which is isotropic. The position and displacement vectors are linearly related. However, the two problems are not related by a coordinate transformation. Olver [37] showed that any planar anisotropic linear elastic solid is equivalent to an orthotropic solid through some linear transformations of coordinates and displacements that are independent.

Lodge [36]'s idea of mapping the boundary-value problem of an anisotropic linearly elastic body to that of an isotropic body can be summarized as follows. The position vector and the displacement field are transformed homogeneously as

$$\mathbf{x}' = \mathbf{A}\mathbf{x}, \quad \mathbf{u}' = \mathbf{A}^{-\mathsf{T}}\mathbf{u}\,. \tag{7.1}$$

We will refer to the transformation $(\mathbf{x}', \mathbf{u}') = (\mathbf{A}\mathbf{x}, \mathbf{A}^{-\mathsf{T}}\mathbf{u})$ as a *Lodge transformation*. Using this pair of linear transformations, strain is transformed as $\boldsymbol{\epsilon}' = \mathbf{A}^{-\mathsf{T}}\boldsymbol{\epsilon}\mathbf{A}^{-1}$. Assuming that under (7.1), $\boldsymbol{\sigma}' : \boldsymbol{\epsilon}' = \boldsymbol{\sigma} : \boldsymbol{\epsilon}$, stress is transformed as $\boldsymbol{\sigma}' = \mathbf{A}\boldsymbol{\sigma}\mathbf{A}^{\mathsf{T}}$. Lodge [36] assumed that body force transforms like a vector, i.e., $\mathbf{b}' = \mathbf{A}\mathbf{b}$. Implicitly, he assumed that mass density transforms like a scalar, i.e., it remains unchanged: $\rho' = \rho$. This is similar to the way mass density transforms under a change of spatial frame. Under these assumptions one can show that

$$\operatorname{div}' \boldsymbol{\sigma}' + \rho'\mathbf{b}' = \mathbf{A}\left(\operatorname{div} \boldsymbol{\sigma} + \rho\mathbf{b}\right)\,. \tag{7.2}$$

The inertial force is transformed as $\rho'\mathbf{a}' = \mathbf{A}^{-\mathsf{T}}(\rho\mathbf{a})$. Therefore, starting from $\operatorname{div} \boldsymbol{\sigma} + \rho\mathbf{b} = \rho\mathbf{a}$, one obtains the balance of linear momentum for the transformed body as

$$\operatorname{div}' \boldsymbol{\sigma}' + \rho' \mathbf{b}' = \mathbf{A}\mathbf{A}^{\mathsf{T}} \rho' \mathbf{a} \,. \tag{7.3}$$

Lodge then concluded that the balance of linear momentum is invariant under the transformation (7.1) when inertial forces are ignored. Under a Lodge transformation the elastic constants, and hence, the anisotropy type changes. Lodge finally calculated the matrix $\mathbf{A}$ such that the transformed body is isotropic. Using this transformation, one can generate equilibrium solutions for certain anisotropic bodies having the corresponding solutions for an isotropic linearly elastic body. When inertial forces are not ignored the balance of linear momentum is invariant only when $\mathbf{A}\mathbf{A}^{\mathsf{T}} = \mathbf{I}$, i.e., $\mathbf{A}$ is a rotation and can be interpreted as a coordinate transformation.

Remark 7.1.1. Fifty years later, not being aware of the work of Lodge, Milton *et al.* [167] for a completely different purpose used the Lodge transformations (7.1) but with a position-dependent $\mathbf{A}$. They assumed a harmonic time dependence and ignored the body forces. Let us examine their Eq.(2.4) for a constant matrix $\mathbf{A}$. Their transformed wave equation in this special case reads

$$\operatorname{div}' \boldsymbol{\sigma}' = -\omega^2 \boldsymbol{\rho}' \mathbf{u}' \,, \tag{7.4}$$

where their matrix-valued mass density is defined as $\boldsymbol{\rho}' = \frac{1}{\det \mathbf{A}} \rho \mathbf{A}\mathbf{A}^{\mathsf{T}}$. They justify a matrix-valued mass density arguing that it has been observed for composites. Two comments are in order here: i) Note that the factor $\frac{1}{\det \mathbf{A}}$ appears when one uses the Piola identity as we will discuss in §8.4.1 and §8.4.2. When $\mathbf{A}$ is a constant matrix, $\operatorname{div}' \boldsymbol{\sigma}'$ is $\frac{1}{\det \mathbf{A}}$ times that of Lodge's. ii) If the body force term $\rho \mathbf{b}$ is kept, this matrix-valued mass density would not work unless one assumes that $\mathbf{b}' = \mathbf{A}^{-\mathsf{T}} \mathbf{b}$, which is different from Lodge's original transformation.

The work of Milton *et al.* [167] on elastodynamic cloaking. The first theoretical study of elastodynamic transformation cloaking is due to Milton *et al.* [167]. They observed that the governing equations of linear elasticity are not invariant under coordinate transformations (or what they called "curvilinear transformations"), and hence, cloaking of elastic waves cannot be achieved using coordinate transformations. One should note that Navier's equations are written with respect to one coordinate system, and hence, do not have the proper geometric structure to be used in studying the transformation properties of the governing equations of linear elasticity under coordinate (or cloaking) transformations. Milton *et al.* [167] start with the wave equation in the setting of linear elasticity, i.e., $\nabla \cdot \boldsymbol{\sigma} + \omega^2 \rho \mathbf{u} = \mathbf{0}$, where $\boldsymbol{\sigma} = \mathsf{C}\nabla\mathbf{u}$, and C is the elasticity tensor, and consider mappings $\mathbf{x} \to \mathbf{x}'(\mathbf{x})$ and $\mathbf{u}(\mathbf{x}) \to \mathbf{u}'(\mathbf{x}')$. They point out that the two mappings can be chosen freely. Denoting the derivative of the map $\mathbf{x}'(\mathbf{x})$ by $\mathbf{A}(\mathbf{x})$, i.e., $A_{i'j} = \partial x'_i/\partial x_j$, they assume that instead of $\mathbf{u}' = \mathbf{A}(\mathbf{x})\mathbf{u}$ (assuming that the change of coordinates is a spatial coordinate transformation), displacement is transformed as $\mathbf{u}' = \mathbf{A}^{-\mathsf{T}}(\mathbf{x})\mathbf{u}$. Milton *et al.* [167] point out that these (Lodge-type) transformations preserve the symmetries of the elasticity tensor. They finally show that under these changes of variables the Cauchy stress and the wave equation transform as

$$\operatorname{div}' \boldsymbol{\sigma}' = -\omega^2 \boldsymbol{\rho}' \mathbf{u}' + \mathbf{D}' \nabla' \mathbf{u}' \quad \text{and} \quad \boldsymbol{\sigma}' = \mathsf{C}' \nabla' \mathbf{u}' + \mathbf{S}' \mathbf{u}', \tag{7.5}$$

where ($J = \det \mathbf{A}$), $C'_{pqrs} = \frac{1}{J} \frac{\partial x'^p}{\partial x^i} \frac{\partial x'^q}{\partial x^j} \frac{\partial x'^r}{\partial x^k} \frac{\partial x'^s}{\partial x^l} C_{ijkl}$, $S'_{pqr} = \frac{1}{J} \frac{\partial x'^p}{\partial x^i} \frac{\partial x'^q}{\partial x^j} \frac{\partial^2 x'^r}{\partial x^k \partial x^l} C_{ijkl} = S'_{qrp}$, $D'_{pqr} = S'_{qrp}$, and the matrix-valued mass density[3] $\rho'_{pq} = \frac{\rho}{J} \frac{\partial x'_p}{\partial x_i} \frac{\partial x'_q}{\partial x_i} + \frac{1}{J} \frac{\partial^2 x'_p}{\partial x_i \partial x_j} C_{ijkl} \frac{\partial^2 x'_q}{\partial x_k \partial x_l}$. Banerjee [193] discusses this in more detail and concludes that: "Therefore, unlike Maxwell's equations the equations of elastodynamics change form under a coordinate transformation." This conclusion is, unfortunately, incorrect; governing equations of linear elasticity are form-invariant under both spatial and referential coordinate transformations. However, as we will show cloaking transformations are not coordinate transformations. It is, nevertheless, correct that transformation cloaking is not possible in classical linear elastodynamics.

Several authors have looked at in-plane waves arguing that in this particular case the governing equations are invariant under coordinate transformations (however, it is not clear what type of coordinate transformations is being considered), and hence, transformation cloaking can be acheived. In

[3] There is a typo in their transformed mass density. The second term does not have the correct physical dimension. The similar expression in [193] has the correct dimension with a factor $1/\omega^2$ but has the opposite sign.

particular, Brun *et al.* [194] observed that for in-plane elastic waves, the balance of linear momentum is invariant under an arbitrary change of coordinates (they do not distinguish between referential and spatial transformations), and following the ideas in electromagnetic cloaking, introduced a cylindrical cloak. They considered an annular cloak of inner and outer radii r_0 and r_1, respectively, and used Pendry *et al.* [142]'s coordinate transformation $(r, \theta) \to (r', \theta') = (r'(r), \theta)$, where

$$r'(r) = \begin{cases} r_0 + \frac{r_1 - r_0}{r_1} r, & r \leq r_1, \\ r, & r \geq r_1. \end{cases} \tag{7.6}$$

This (singular) transformation maps a disk of radius r_1 to an annulus of inner and outer radii r_0 and r_1, respectively, and is the identity transformation outside the disk. These authors observed that assuming $\mathbf{u}' = \mathbf{u}$, Navier's equations are form-invariant, and mass density transforms as $\rho'(r) = \frac{r - r_0}{r} \left(\frac{r_1}{r_1 - r_0} \right)^2 \rho(r)$. In the transformed coordinates, they obtained an elasticity tensor that does not possess the minor symmetries. Then their recourse is to argue that the transformed body is made of a Cosserat solid.[4] We will show in this chapter that the map (7.6), which has been borrowed from the literature of electromagnetic cloaking is not admissible for elastodynamic cloaking; the derivative of this map is not the identity on the outer boundary of the cloak, i.e., $f'(r_1) \neq 1$ (see §7.5.2). Instead of assuming that the cloak is made of a Cosserat solid without even discussing its elastic constants, we believe one should assume that both the physical and virtual bodies (that will be defined in §8.4) are made of Cosserat solids and see if a Cosserat cloak can be designed while all the balance laws are respected in both bodies. We will show that the minor symmetries of the elastic constants are not preserved under cloaking transformations (that will be defined in §8.4.1). Our conclusion is that classical linear elasticity is not flexible enough to allow for cloaking. A possible solution may be linearized elasticity with respect to a pre-stressed configuration, i.e., the small-on-large theory of Green *et al.* [195]. We will show that cloaking is not possible in this more general framework either. However, we will see that a cylindrical hole can be cloaked for in-plane

[4]Surprisingly, in none of the works that accept non-symmetric Cauchy stresses in the cloak is there any mention of the balance of angular momentum and the distribution of couple stresses. There is also no discussion on what the extra elastic constants of the Cosserat cloak should be.

deformations provided that the cloak is pre-stressed. This, however, is not possible for the anti-plane deformations. These observations would force one to start from some kind of a solid with microstructure. In addition to finding the classical elastic constants in the cloak, the non-classical elastic constants must be calculated as well. This has not been discussed in the literature to this date. This is a non-trivial calculation that will be discussed in §7.5. In the literature, it has been implicitly assumed that the material outside the cloak is a classical linear elastic solid. We will see that this is not possible (Remark 7.5.5).

Starting from linear elasticity, Norris and Shuvalov [196] tried to find the governing equations in a transformed domain using Lodge-type transformations. Similar to the work of Milton *et al.* [167], they assumed that the displacement field does not transform the way it does under a spatial coordinate transformation. They refer to this as a linear gauge change. Unlike that of Milton *et al.* [167], their displacement transformation is completely independent of the coordinate transformation. This is similar to the transformations that had earlier been used by Olver [37]. They discussed several possibilities for the displacement transformation and observed a loss of minor symmetries of the elastic constants in the cloak and assumed that it is made of a Cosserat solid. However, the balance of angular momentum and the calculation of the non-classical elastic constants of the Cosserat elastic cloak were not discussed.

Olsson and Wall [197] studied time-harmonic cloaking of a finite rigid body that is fixed, i.e., cannot move, and is embedded in an elastic matrix. In the case of both a rigid circular disk and a rigid spherical ball they assumed that the matrix is an elastic medium with inextensible radial fibers. To our best knowledge, this is the first paper on elastic cloaking that actually discusses boundary conditions. In the physical and virtual bodies (that we will define in §8.4) in both 2D and 3D they assumed the following relation between displacement vectors: $\mathbf{U}/R = \tilde{\mathbf{U}}/\tilde{R}$. They observed that the balance of linear momentum of the two configurations have the same form, and that the elastic constants retain their full symmetries. Khlopotin *et al.* [198] investigated cloaking a finite rigid body embedded in an elastic medium. Their motivation was cloaking an object in a soft matrix from surface elastic waves. They assumed that the matrix is a micropolar solid and discussed boundary conditions. However, in this work there is no discussion on how the non-classical elastic constants of the cloak should be calculated. In particular, there is no mention of how the couple stiffness tensor and the couple stress tensor of the micropolar medium are transformed under a cloaking transformation. We believe that a proper formulation of linear elastodynamic cloaking should consider both the physical and virtual bodies to be made of generalized Cosserat solids and all the elastic constants of the cloak must be calculated. This will be discussed in §7.5.

Parnell [199] and Parnell *et al.* [200] (see also [201] for in-plane deformations) first considered anti-plane deformations of an isotropic linear elastic solid for which the displacement field in cylindrical coordinates has the form $(0, 0, W(R, \Theta))$. They concluded that the balance of linear momentum (wave equation) is form-invariant under coordinate transformations. The transformed mass density is identical to that of Brun *et al.* [194]. Next, Parnell considered the wave equation in the small-on-large theory of Green *et al.* [195], which is simply linearization about a finitely-deformed (and stressed) configuration [54]. He considered a body with a small hole made of an incompressible neo-Hookean solid. Using a (static) applied internal pressure the hole is inflated. Parnell showed that the incremental wave equation with respect to this pre-stressed configuration has anisotropic shear moduli. However, they are different from those of a cloak. One should also note that inflating an initially small hole in a body the entire body would deform. In other words, the small-on-large elasticity of such a body cannot be identical to that of a stress-free and homogeneous linear elastic body outside any finite region. In §7.4.4 we will show that in the setting of small-on-large theory cloaking a cylindrical hole for anti-plane deformations is not possible. However, it is possible to cloak a cylindrical hole for in-plane deformations.

There have been other efforts in the literature on guiding elastic waves in structures. We should mention Amirkhizi *et al.* [202] who proposed the idea of redirecting stress waves by smoothly changing anisotropy of a structure. In particular, they experimentally and numerically showed that when the direction of a stress wave is known and is fixed its propagation in a structure made of a transversely isotropic material with a varying axis of anisotropy can be guided. Of course, their construction is restricted and useful only for one specific direction of wave propagation. Using a pre-stressed neo-

Hookean solid Chang *et al.* [203] demonstrated the possibility of manipulating shear waves. Sklan *et al.* [204] proposed a symmetrized elastic cloak by symmetrizing the non-symmetric elastic constants induced from a cloaking map. They used the symmetrized elastic constants in a cloaking region around a cylindrical hole in order to shield an object against elastic waves. In their numerical examples for certain frequencies they observed reduction of average displacements and elastic energy in the cloaking region. This approach based on brute-force symmetrization has clear limitations. In particular, any given geometry and loading condition needs to be numerically checked.

Contributions of this chapter. In this chapter, we investigate the problem of hiding a hole from elastic waves in both nonlinear and linear elastodynamics. It should be emphasized that we do not consider the so-called metamaterials [38, 39]. We start by discussing the invariance of the governing equations of nonlinear and linearized (with respect to any finitely-deformed configuration) elasticity under both arbitrary time-dependent spatial changes of frame (coordinate transformations) and arbitrary time-independent referential changes of frame. We, however, note that cloaking cannot be achieved using either (or both) spatial or referential coordinate transformations. We define a cloaking map to be a mapping that transforms the boundary-value problem of an elastic body with a hole reinforced by a cloak (physical body) to that of a homogeneous and isotropic body with an infinitesimal hole (virtual body). The cloak needs to be designed while the loads and boundary conditions in the virtual body are not known a priori. We define a cloaking transformation to be a map between the boundary-value problems of an elastic body to be designed and a virtual elastic body that has some desired mechanical response. The main contributions of this work can be summarized as follows:

- We provide a geometric formulation of transformation cloaking in nonlinear elasticity. It is shown that nonlinear elastodynamic transformation cloaking is not possible (Proposition 7.4.7).
- It is shown that nonlinear elastostatic transformation cloaking may be possible for special deformations. This is somehow a nonlinear analogue of Mansfield's neutral holes. We provide one such example, namely radial deformations in an infinitely long solid cylinder with a cylindrical hole or a finite spherical ball with a spherical cavity.

- Classical linear elasticity is not flexible enough to allow for transformation cloaking (Proposition 7.4.8). More specifically, linear elastodynamic cloaking cannot be achieved because the elastic constants in the cloak lose their minor symmetries. This is true for a hole of any shape reinforced by a cloak with an arbitrary shape.

- Transformation cloaking is not possible even in the small-on-large theory, i.e., linearized elasticity with respect to a pre-stressed configuration (Proposition 7.4.13). This is true for a hole with an any shape. We show that a cylindrical hole can be cloaked for in-plane deformations while it is not possible to cloak it for anti-plane deformations.

- Assuming that the virtual body is isotropic and centro-symmetric, elastodynamic transformation cloaking is not possible in the setting of gradient elasticity (Proposition 7.5.3). This result is independent of the shape of the hole.

- Elastodynamic transformation cloaking is not possible in the setting of generalized Cosserat elasticity in dimension two (Proposition 7.5.7). In particular, in dimension two transformation cloaking cannot be achieved in Cosserat elasticity (with rigid directors) either. This result is independent of the shapes of the hole and the cloak. Elastodynamic transformation cloaking is not possible for a spherical cavity using a spherical cloak in the setting of generalized Cosserat elasticity (Proposition 7.5.9). We conjecture that this result in dimension three is independent of the shapes of the cavity and the cloak (Conjecture 7.5.11).

This chapter is structured as follows. In §8.2, we revisit nonlinear elasticity and discuss the invariance of its governing equations under both arbitrary time-dependent transformations of the ambient space (current configuration) and arbitrary time-independent transformations of the reference configuration. In particular, structural tensors are discussed in some detail. In §8.3, linearization of nonlinear elasticity is discussed in detail. Then spatial and referential covariance of the governing equations of linearized elasticity are investigated. The problems of cloaking for both nonlinear elastodynamics and elastostatics are formulated in §8.4.1 and §7.4.2, respectively. Transformation cloaking in classical linear elasticity is investigated in §8.4.2. In §7.4.4, transformation cloaking in the small-on-large theory is formulated. In §7.5, elastodynamic transformation cloaking in solids with microstructure is investigated. The impossibility of transformation cloaking in linearized gradient elasticity is discussed in §7.5.1. Elastodynamic transformation cloaking in generalized Cosserat solids is formulated and discussed in §7.5.2. Note that the results of this section have been previously reported in our published work [41].

7.2 Nonlinear Elastodynamics

Kinematics. In nonlinear elasticity, motion is a time-dependent mapping between a reference configuration (or natural configuration) and the ambient space (see Fig.7.1). Geometrically, we write this as $\varphi_t : \mathcal{B} \to \mathcal{S}$, where $(\mathcal{B}, \mathbf{G})$ and $(\mathcal{S}, \mathbf{g})$ are the material and the ambient space Riemannian manifolds, respectively [43]. Here, $\mathbf{G}$ is the material metric (that allows one to measure distances in a natural stress-free configuration) and g is the background metric of the ambient space. The Levi-Civita connections associated with the metrics $\mathbf{G}$ and $\mathbf{g}$ are denoted as $\nabla^{\mathbf{G}}$ and $\nabla^{\mathbf{g}}$, respectively. The corresponding Christoffel symbols of $\nabla^{\mathbf{G}}$ and $\nabla^{\mathbf{g}}$ in the local coordinate charts $\{X^A\}$ and $\{x^a\}$ are denoted by $\Gamma^A{}_{BC}$ and $\gamma^a{}_{bc}$, respectively. These can be directly expressed in terms of the metric components as

$$\gamma^a{}_{bc} = \frac{1}{2} g^{ak} \left(g_{kb,c} + g_{kc,b} - g_{bc,k} \right) , \qquad \Gamma^A{}_{BC} = \frac{1}{2} G^{AK} \left(G_{KB,C} + G_{KC,B} - G_{BC,K} \right) . \tag{7.7}$$

The deformation gradient $\mathbf{F}$ is the tangent map of φ_t, which is defined as $\mathbf{F}(X,t) = T\varphi_t(X) : T_X\mathcal{B} \to T_{\varphi_t(X)}\mathcal{S}$. The transpose of $\mathbf{F}$ is denoted by $\mathbf{F}^{\mathsf{T}}$, where

$$\mathbf{F}^{\mathsf{T}}(X,t) : T_{\varphi_t(X)}\mathcal{S} \to T_X\mathcal{B} \,, \qquad \langle\!\langle \mathbf{W}, \mathbf{F}^{\mathsf{T}}\mathbf{w} \rangle\!\rangle_{\mathbf{G}} = \langle\!\langle \mathbf{F}\mathbf{W}, \mathbf{w} \rangle\!\rangle_{\mathbf{g}}, \;\; \forall\, \mathbf{W} \in T_X\mathcal{B},\; \mathbf{w} \in T_{\varphi_t(X)}\mathcal{S} \,. \tag{7.8}$$

In components, $(F^{\mathsf{T}})^A{}_a = G^{AB} F^b{}_B g_{ab}$. The right Cauchy-Green deformation tensor[5] is defined as $\mathbf{C} = \mathbf{F}^{\mathsf{T}}\mathbf{F} : T_X\mathcal{B} \to T_X\mathcal{B}$, which in components reads $C^A{}_B = F^a{}_L F^b{}_B g_{ab} G^{AL}$.

The material velocity of the motion is the mapping $\mathbf{V} : \mathcal{B} \times \mathbb{R}^+ \to T\mathcal{S}$, where $\mathbf{V}(X,t) \in T_{\varphi_t(X)}\mathcal{S}$,

[5]Note that $\mathbf{C}^\flat$ agrees with the pull-back of the ambient space metric by φ_t, i.e., $\mathbf{C}^\flat = \varphi_t^*\mathbf{g}$.

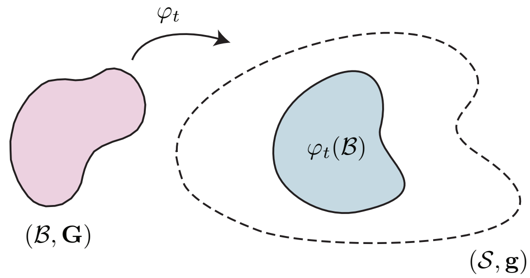


Figure 7.1: *Motion in nonlinear elasticity is a time-dependent mapping between two Riemannian manifolds.*

and in components, $V^a(X,t) = \frac{\partial \varphi^a}{\partial t}(X,t)$. The spatial velocity is defined as $\mathbf{v} : \varphi_t(\mathcal{B}) \times \mathbb{R}^+ \to T\mathcal{S}$ such that $\mathbf{v}_t(x) = \mathbf{V}_t \circ \varphi_t^{-1}(x) \in T_x\mathcal{S}$, where $x = \varphi_t(X)$. The convected velocity is defined as $\boldsymbol{\mathscr{V}}_t = \varphi_t^*(\mathbf{v}_t) = T\varphi_t^{-1} \circ \mathbf{v}_t \circ \varphi_t = \mathbf{F}^{-1} \cdot \mathbf{V}$. The material acceleration is a mapping $\mathbf{A} : \mathcal{B} \times \mathbb{R}^+ \to T\mathcal{S}$ defined as[6] $\mathbf{A}(X,t) := D_t^{\mathbf{g}}\mathbf{V}(X,t) = \nabla^{\mathbf{g}}_{\mathbf{V}(X,t)}\mathbf{V}(X,t) \in T_{\varphi_t(X)}\mathcal{S}$, where $D_t^{\mathbf{g}}$ denotes the covariant derivative along the curve $\varphi_t(X)$ in $\mathcal{S}$. In components, $A^a = \frac{\partial V^a}{\partial t} + \gamma^a{}_{bc}V^bV^c$. To motivate this definition note that the time derivative of the kinetic energy density is calculated as[7]

$$\frac{d}{dt}\frac{1}{2}\rho_0(X)\langle\!\langle \mathbf{V}(X,t), \mathbf{V}(X,t)\rangle\!\rangle_{\mathbf{g}} = \rho_0(X)\langle\!\langle \mathbf{V}(X,t), D_t^{\mathbf{g}}\mathbf{V}(X,t)\rangle\!\rangle_{\mathbf{g}} = \rho_0(X)\langle\!\langle \mathbf{V}(X,t), \mathbf{A}(X,t)\rangle\!\rangle_{\mathbf{g}}. \tag{7.9}$$

Therefore, $\mathbf{A}(X,t)$ is the covariant time derivative of the velocity vector field. The spatial acceleration is defined as $\mathbf{a} : \varphi_t(\mathcal{B}) \times \mathbb{R}^+ \to T\mathcal{S}$ such that $\mathbf{a}_t(x) = \mathbf{A}_t \circ \varphi_t^{-1}(x) \in T_x\mathcal{S}$. In components, it reads $a^a = \frac{\partial v^a}{\partial t} + \frac{\partial v^a}{\partial x^b}v^b + \gamma^a{}_{bc}v^bv^c$. The spatial acceleration can also be expressed as the material time derivative of $\mathbf{v}$, i.e., $\mathbf{a} = \dot{\mathbf{v}} = \frac{\partial \mathbf{v}}{\partial t} + \nabla^{\mathbf{g}}_{\mathbf{v}}\mathbf{v}$. The convected acceleration is defined as [205]

$$\boldsymbol{\mathscr{A}}_t = \varphi_t^*(\mathbf{a}_t) = \frac{\partial \boldsymbol{\mathscr{V}}_t}{\partial t} + \nabla^{\varphi_t^*\mathbf{g}}_{\boldsymbol{\mathscr{V}}_t}\boldsymbol{\mathscr{V}}_t = \frac{\partial \boldsymbol{\mathscr{V}}_t}{\partial t} + \nabla^{\mathbf{C}^\flat}_{\boldsymbol{\mathscr{V}}_t}\boldsymbol{\mathscr{V}}_t\,. \tag{7.10}$$

Balance laws. The balance of linear momentum in spatial and material forms reads

$$\operatorname{div}_{\mathbf{g}}\boldsymbol{\sigma} + \rho\mathbf{b} = \rho\mathbf{a}, \quad \operatorname{Div}\mathbf{P} + \rho_0\mathbf{B} = \rho_0\mathbf{A}, \tag{7.11}$$

[6] Not to be confused with a Lodge transformation.

[7] Note that if a connection ∇ is $\mathbf{G}$-compatible, then $\frac{d}{dt}\langle\!\langle \mathbf{X}, \mathbf{Y}(X,t)\rangle\!\rangle_{\mathbf{G}} = \langle\!\langle D_t\mathbf{X}, \mathbf{Y}\rangle\!\rangle_{\mathbf{G}} + \langle\!\langle \mathbf{X}, D_t\mathbf{Y}\rangle\!\rangle_{\mathbf{G}}$, where D_t is the covariant time derivative.

where $\boldsymbol{\sigma}$ and $\mathbf{P}$ are the Cauchy stress and the first Piola-Kirchhoff stress, respectively. ρ_0, $\mathbf{B}$, and $\mathbf{A}$ are the material mass density, material body force, and material acceleration, respectively, and ρ, $\mathbf{b}$, and $\mathbf{a}$ are their corresponding spatial counterparts. Note that $\operatorname{div}_{\mathbf{g}} \boldsymbol{\sigma}$ and $\operatorname{Div} \mathbf{P}$ have the following coordinate expressions

$$
\begin{aligned}
\operatorname{div}_{\mathbf{g}} \boldsymbol{\sigma} &= \sigma^{ab}{}_{|b} \frac{\partial}{\partial x^a} = \left(\frac{\partial \sigma^{ab}}{\partial x^b} + \sigma^{ac} \gamma^b{}_{cb} + \sigma^{cb} \gamma^a{}_{cb} \right) \frac{\partial}{\partial x^a} \,, \\
\operatorname{Div} \mathbf{P} &= P^{aA}{}_{|A} \frac{\partial}{\partial x^a} = \left(\frac{\partial P^{aA}}{\partial X^A} + P^{aB} \Gamma^A{}_{AB} + P^{cA} F^b{}_A \gamma^a{}_{bc} \right) \frac{\partial}{\partial x^a} \,.
\end{aligned}
\tag{7.12}
$$

In coordinates, $J\sigma^{ab} = F^a{}_A P^{bA}$, where J is the Jacobian of deformation that relates the deformed and undeformed Riemannian volume elements as $dv(x, \mathbf{g}) = J dV(X, \mathbf{G})$, and

$$
J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}} \det \mathbf{F}. \tag{7.13}
$$

One can pull back the balance of linear momentum to the reference configuration, i.e., $\varphi_t^* \left(\operatorname{div}_{\mathbf{g}} \boldsymbol{\sigma} \right) + \varphi_t^* \left(\rho \mathbf{b} \right) = \varphi_t^* \left(\rho \mathbf{a} \right)$. This can be written as

$$
\operatorname{div}_{\mathbf{C}^\flat} \boldsymbol{\Sigma} + \varrho \mathscr{B}_t = \varrho \mathscr{A}_t, \tag{7.14}
$$

where $\boldsymbol{\Sigma} = \varphi_t^* \boldsymbol{\sigma}$ is the convected stress, $\mathscr{B}_t = \varphi_t^* \mathbf{b}$ is the convected body force, and $\varrho = \rho \circ \varphi_t$.

Identifying a material point with its position in the material manifold $X \in \mathcal{B}$, we have $x = \varphi_t(X)$. When the ambient space is Euclidean one defines the material displacement field as $\mathbf{U} = \varphi_t(X) - X$. The spatial displacement field is denoted by $\mathbf{u} = \mathbf{U} \circ \varphi_t^{-1}$.

Balance of angular momentum in local form reads $\boldsymbol{\sigma}^{\mathsf{T}} = \boldsymbol{\sigma}$ or $\mathbf{F}\mathbf{P}^\star = \mathbf{P}\mathbf{F}^\star$, where $\mathbf{P}^\star$ and $\mathbf{F}^\star$ are duals of $\mathbf{P}$ and $\mathbf{F}$, respectively, and are defined as

$$
\begin{aligned}
\mathbf{F} &= F^a{}_A \frac{\partial}{\partial x^a} \otimes dX^A, \qquad & \mathbf{F}^\star &= F^a{}_A dX^A \otimes \frac{\partial}{\partial x^a} \,, \\
\mathbf{P} &= P^{aA} \frac{\partial}{\partial x^a} \otimes \frac{\partial}{\partial X^A}, \qquad & \mathbf{P}^\star &= P^{aA} \frac{\partial}{\partial X^A} \otimes \frac{\partial}{\partial x^a} \,.
\end{aligned}
\tag{7.15}
$$

Note that $\mathbf{F}^{\star}: T^{*}_{\varphi_t(X)}\mathcal{S} \to T^{*}_{X}\mathcal{B}$, where $T^{*}_{\varphi_t(X)}\mathcal{S}$ and $T^{*}_{X}\mathcal{B}$ denote the cotangent spaces of $T_{\varphi_t(X)}\mathcal{S}$ and $T_X\mathcal{B}$, respectively.

Conservation of mass implies that $\rho dv = \rho_0 dV$ or $\rho J = \rho_0$, where ρ_o and ρ denote the material and spatial mass densities, respectively. In terms of Lie derivatives, conservation of mass can be written as $\mathbf{L}_{\mathbf{v}}\rho = 0$ [43].

Constitutive equations. In nonlinear elasticity, the energy function (per unit undeformed volume) of an inhomogeneous anisotropic hyperelastic material at a material point X is written in the following general form

$$W = \hat{W}(X, \mathbf{C}^{\flat}, \mathbf{G}, \boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_n)\,, \tag{7.16}$$

where $\boldsymbol{\zeta}_i, i = 1, \ldots, n$ are a collection of the so called *structural tensors* characterizing the material symmetry group at the point X. The inclusion of the structural tensors, along with $\mathbf{C}^{\flat}$ in the energy function as shown in (7.16) constructs an isotropic function, i.e., it is invariant under the orthogonal group [51]. Therefore, (7.16) can be treated as the energy function of an isotropic material, and hence, the second Piola-Kirchhoff stress tensor is given as

$$\mathbf{S} = 2\frac{\partial \hat{W}}{\partial \mathbf{C}^{\flat}}\,. \tag{7.17}$$

Alternatively, by the Doyle-Ericksen formula [56], the Cauchy, the first Piola-Kirchhoff, and the convected stress tensors are expressed as

$$\boldsymbol{\sigma} = \frac{2}{J}\frac{\partial \hat{W}}{\partial \mathbf{g}}\,, \quad \mathbf{P} = \mathbf{g}^{\sharp}\frac{\partial \hat{W}}{\partial \mathbf{F}}, \quad \boldsymbol{\Sigma} = \frac{2}{J}\frac{\partial \hat{W}}{\partial \mathbf{C}^{\flat}}\,, \tag{7.18}$$

where, with a slight abuse of notation, one may write

$$\hat{W}\big(x, \mathbf{G}\circ\varphi^{-1}, \mathbf{g}, \mathbf{F}, \boldsymbol{\zeta}_1\circ\varphi^{-1}, \ldots, \boldsymbol{\zeta}_n\circ\varphi^{-1}\big) = \hat{W}\big(X, \mathbf{G}, \mathbf{g}\circ\varphi, \mathbf{F}, \boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_n\big) = \hat{W}\big(X, \mathbf{G}, \mathbf{C}^{\flat}, \boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_n\big)\,. \tag{7.19}$$

According to Hilbert's theorem, for any finite number of tensors, there exists a finite number of isotropic invariants forming a basis called *integrity basis* for the space of isotropic invariants of the collection of tensors.[8] Thus, if $I_j, j = 1, \ldots, m$, form an integrity basis for the set of tensors in (7.16), one has $W = W(X, I_1, ..., I_m)$. Hence, using (7.17), one obtains

$$\mathbf{S} = \sum_{j=1}^{m} 2W_{I_j} \frac{\partial I_j}{\partial \mathbf{C}^\flat}\,, \quad W_{I_j} := \frac{\partial W}{\partial I_j}\,, \quad j = 1, \ldots, m\,. \tag{7.20}$$

If the material is isotropic, i.e., $W = W(X, I_1, I_2, I_3)$, where $I_1 = \operatorname{tr}\mathbf{C}$, $I_2 = \det\mathbf{C}\operatorname{tr}\mathbf{C}^{-1}$, and $I_3 = \det\mathbf{C}$ are the principal invariants of the right Cauchy-Green deformation tensor, it follows from (7.20) that

$$\mathbf{S} = 2\left\{W_{I_1}\mathbf{G}^\sharp + W_{I_2}(I_2\mathbf{C}^{-1} - I_3\mathbf{C}^{-2}) + W_{I_3}I_3\mathbf{C}^{-1}\right\}\,. \tag{7.21}$$

7.2.1 Spatial Covariance of the Governing Equations of Nonlinear Elasticity

It turns out that in continuum mechanics (and even discrete systems) one can obtain all the balance laws using the energy balance and postulating its invariance under some groups of transformations. This idea was introduced by Green and Rivlin [206] in the case of Euclidean ambient spaces and was later extended to manifolds by Hughes and Marsden [180]. See also Marsden and Hughes [43], Simo and Marsden [55], Yavari *et al.* [46], Yavari and Ozakin [82], Yavari [207], and Yavari and Marsden [208, 209] for applications of covariance ideas in different continuous and discrete systems.

Consider an arbitrary time-dependent spatial diffeomorphism $\xi_t : \mathcal{S} \to \mathcal{S}$ (see Fig.7.2). Denoting the fields in the transformed configuration by primes, we know that [43, 46]

$$R' = R,\;\; H' = H,\;\; \rho_0' = \rho_0,\;\; \mathbf{T}' = \xi_{t*}\mathbf{T},\;\; \mathbf{V}' = \xi_{t*}\mathbf{V} + \mathbf{w}\circ\varphi_t, \tag{7.22}$$

where $\mathbf{w} = \frac{\partial}{\partial t}\xi_t \circ \varphi_t$ is the velocity of the change of frame and $\mathbf{T}$ is the traction vector. Note that

[8] See [49] for a detailed discussion on integrity basis for a finite set of tensors.

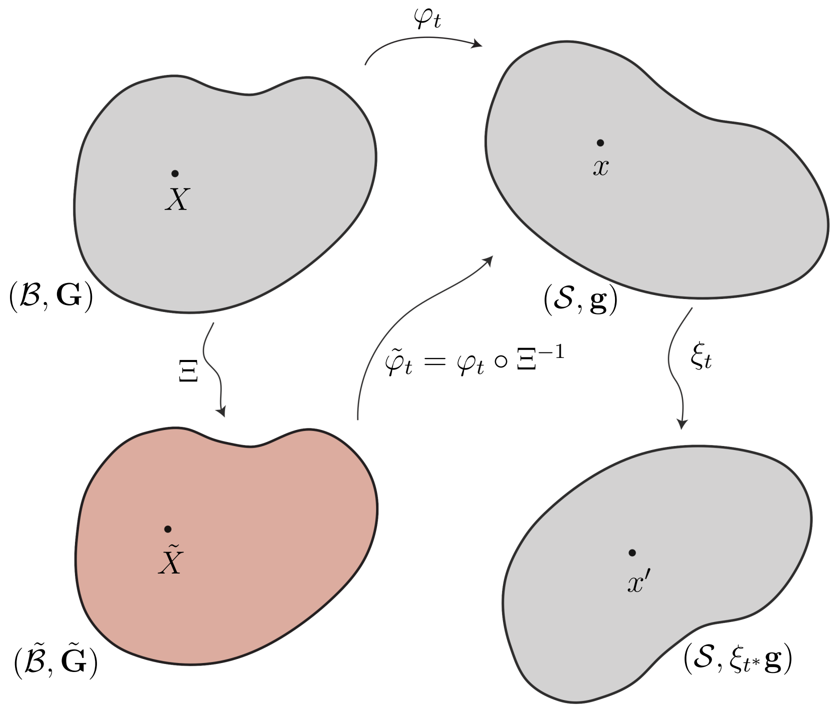


Figure 7.2: *Motion of a nonlinear elastic body and spatial and material changes of frame. ξ_t is a time-dependent spatial change of frame and Ξ is a time-independent material (referential) change of frame.*

$\mathbf{A} = \nabla^{\mathbf{g}}_{\mathbf{V}}\mathbf{V}$, and hence

$$
\begin{aligned}
\mathbf{A}' = \nabla^{\mathbf{g}'}_{\mathbf{V}'}\mathbf{V}' &= \nabla^{\xi_{t*}\mathbf{g}}_{\xi_{t*}\mathbf{V}+\mathbf{w}\circ\varphi_t}\left(\xi_{t*}\mathbf{V} + \mathbf{w}\circ\varphi_t\right) \\
&= \xi_{t*}\left(\nabla^{\mathbf{g}}_{\mathbf{V}}\mathbf{V} + \nabla^{\mathbf{g}}_{\mathbf{W}}\mathbf{W} + \nabla^{\mathbf{g}}_{\mathbf{V}}\mathbf{W} + \nabla^{\mathbf{g}}_{\mathbf{W}}\mathbf{V}\right) \\
&= \xi_{t*}\left(\mathbf{A} + \nabla^{\mathbf{g}}_{\mathbf{W}}\mathbf{W} + 2\nabla^{\mathbf{g}}_{\mathbf{V}}\mathbf{W} + [\mathbf{W}, \mathbf{V}]\right).
\end{aligned}
\tag{7.23}
$$

It is assumed that body forces are transformed such that [43] $\mathbf{B}' - \mathbf{A}' = \xi_{t*}(\mathbf{B} - \mathbf{A})$. Similar transformations hold for the spatial quantities. It can be shown that [43, 46]

$$
\begin{aligned}
\mathrm{div}'\,\boldsymbol{\sigma}' + \rho'\mathbf{b}' - \rho'\mathbf{a}' &= \xi_{t*}\left(\mathrm{div}\,\boldsymbol{\sigma} + \rho\mathbf{b} - \rho\mathbf{a}\right), \\
\mathrm{Div}'\,\mathbf{P}' + \rho_0'\mathbf{B}' - \rho_0'\mathbf{A}' &= \xi_{t*}\left(\mathrm{Div}\,\mathbf{P} + \rho_0\mathbf{B} - \rho_0\mathbf{A}\right),
\end{aligned}
\tag{7.24}
$$

i.e., the balance of linear momentum is spatially covariant provided that the Doyle-Ericksen formula $(7.18)_1$ is satisfied. This in turn restricts the body to be isotropic. A way out to have a covariant elasticity theory for anisotropic bodies is to include structural tensors in the energy function. Note that the balance of angular momentum (symmetry of the Cauchy stress) is covariant as well, i.e., $\boldsymbol{\sigma}'^{\mathsf{T}} = \boldsymbol{\sigma}'$.

The main idea in covariant elasticity is that in the ambient space the physical laws (here, the balance of energy or the first law of thermodynamics) should be observer-independent.[9] Yavari *et al.* [46] investigated the possibility of the covariance of the energy balance under diffeomorphisms of the reference configuration. Their motivation was to see if there was any connection between material covariance and balance of the so-called configurational forces. It was observed that the energy balance is not invariant under material diffeomorphisms, in general. They obtained a transformation equation for the balance of energy. This is discussed next.

7.2.2 Material Covariance of the Governing Equations of Nonlinear Elasticity

Note that a spatial diffeomorphism is nothing but a change of observer. In other words, given an elastic body in (dynamic) equilibrium, a spatial diffeomorphism is simply representing the same configuration in another frame. For cloaking applications, one needs to know what the elastic properties of the cloak should be in order to make a given cavity invisible to elastic waves. This means that given a reference configuration, one should be looking at material (referential) diffeomorphisms. This is the motivation for the following discussion.

In this section, we discuss the transformation of the governing equations of nonlinear elasticity for an anisotropic and inhomogenous body under a time-independent material diffeomorphism. More specifically, consider a diffeomorphism $\Xi : \mathcal{B} \to \tilde{\mathcal{B}}$. We use the coordinate charts $\{X^A\}$, $\{\tilde{X}^{\tilde{A}}\}$, and $\{x^a\}$ for the reference configuration, the transformed reference configuration, and the ambient space, respectively, see Fig 7.2. The deformation map $\varphi_t : \mathcal{B} \to \mathcal{S}$ is transformed under the material diffeomorphism to $\tilde{\varphi}_t : \mathcal{B} \to \mathcal{S}$, where $\tilde{\varphi}_t = \varphi_t \circ \Xi^{-1}$. Material velocity with respect to the new reference configuration reads

$$\tilde{\mathbf{V}}(\tilde{X}, t) = \frac{\partial}{\partial t}\tilde{\varphi}(\tilde{X}, t) = \frac{\partial}{\partial t}\varphi(\Xi^{-1}(\tilde{X}), t) = \mathbf{V}(\Xi^{-1}(\tilde{X}), t). \tag{7.25}$$

[9] This is also known as the principle of material objectivity (see, e.g., [130]).

Thus, $\tilde{\mathbf{V}} = \mathbf{V} \circ \Xi^{-1}$. Material acceleration with respect to the new reference configuration reads

$$\tilde{\mathbf{A}}(\tilde{X}, t) = \nabla^{\mathbf{g}}_{\frac{\partial}{\partial t}} \tilde{\mathbf{V}}(\tilde{X}, t) = \nabla^{\mathbf{g}}_{\frac{\partial}{\partial t}} \mathbf{V}(\Xi^{-1}(\tilde{X}), t) = \nabla^{\mathbf{g}}_{\frac{\partial}{\partial t}} \mathbf{V}_t \circ \Xi^{-1}(\tilde{X}) = \mathbf{A}_t \circ \Xi^{-1}(\tilde{X}). \tag{7.26}$$

The deformation gradient $\mathbf{F} = T\varphi_t : T_X\mathcal{B} \to T_{\varphi_t(X)}\mathcal{S}$, under Ξ is transformed to $\tilde{\mathbf{F}} = T\tilde{\varphi}_t : T_{\tilde{X}}\mathcal{B} \to T_{\tilde{\varphi}_t(\tilde{X})}\mathcal{S} = T_{\varphi_t(X)}\mathcal{S}$, where

$$\tilde{\mathbf{F}} = T\tilde{\varphi}_t = T(\varphi_t \circ \Xi^{-1}) = T\varphi_t \circ T\Xi^{-1} = \mathbf{F} \circ \overset{\Xi}{\mathbf{F}}^{-1}, \tag{7.27}$$

and $\overset{\Xi}{\mathbf{F}} = T\Xi$. Moreover, note that $\tilde{\mathbf{G}} = \Xi_*\mathbf{G}$. In coordinates, $\tilde{G}_{\tilde{A}\tilde{B}} = (\overset{\Xi}{F}{}^{-1})^A{}_{\tilde{A}}(\overset{\Xi}{F}{}^{-1})^B{}_{\tilde{B}}G_{AB}$. The right Cauchy-Green deformation tensor is written as $\mathbf{C}^\flat = {\varphi_t}^*\mathbf{g}$. Hence, one notes that

$$\tilde{\mathbf{C}}^\flat = {\tilde{\varphi}_t}{}^*\mathbf{g} = \left(\varphi_t \circ \Xi^{-1}\right)^* \mathbf{g} = \Xi_* \circ {\varphi_t}^*\mathbf{g} = \Xi_*\left({\varphi_t}^*\mathbf{g}\right) = \Xi_*\mathbf{C}^\flat. \tag{7.28}$$

In coordinates, $\tilde{C}_{\tilde{A}\tilde{B}} = (\overset{\Xi}{F}{}^{-1})^A{}_{\tilde{A}}(\overset{\Xi}{F}{}^{-1})^B{}_{\tilde{B}}C_{AB}$.

Material symmetry. The material symmetry group $\mathcal{G}_X$ associated with an elastic body made of a simple material[10] with the response function $\mathscr{R}$[11] at a point X with respect to the reference configuration $(\mathcal{B}, \mathbf{G})$ is defined as

$$\mathscr{R}(\mathbf{FK}) = \mathscr{R}(\mathbf{F}), \qquad \forall\ \mathbf{K} \in \mathcal{G}_X, \tag{7.29}$$

for all deformation gradients $\mathbf{F}$, where $\mathbf{K} : T_X\mathcal{B} \to T_X\mathcal{B}$ is an invertible linear transformation. For a hyperelastic solid, objectivity requires that the energy function depend on the deformation through the right Cauchy-Green deformation tensor $\mathbf{C}^\flat$, i.e., $W = W(X, \mathbf{C}^\flat, \mathbf{G})$ at a material point X. Therefore, the material symmetry group $\mathcal{G}_X$ for a hyperelastic solid is defined to be the subgroup of $\mathbf{G}$-orthogonal

[10] The response of a *simple* material at any material point depends only on the first deformation gradient (and its evolution) at that point [130].

[11] Here we assume that $\mathscr{R}$ is the energy function. Response function may be any measure of stress as well.

transformations Orth$(\mathbf{G})$ such that[12] [131]

$$W(X, \mathbf{Q}^{-\star}\mathbf{C}^{\flat}\mathbf{Q}^{-1}, \mathbf{G}) = W(X, \mathbf{C}^{\flat}, \mathbf{G})\,, \qquad \forall\ \mathbf{Q} \in \mathcal{G}_X \leqslant \mathrm{Orth}\,(\mathbf{G})\ . \tag{7.30}$$

The symmetry group of the material relative to a transformed reference configuration $(\tilde{\mathcal{B}}, \tilde{\mathbf{G}})$ is denoted by $\widetilde{\mathcal{G}}$. According to Noll's rule [130, 210, 211, 212], the relation between the two groups is

$$\widetilde{\mathcal{G}} = \Xi_* \mathcal{G} = \bar{\bar{\bar{\mathbf{F}}}}\, \mathcal{G}\, \bar{\bar{\bar{\mathbf{F}}}}^{-1}\,. \tag{7.31}$$

In other words, in the sense of group theory, at each material point X, $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ are conjugate subgroups of the general linear group, and hence, isomorphic ($\widetilde{\mathcal{G}} \cong \mathcal{G}$). Note that if $\bar{\bar{\bar{\mathbf{F}}}} \in \mathcal{G}$, then $\mathcal{G} = \widetilde{\mathcal{G}}$. Also, the symmetry group is not affected by a change of reference configuration ($\widetilde{\mathcal{G}} = \mathcal{G}$) if $\bar{\bar{\bar{\mathbf{F}}}} = \alpha \mathbf{I}$ (pure dilatation) for some positive scalar α [54]. More generally, $\widetilde{\mathcal{G}} = \mathcal{G}$ if and only if $\bar{\bar{\bar{\mathbf{F}}}}$ belongs to the normalizer group[13] of $\mathcal{G}$ within the general linear group. It is straightforward to see that (7.31) is satisfied if and only if it holds for all the generators of the group $\mathcal{G}$. Therefore, if $\mathcal{G}$ is finitely generated, the elements of the generating sets of $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ denoted by $\{\mathbf{Q}_1, \ldots, \mathbf{Q}_m\}$ and $\{\tilde{\mathbf{Q}}_1, \ldots, \tilde{\mathbf{Q}}_m\}$, respectively, are related as $\tilde{\mathbf{Q}}_j = \bar{\bar{\bar{\mathbf{F}}}}\mathbf{Q}_j \bar{\bar{\bar{\mathbf{F}}}}^{-1}$, $j = 1, \ldots, m$. The symmetry group can be characterized using a finite collection of structural tensors[14] $\boldsymbol{\zeta}_i$ of order μ_i, $i = 1, \ldots, n$, as follows [51, 213, 52, 214, 53, 132]

$$\mathbf{Q} \in \mathcal{G} \leqslant \mathrm{Orth}\,(\mathbf{G}) \iff \langle \mathbf{Q} \rangle_{\mu_1} \boldsymbol{\zeta}_1 = \boldsymbol{\zeta}_1\,, \ldots\,, \langle \mathbf{Q} \rangle_{\mu_n} \boldsymbol{\zeta}_n = \boldsymbol{\zeta}_n\,, \tag{7.32}$$

where the μ-th power Kronecker product $\langle \mathbf{Q} \rangle_\mu$ of a $\mathbf{G}$-orthogonal transformation $\mathbf{Q}$ for any μ-th order tensor $\boldsymbol{\zeta}$ is defined as[15]

$$(\langle \mathbf{Q} \rangle_\mu \zeta)^{\bar{A}_1 \ldots \bar{A}_\mu} = Q^{\bar{A}_1}{}_{A_1} \ldots Q^{\bar{A}_\mu}{}_{A_\mu} \zeta^{A_1 \ldots A_\mu}. \tag{7.33}$$

[12]Note that Orth$(\mathbf{G}) = \left\{ \mathbf{Q} : T_X\mathcal{B} \to T_X\mathcal{B} \ \ | \ \ \mathbf{Q}^\top = \mathbf{Q}^{-1} \right\}$. We use the notation $\mathscr{G} \leqslant \mathscr{H}$ when $\mathscr{G}$ is a subgroup of $\mathscr{H}$.

[13]The normalizer group $\mathcal{N}_{\mathscr{G}}(\mathcal{Q})$ of a subgroup $\mathcal{Q}$ of $\mathscr{G}$ ($\mathcal{Q} \leqslant \mathscr{G}$) is defined as $\mathcal{N}_{\mathscr{G}}(\mathcal{Q}) = \{g_i \in \mathscr{G} : g_i \mathcal{Q} g_i^{-1} = \mathcal{Q}\}$.

[14]Note that such a collection forms a basis for the space of tensors that are invariant under the action of $\mathcal{G}$.

[15]Note that $\langle \mathbf{Q} \rangle_m (\mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_m) = \mathbf{Q}\mathbf{v}_1 \otimes \cdots \otimes \mathbf{Q}\mathbf{v}_m$, where $\mathbf{v}_i \in T_X\mathcal{B}$, $i = 1, \ldots, m$, are arbitrary vectors.

Note that (7.32) suggests that the material symmetry group $\mathcal{G}$ is the invariance group of the set of the structural tensors $\boldsymbol{\zeta}_i$, $i = 1, \ldots, n$. Using (7.31) and (7.32), one obtains the following relation for the transformed structural tensors under the material diffeomorphism, which characterize the transformed symmetry group $\widetilde{\mathcal{G}}$

$$\tilde{\mathbf{Q}} \in \widetilde{\mathcal{G}} \leqslant \mathrm{Orth}(\tilde{\mathbf{G}}) \iff \langle \tilde{\mathbf{Q}} \rangle_{\mu_1} \tilde{\boldsymbol{\zeta}}_1 = \tilde{\boldsymbol{\zeta}}_1 \,, \ldots \,, \langle \tilde{\mathbf{Q}} \rangle_{\mu_n} \tilde{\boldsymbol{\zeta}}_n = \tilde{\boldsymbol{\zeta}}_n \,, \tag{7.34}$$

where $\tilde{\mathbf{Q}} = \Xi_* \mathbf{Q} = \overset{\Xi}{\mathbf{F}} \, \mathbf{Q} \, \overset{\Xi}{\mathbf{F}}^{-1}$ and $\tilde{\boldsymbol{\zeta}}_i = \Xi_* \boldsymbol{\zeta}_i$, $i = 1, \ldots, n$. Therefore, the type of the symmetry group of the material is preserved under a material change of frame Ξ.

Balance of energy. Yavari *et al.* [46] showed that the balance of energy is not invariant under an arbitrary time-dependent material diffeomorphism. However, it can be shown that the balance of energy is always invariant under time-independent material diffeomorphisms. The strain energy function satisfies (7.30) if and only if it is represented as an isotropic function (invariant under the special orthogonal group) of the structural tensors and $\mathbf{C}^\flat$ at a material point X (see (7.35)) [215, 216, 131]. Thus, we write the general form of the energy function (per unit undeformed volume) of an inhomogeneous anisotropic hyperelastic material with a set of structural tensors $\boldsymbol{\zeta}_i$, $i = 1, \ldots, n$ (cf. (7.32)) characterizing the material symmetry group (at a referential point X) as

$$W = \hat{W}(X, \mathbf{C}^\flat, \mathbf{G}, \boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_n) \,. \tag{7.35}$$

Similarly, using (7.28) and (7.34), one obtains

$$\tilde{W} = \widetilde{\hat{W}}(\tilde{X}, \tilde{\mathbf{C}}^\flat, \tilde{\mathbf{G}}, \tilde{\boldsymbol{\zeta}}_1, \ldots, \tilde{\boldsymbol{\zeta}}_n) = \hat{W}(\Xi(X), \Xi_* \mathbf{C}^\flat, \Xi_* \mathbf{G}, \Xi_* \boldsymbol{\zeta}_1, \ldots, \Xi_* \boldsymbol{\zeta}_n) = \Xi_* W \,. \tag{7.36}$$

Using the theory of invariants [49, 50], the energy function can be represented in terms of a finite set of isotropic invariants. It can be shown that these invariants do not change under material diffeomorphisms, which in turn implies the material covariance of the energy function, i.e., (see also [53])

$$\tilde{W} = W \circ \Xi^{-1} \,. \tag{7.37}$$

Balance of linear momentum. The balance of linear momentum in terms of the first Piola-Kirchhoff stress reads $\operatorname{Div}\mathbf{P} + \rho_0\mathbf{B} = \rho_0\mathbf{A}$. We now examine the transformed first and second Piola-Kirchhoff stress tensors, denoted, respectively, by $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{S}}$, and the material acceleration $\tilde{\mathbf{A}}$ pertaining to the transformed configuration $(\tilde{\mathcal{B}}, \tilde{\mathbf{G}})$ with the deformation $\tilde{\varphi}_t = \varphi_t \circ \Xi^{-1}$. The second Piola-Kirchhoff stress tensor is written as $\mathbf{S} = 2\frac{\partial \hat{W}}{\partial \mathbf{C}^\flat}$. Therefore, using (7.36), one obtains

$$\tilde{\mathbf{S}} = 2\frac{\partial \widetilde{\hat{W}}}{\partial \tilde{\mathbf{C}}^\flat} = 2\frac{\partial(\Xi_*\hat{W})}{\partial(\Xi_*\mathbf{C}^\flat)} \,. \tag{7.38}$$

Moreover, employing (7.37) and (7.38), one can write

$$\begin{aligned}
(\Xi_* S)^{\tilde{A}\tilde{B}} &= \bar{\bar{\bar{F}}}^{\tilde{A}}{}_A \bar{\bar{\bar{F}}}^{\tilde{B}}{}_B \, S^{AB} \circ \Xi^{-1} = 2\bar{\bar{\bar{F}}}^{\tilde{A}}{}_A \bar{\bar{\bar{F}}}^{\tilde{B}}{}_B \, \frac{\partial \hat{W}}{\partial C_{AB}} \circ \Xi^{-1} \\
&= 2\bar{\bar{\bar{F}}}^{\tilde{A}}{}_A \bar{\bar{\bar{F}}}^{\tilde{B}}{}_B \frac{\partial \widetilde{\hat{W}}}{\partial \tilde{C}_{\tilde{D}\tilde{H}}} \frac{\partial \tilde{C}_{\tilde{D}\tilde{H}} \circ \Xi}{\partial C_{AB}} \\
&= 2\bar{\bar{\bar{F}}}^{\tilde{A}}{}_A \bar{\bar{\bar{F}}}^{\tilde{B}}{}_B (\bar{\bar{\bar{F}}}^{-1})^A{}_{\tilde{D}} (\bar{\bar{\bar{F}}}^{-1})^B{}_{\tilde{H}} \, \frac{\partial \widetilde{\hat{W}}}{\partial \tilde{C}_{\tilde{D}\tilde{H}}} = 2\frac{\partial \widetilde{\hat{W}}}{\partial \tilde{C}_{\tilde{A}\tilde{B}}} = \tilde{S}^{\tilde{A}\tilde{B}} \,.
\end{aligned} \tag{7.39}$$

Hence, it immediately follows that $\tilde{\mathbf{S}} = \Xi_*\mathbf{S}$. Under the diffeomorphism $\Xi : \mathcal{B} \to \tilde{\mathcal{B}}$, the two-point tensor $\mathbf{P}$ is transformed to $\Xi_*\mathbf{P}$, where in components

$$\tilde{P}^{a\tilde{A}} = \frac{\partial \tilde{X}^{\tilde{A}}}{\partial X^A} P^{aA} = \bar{\bar{\bar{F}}}^{\tilde{A}}{}_A P^{aA} \circ \Xi^{-1}. \tag{7.40}$$

We note that, using (7.27), one has $\tilde{\mathbf{P}} = \tilde{\mathbf{F}}\tilde{\mathbf{S}} = (\mathbf{F} \circ \bar{\bar{\bar{\mathbf{F}}}}^{-1})\tilde{\mathbf{S}}$. Therefore, in components (cf. (7.40) and $\tilde{\mathbf{S}} = \Xi_*\mathbf{S}$)

$$\begin{aligned}
\tilde{P}^{a\tilde{A}} =& F^a{}_A (\bar{\bar{\bar{F}}}^{-1})^A{}_{\tilde{B}} \, \tilde{S}^{\tilde{B}\tilde{A}} = F^a{}_A (\bar{\bar{\bar{F}}}^{-1})^A{}_{\tilde{B}} \bar{\bar{\bar{F}}}^{\tilde{B}}{}_C \bar{\bar{\bar{F}}}^{\tilde{A}}{}_B \, S^{CB} \circ \Xi^{-1} \\
=& F^a{}_A \bar{\bar{\bar{F}}}^{\tilde{A}}{}_B \, S^{AB} \circ \Xi^{-1} = \bar{\bar{\bar{F}}}^{\tilde{A}}{}_B P^{aB} \circ \Xi^{-1} = (\Xi_* P)^{a\tilde{A}} \,,
\end{aligned} \tag{7.41}$$

that is, $\tilde{\mathbf{P}} = \Xi_*\mathbf{P}$. The divergence term has components $(\mathrm{Div}\,\mathbf{P})^a = \partial P^{aA}/\partial X^A + \Gamma^A{}_{AB}P^{aB} + (\gamma^a{}_{bc}\circ\varphi_t)\,F^b{}_A P^{cA}$. Thus, $(\widetilde{\mathrm{Div}}\tilde{\mathbf{P}})^a = \partial\tilde{P}^{a\tilde{A}}/\partial\tilde{X}^{\tilde{A}} + \tilde{\Gamma}^{\tilde{A}}{}_{\tilde{A}\tilde{B}}\tilde{P}^{a\tilde{B}} + (\gamma^a{}_{bc}\circ\tilde{\varphi}_t)\,\tilde{F}^b{}_{\tilde{A}}\tilde{P}^{c\tilde{A}}$. Note that $\tilde{\mathbf{F}} = T\tilde{\varphi}_t = T\left(\varphi_t\circ\Xi^{-1}\right) = T\varphi_t\circ(T\Xi)^{-1} = \mathbf{F}\circ\overset{\Xi}{\mathbf{F}}{}^{-1}$. Thus, $\tilde{\mathbf{F}}\circ\tilde{\mathbf{P}} = \mathbf{F}\circ\mathbf{P}\circ\Xi^{-1}$. It can be shown that $\partial\tilde{P}^{a\tilde{A}}/\partial\tilde{X}^{\tilde{A}} = \partial P^{aA}/\partial X^A\circ\Xi^{-1}$. Using the transformation of connection coefficients, it is straightforward to show that $\tilde{\Gamma}^{\tilde{A}}{}_{\tilde{A}\tilde{B}} = (\overset{\Xi}{F}{}^{-1})^B{}_{\tilde{B}}\Gamma^A{}_{AB}\circ\Xi^{-1}$. Therefore, $\widetilde{\mathrm{Div}}\tilde{\mathbf{P}} = \mathrm{Div}\,\mathbf{P}\circ\Xi^{-1}$. From conservation of mass $\rho_0 dV = \tilde{\rho}_0 d\tilde{V} = \tilde{\rho}_0 J_\Xi dV$, where

$$J_\Xi = \sqrt{\frac{\det\tilde{\mathbf{G}}}{\det\mathbf{G}}}\,\det\overset{\Xi}{\mathbf{F}}. \tag{7.42}$$

Note that $\tilde{\mathbf{G}} = \Xi_*\mathbf{G}$ and $\det\tilde{\mathbf{G}} = \det\mathbf{G}\,(\det\overset{\Xi}{\mathbf{F}})^{-2}$. Hence, $J_\Xi = 1$, and therefore, $\tilde{\rho}_0 = \rho_0\circ\Xi^{-1}$. For the spatial mass density, $\tilde{\rho} = \rho$.

Body force is a vector field in the ambient space, i.e., $\mathbf{B}_X\in T_{\varphi_t(X)}\mathcal{S}$, and hence, $\tilde{\mathbf{B}} = \Xi_*\mathbf{B} = \mathbf{B}\circ\Xi^{-1}$. For a time-independent material diffeomprphism, $\tilde{\mathbf{V}} = \mathbf{V}\circ\Xi^{-1}$, and hence, acceleration transforms as $\tilde{\mathbf{A}} = \mathbf{A}\circ\Xi^{-1}$. Therefore, the balance of linear momentum is invariant under the diffeomorphism $\Xi:\mathcal{B}\to\tilde{\mathcal{B}}$, i.e.,

$$\widetilde{\mathrm{Div}}\tilde{\mathbf{P}} + \tilde{\rho}_0\tilde{\mathbf{B}} - \tilde{\rho}_0\tilde{\mathbf{A}} = \Xi_*\left(\mathrm{Div}\,\mathbf{P} + \rho_0\mathbf{B} - \rho_0\mathbf{A}\right) = \left(\mathrm{Div}\,\mathbf{P} + \rho_0\mathbf{B} - \rho_0\mathbf{A}\right)\circ\Xi^{-1}. \tag{7.43}$$

This is identical to what Mazzucato and Rachele [132] proved.

Balance of angular momentum. The balance of angular momentum in terms of the first Piola-Kirchhoff stress reads $P^{aA}F^b{}_A - P^{bA}F^a{}_A = 0$. It is straightforward to show that $\tilde{P}^{a\tilde{A}}\tilde{F}^b{}_{\tilde{A}} = P^{aA}F^b{}_A\circ\Xi^{-1}$. Therefore, $\tilde{P}^{a\tilde{A}}\tilde{F}^b{}_{\tilde{A}} - \tilde{P}^{b\tilde{A}}\tilde{F}^a{}_{\tilde{A}} = \left(P^{aA}F^b{}_A - P^{bA}F^a{}_A\right)\circ\Xi^{-1}$, i.e., the balance of angular momentum is invariant under material diffeomorphisms.

Conservation of mass. Conservation of mass in local form reads $\rho_0 - J\rho \circ \varphi = 0$, where the Jacobian is written as $J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}} \det \mathbf{F}$. Thus

$$\tilde{J} = \sqrt{\frac{\det \tilde{\mathbf{g}}}{\det \tilde{\mathbf{G}}}} \det \tilde{\mathbf{F}} = J \circ \Xi^{-1}. \tag{7.44}$$

Therefore, $\tilde{\rho}_0 - \tilde{J}\tilde{\rho} \circ \tilde{\varphi} = \Xi_*(\rho_0 - J\rho \circ \varphi) = (\rho_0 - J\rho \circ \varphi) \circ \Xi^{-1}$, i.e., conservation of mass is materially covariant.

We next observe that under the diffeomorphism $\Xi : \mathcal{B} \to \tilde{\mathcal{B}}$, the Cauchy stress tensor remains unchanged and is transformed as $\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} \circ \Xi^{-1}$. To see this, note that (cf. (7.40) and (7.44))

$$\tilde{\boldsymbol{\sigma}} = \tilde{J}^{-1}\tilde{\mathbf{F}}\tilde{\mathbf{P}}^{\star} = (J^{-1}\mathbf{F}\overset{\Xi}{\mathbf{F}}^{-1}\overset{\Xi}{\mathbf{F}}\mathbf{P}^{\star}) \circ \Xi^{-1} = \boldsymbol{\sigma} \circ \Xi^{-1}\,. \tag{7.45}$$

7.3 Linearized Elastodynamics

Classical linear elasticity can be derived from nonlinear elasticity if one linearizes the governing equations of nonlinear elasticity with respect to a stress-free equilibrium configuration. More generally, nonlinear elasticity can be linearized with respect to any stressed and finitely-deformed (in either static or dynamic equilibrium) configuration. This is the so-called small-on-large theory of Green *et al.* [195]. In the language of geometric mechanics, Marsden and Hughes [43] presented a geometric linearization of nonlinear elasticity. In particular, in their formulation elastic constants are properly defined in terms of two-point tensors. See also Yavari and Ozakin [82] for a discussion on covariance in linearized elasticity.

Variation of a map $\varphi_t : \mathcal{B} \to \mathcal{S}$ is a map $\Phi_t : \mathcal{B} \times I$, where $I = (-a, a)$ is some interval, such that $\Phi_t(X, 0) = \mathring{\varphi}_t$, which we call the reference motion. Let us denote $\varphi_{t,\epsilon}(X) = \Phi_t(X, \epsilon)$, and hence, $\varphi_{t,0} = \mathring{\varphi}_t$. The variation field is defined as

$$\delta\varphi_t(X) = \mathbf{U}(X, t) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \varphi_{t,\epsilon}(X). \tag{7.46}$$

The spatial variation (displacement) field is defined as $\mathbf{u} = \mathbf{U} \circ \mathring{\varphi}_t^{-1}$ or $u^a(x, t) = U^a(X, t)$. Note that $\delta\varphi \in \Gamma(\mathring{\varphi}^{-1}T\mathcal{S})$ (see the appendix for the definition of the induced bundle). The tangent space $T_{(X,\epsilon)}(\mathcal{B} \times I)$ of the product manifold $\mathcal{B} \times I$ at (X, ϵ) is identified with $T_X\mathcal{B} \otimes T_\epsilon I$.

Linearization of the material velocity and acceleration. Material velocity is defined as $\mathbf{V}_\epsilon(X,t) = \frac{\partial \varphi_{t,\epsilon}(X)}{\partial t}$. Note that $\mathbf{V}_\epsilon(X,t) \in T_{\varphi_{t,\epsilon}(X)}\mathcal{S}$, i.e., for different values of ϵ, velocity lies in different tangent spaces, and hence, a covariant derivative along the curve $\epsilon \mapsto \varphi_{t,\epsilon}(X)$ should be used to find the linearization of velocity [43]. Therefore

$$\delta\mathbf{V}(X,t) = \nabla_{\frac{\partial}{\partial\epsilon}} \frac{\partial \varphi_{t,\epsilon}(X)}{\partial t}\Big|_{\epsilon=0} = \nabla_{\frac{\partial}{\partial t}} \frac{\partial \varphi_{t,\epsilon}(X)}{\partial \epsilon}\Big|_{\epsilon=0} = \nabla_t \delta\varphi_t(X) = D_t\mathbf{U}(X,t) = \nabla_{\mathring{\mathbf{V}}}\mathbf{U} =: \dot{\mathbf{U}}, \tag{7.47}$$

i.e., variation of the velocity field is the covariant time derivative of the displacement field. In the above calculation in the second equality the symmetry lemma of Riemannian geometry [217] was used. In components, $(D_t\mathbf{U}(X,t))^a = \dot{U}^a = \frac{\partial U^a}{\partial t} + \gamma^a{}_{bc}\mathring{V}^b U^c$. Note that $\frac{\partial U^a}{\partial t} = \frac{\partial u^a}{\partial t} + \frac{\partial u^a}{\partial x^b}\frac{\partial x^b}{\partial t}$. Therefore, $\delta\mathbf{v} = \frac{\partial \mathbf{u}}{\partial t} + \nabla^{\mathbf{g}}_{\mathring{\mathbf{v}}}\mathbf{u} =: \dot{\mathbf{u}}$.

Material acceleration is the covariant time derivative of velocity, i.e., $\mathbf{A} = D_t\mathbf{V} = \nabla_{\mathbf{V}}\mathbf{V}$, which in coordinates reads [43] $A^a = \frac{\partial V^a}{\partial t} + \gamma^a{}_{bc}V^bV^c$. Therefore

$$\begin{aligned}
\delta\mathbf{A}(X,t) &= \nabla_{\frac{\partial}{\partial\epsilon}}\nabla_{\frac{\partial}{\partial t}} \frac{\partial \varphi_{t,\epsilon}(X)}{\partial t}\Big|_{\epsilon=0} \\
&= \nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial \epsilon}} \frac{\partial \varphi_{t,\epsilon}(X)}{\partial t}\Big|_{\epsilon=0} + \nabla_{[\frac{\partial}{\partial\epsilon},\frac{\partial}{\partial t}]} \frac{\partial \varphi_{t,\epsilon}(X)}{\partial t}\Big|_{\epsilon=0} + \boldsymbol{\mathcal{R}}_{\mathbf{g}}\left(\frac{\partial}{\partial\epsilon},\frac{\partial}{\partial t}\right)\frac{\partial \varphi_{t,\epsilon}(X)}{\partial t}\Big|_{\epsilon=0} \\
&= \nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial t}} \frac{\partial \varphi_{t,\epsilon}(X)}{\partial \epsilon}\Big|_{\epsilon=0} + \nabla_{[\mathbf{U},\mathring{\mathbf{V}}]}\mathring{\mathbf{V}} + \boldsymbol{\mathcal{R}}_{\mathbf{g}}\left(\mathbf{U},\mathring{\mathbf{V}},\mathring{\mathbf{V}}\right) \\
&= D_tD_t\mathbf{U} + \nabla_{[\mathbf{U},\mathring{\mathbf{V}}]}\mathring{\mathbf{V}} + \boldsymbol{\mathcal{R}}_{\mathbf{g}}(\mathbf{U},\mathring{\mathbf{V}},\mathring{\mathbf{V}}) \\
&= \nabla_{\mathring{\mathbf{V}}}\nabla_{\mathring{\mathbf{V}}}\mathbf{U} + \nabla_{[\mathbf{U},\mathring{\mathbf{V}}]}\mathring{\mathbf{V}} + \boldsymbol{\mathcal{R}}_{\mathbf{g}}(\mathbf{U},\mathring{\mathbf{V}},\mathring{\mathbf{V}}) \\
&= \ddot{\mathbf{U}} + \nabla_{[\mathbf{U},\mathring{\mathbf{V}}]}\mathring{\mathbf{V}} + \boldsymbol{\mathcal{R}}_{\mathbf{g}}(\mathbf{U},\mathring{\mathbf{V}},\mathring{\mathbf{V}}),
\end{aligned} \tag{7.48}$$

where $\boldsymbol{\mathcal{R}}_{\mathbf{g}}$ is the curvature tensor of the metric $\mathbf{g}$ and $\ddot{\mathbf{U}} = D_tD_t\mathbf{U}$ is the second covariant time derivative of the displacement field. Note that $\delta\mathbf{a}_t = \delta\mathbf{A}_t \circ \mathring{\varphi}_t^{-1} = \ddot{\mathbf{U}} \circ \mathring{\varphi}_t^{-1} + \nabla^{\mathbf{g}}_{[\mathbf{u},\mathring{\mathbf{v}}]}\mathring{\mathbf{v}} + \boldsymbol{\mathcal{R}}_{\mathbf{g}}\left(\mathbf{u},\mathring{\mathbf{v}},\mathring{\mathbf{v}}\right)$.

Linearization of the right Cauchy-Green strain, the Jacobian, and the deformation gradient. The right Cauchy-Green strain of the motion $\varphi_{t,\epsilon}$ is defined as $\mathbf{C}^{\flat}_{\epsilon} = \varphi^{*}_{t,\epsilon}\mathbf{g} \circ \varphi_{t,\epsilon}$. Note that $\mathbf{C}^{\flat}_{\epsilon} \in \Gamma(\mathcal{B}, T^{*}\mathcal{B} \otimes T^{*}\mathcal{B})$ for all ϵ, where $\Gamma(\mathcal{B}, T^{*}\mathcal{B} \otimes T^{*}\mathcal{B})$ is the set of $\binom{0}{2}$-tensors on $\mathcal{B}$. Linearization of $\mathbf{C}^{\flat}$ is calculated as

$$\delta \mathbf{C}^{\flat} = \frac{d}{d\epsilon}\mathbf{C}^{\flat}_{\epsilon}\Big|_{\epsilon=0} = \frac{d}{d\epsilon}\varphi^{*}_{t,\epsilon}\mathbf{g}\Big|_{\epsilon=0} = \mathring{\varphi}^{*}_{t}\left(\mathbf{L}_{\mathbf{u}}\mathbf{g}\right) = \mathring{\varphi}^{*}_{t}\left(\mathfrak{L}_{\mathbf{u}}\mathbf{g}\right) = \mathring{\varphi}^{*}_{t}\left(\nabla^{\mathbf{g}}\mathbf{u}^{\flat} + (\nabla^{\mathbf{g}}\mathbf{u}^{\flat})^{\star}\right) = 2\mathring{\varphi}^{*}_{t}\boldsymbol{\epsilon}, \quad (7.49)$$

where $\mathbf{L}_{\mathbf{u}}$ is Lie derivative with respect to the vector field $\mathbf{u}$ and $\mathfrak{L}_{\mathbf{u}}$ is the autonomous Lie derivative with respect to $\mathbf{u}$ (see the appendix). The linearized strain $\boldsymbol{\epsilon}$ has the components $2\epsilon_{ab} = u_{a|b} + u_{b|a}$.

The Jacobian for the perturbed motion is written as

$$J_{\epsilon} = \sqrt{\frac{\det \mathbf{g} \circ \varphi_{t,\epsilon}}{\det \mathbf{G}}}\det \mathbf{F}_{\epsilon} = \sqrt{\frac{\det \mathbf{C}^{\flat}_{\epsilon}}{\det \mathbf{G}}}. \quad (7.50)$$

Hence

$$\delta J = \frac{d}{d\epsilon}J_{\epsilon}\Big|_{\epsilon=0} = \frac{1}{2\sqrt{\det \mathbf{G} \det \mathring{\mathbf{C}}^{\flat}}}\frac{d}{d\epsilon}\det \mathbf{C}^{\flat}_{\epsilon}\Big|_{\epsilon=0}. \quad (7.51)$$

Using Jacobi's formula we know that

$$\frac{d}{d\epsilon}\det \mathbf{C}^{\flat}_{\epsilon}\Big|_{\epsilon=0} = \det \mathring{\mathbf{C}}^{\flat}\,\mathrm{tr}\left[(\mathring{\mathbf{C}}^{\flat})^{-1}2\mathring{\varphi}^{*}_{t}\boldsymbol{\epsilon}\right]. \quad (7.52)$$

Note that $(\mathring{\mathbf{C}}^{\flat})^{-1} = \mathring{\varphi}^{*}_{t}\mathbf{g}^{\sharp}$, and hence, the right-hand side of (7.52) is simplified to read $(\det \mathring{\mathbf{C}}^{\flat})\mathbf{g}^{\sharp}\!:\!\boldsymbol{\epsilon}$. Therefore, $\delta J = \mathring{J}\mathbf{g}^{\sharp}\!:\!\boldsymbol{\epsilon} = \mathring{J}\,\mathrm{div}_{\mathbf{g}}\,\mathbf{u}$.

Deformation gradient of the motion $\varphi_{t,\epsilon}$ is defined as $\mathbf{F}_{t,\epsilon} = \frac{\partial \varphi_{t,\epsilon}}{\partial X}$. Consider the vector fields $\left(\frac{\partial}{\partial X^{A}}, 0\right)$ and $\left(0, \frac{\partial}{\partial \epsilon}\right)$ on $\mathcal{B} \times I$, and note that $\left[\left(\frac{\partial}{\partial X^{A}}, 0\right), \left(0, \frac{\partial}{\partial \epsilon}\right)\right] = 0$. Thus, from (C.8) one can write

$$\nabla_{\left(0,\frac{\partial}{\partial\epsilon}\right)}\varphi_{t,\epsilon*}\left(\frac{\partial}{\partial X^{A}}, 0\right) = \nabla_{\left(\frac{\partial}{\partial X^{A}},0\right)}\varphi_{t,\epsilon*}\left(0, \frac{\partial}{\partial \epsilon}\right). \quad (7.53)$$

Or

$$\nabla_{\frac{\partial}{\partial\epsilon}}\frac{\partial \varphi^{a}_{\epsilon}}{\partial X^{A}} = \nabla_{\frac{\partial}{\partial X^{A}}}\frac{\partial \varphi^{a}_{\epsilon}}{\partial \epsilon}. \quad (7.54)$$

Therefore, $\delta F^a{}_A = \varphi^a{}_{|A} = U^a{}_{|A} = F^b{}_A U^a{}_{|b}$. Note that $U^a{}_{|b} = \frac{\partial U^b}{\partial x^b} + \gamma^a{}_{bc} U^c$.

Linearization of the first and the second Piola-Kirchhoff stresses. For a hyperelastic solid, given an energy function $W = W(X, \mathbf{F}, \mathbf{G}, \mathbf{g} \circ \varphi, \boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_n)$, the first Piola-Kirchhoff stress is given as $\mathbf{P} = \mathbf{g}^\sharp \frac{\partial W}{\partial \mathbf{F}}$. In components, $P^{aA} = g^{ab} \frac{\partial W}{\partial F^b{}_A}$. For the perturbed motion, $W_\epsilon = W(X, \mathbf{F}_\epsilon, \mathbf{G}, \mathbf{g} \circ \varphi_\epsilon, \boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_n)$, and hence

$$\delta \mathbf{P} = \mathbf{g}^\sharp \nabla_{\frac{\partial}{\partial \epsilon}} \frac{\partial W_\epsilon}{\partial \mathbf{F}_\epsilon}\bigg|_{\epsilon=0} = \mathbf{g}^\sharp \frac{\partial^2 W_\epsilon}{\partial \mathbf{F}_\epsilon \partial \mathbf{F}_\epsilon} : \nabla_{\frac{\partial}{\partial \epsilon}} \mathbf{F}_\epsilon \bigg|_{\epsilon=0} + \mathbf{g}^\sharp \frac{\partial^2 W_\epsilon}{\partial \mathbf{g} \circ \varphi_\epsilon \partial \mathbf{F}_\epsilon} : \nabla_{\frac{\partial}{\partial \epsilon}} \mathbf{g} \circ \varphi_\epsilon \bigg|_{\epsilon=0} = \mathbf{g}^\sharp \frac{\partial^2 W}{\partial \mathbf{F} \partial \mathbf{F}} : \nabla \mathbf{U}. \tag{7.55}$$

In the above derivation, in the second equality the geometric ω-lemma [43], and in the last equality the metric compatibility of the Levi-Civita connection ($\nabla^{\mathbf{g}} \mathbf{g} = \mathbf{0}$) was used. Therefore, $\delta \mathbf{P} = \mathsf{A} : \nabla \mathrm{U}$, where A is the first elasticity tensor [43] with components

$$\mathsf{A}^{aA}{}_b{}^B = \frac{\partial P^{aA}}{\partial F^b{}_B} = g^{ac} \frac{\partial^2 W}{\partial F^c{}_A \partial F^b{}_B}. \tag{7.56}$$

Thus, in components $\delta P^{aA} = \mathsf{A}^{aA}{}_b{}^B U^b{}_{|B}$.

The second Piola-Kirchhoff stress is defined as $S^{AB} = (F^{-1})^A{}_a P^{aB}$. Using the material frame-indifference (objectivity) the energy function is written as $W = \hat{W}(X, \mathbf{G}, \mathbf{C}^\flat, \boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_n)$. The second Piola-Kirchhoff stress is written as $\mathbf{S} = 2 \frac{\partial \hat{W}}{\partial \mathbf{C}^\flat}$. For the one-parameter family of motions $\varphi_{t,\epsilon}$, one has $\mathbf{S}_\epsilon = 2 \frac{\partial \hat{W}_\epsilon}{\partial \mathbf{C}^\flat_\epsilon}$, where $\hat{W}_\epsilon = \hat{W}(X, \mathbf{G}, \mathbf{C}^\flat_\epsilon, \boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_n)$. Therefore

$$\delta \mathbf{S} = 2 \frac{\partial^2 \hat{W}_\epsilon}{\partial \mathbf{C}^\flat_\epsilon \partial \mathbf{C}^\flat_\epsilon} : \frac{d}{d\epsilon} \mathbf{C}^\flat_\epsilon \bigg|_{\epsilon=0} = 4 \frac{\partial^2 \hat{W}}{\partial \mathring{\mathbf{C}}^\flat \partial \mathring{\mathbf{C}}^\flat} : \mathring{\varphi}^* \boldsymbol{\epsilon} = \mathsf{C} : \mathring{\varphi}^* \boldsymbol{\epsilon}, \tag{7.57}$$

where C is the second elasticity tensor with components

$$\mathsf{C}^{ABCD} = 4 \frac{\partial^2 \hat{W}}{\partial C_{AB} \partial C_{CD}}. \tag{7.58}$$

In components, $\delta S^{AB} = \mathsf{C}^{ABCD} (\mathring{\varphi}^* \epsilon)_{CD}$. It is straightforward to show that the first and the second elasticity tensors are related as

$$\mathsf{A}_a{}^A{}_b{}^B = \mathsf{C}^{AMBN} \mathring{F}^m{}_M \mathring{F}^n{}_N g_{am} g_{bn} + \mathring{S}^{AB} g_{ab}. \tag{7.59}$$

Or

$$\mathsf{A}^{aAbB} = \mathsf{C}^{AMBN} \mathring{F}^a{}_M \mathring{F}^b{}_N + \mathring{S}^{AB} g^{ab}. \tag{7.60}$$

Similarly, the spatial elasticity tensors are defined as $\mathsf{a} = \frac{1}{J}\varphi_* \mathsf{A}$, and $\mathsf{c} = \frac{1}{J}\varphi_* \mathsf{C}$. In components

$$\mathsf{a}^{ac}{}_b{}^d = \frac{1}{J} F^c{}_A F^d{}_B \mathsf{A}^{aA}{}_b{}^B, \quad \mathsf{c}^{abcd} = \frac{\partial \sigma^{ab}}{\partial g_{cd}} = \frac{1}{J} F^a{}_A F^b{}_B F^c{}_C F^d{}_D \mathsf{C}^{ABCD}. \tag{7.61}$$

The relation analogous to (7.60) is given by $\mathsf{a}^{ac}{}_b{}^d = \sigma^{cd}\delta^a_b + \mathsf{c}^{aced} g_{eb}$. Note that the spatial elasticity tensors have the following symmetries $\mathsf{a}^{acbd} = \mathsf{a}^{bdac}$, and $\mathsf{c}^{abcd} = \mathsf{c}^{bacd} = \mathsf{c}^{abdc} = \mathsf{c}^{cdab}$.

Linearization of conservation of mass. For the family of motions $\varphi_{t,\epsilon}$, conservation of mass is locally written as $\rho_0(X) = J_\epsilon \; \rho \circ \varphi_{t,\epsilon}(X)$. Taking derivatives of both sides with respect to ϵ and evaluating at $\epsilon = 0$, one obtains $0 = \delta J \; \rho \circ \mathring{\varphi}_t(X) + \mathring{J} \; \mathbf{d}\mathring{\rho} \cdot \delta\varphi$, where $\mathbf{d}\mathring{\rho}$ is the exterior derivative of the mass density of the reference motion (a 1-form). Thus, $\mathbf{d}\mathring{\rho} \cdot \mathbf{u} + \mathring{\rho}\mathbf{g}^\sharp : \boldsymbol{\epsilon} = 0$. Note that $\mathbf{g}^\sharp : \boldsymbol{\epsilon} = \mathbf{g}^\sharp : \nabla \mathbf{u}^\flat$. Hence, the linearized conservation of mass can be written as, $\mathbf{d}\mathring{\rho} \cdot \mathbf{u} + \mathring{\rho}\mathbf{g}^\sharp : \nabla^{\mathbf{g}} \mathbf{u}^\flat = 0$. Therefore

$$\delta\rho = -\mathring{\rho}\, \boldsymbol{\epsilon} : \mathbf{g}^\sharp = -\mathring{\rho}\, \mathrm{div}_{\mathbf{g}}\, \mathbf{u}. \tag{7.62}$$

Linearization of the balance of linear momentum. The balance of linear momentum in terms of the first Piola-Kirchhoff stress reads $\mathrm{Div}\, \mathbf{P} + \rho_0 \mathbf{B} = \rho_0 \mathbf{A}$. Note that the operator Div depends on both $\mathbf{G}$ and g and is defined through the Piola identity (see the appendix) : $\mathrm{Div}\, \mathbf{P} = J\, \mathrm{div}_{\mathbf{g}}\, \boldsymbol{\sigma}$. The balance of linear momentum in terms of the Cauchy stress reads $\mathrm{div}_{\mathbf{g}}\, \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \mathbf{a}$. Using the Doyle-Ericksen formula, the Cauchy stress is written as [56] $\boldsymbol{\sigma} = \frac{2}{J}\frac{\partial W}{\partial \mathbf{g}}$, where $W = W(X, \mathbf{F}, \mathbf{g} \circ \varphi, \mathbf{G}, \boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_n)$ is the energy density per unit undeformed volume. The convected stress tensor $\boldsymbol{\Sigma} = \varphi_t^* \boldsymbol{\sigma}$ can be written as [205] $\boldsymbol{\Sigma} = \frac{2}{J}\frac{\partial \hat{W}}{\partial \mathbf{C}^\flat}$, where $W = \hat{W}(X, \mathbf{C}^\flat, \mathbf{G}, \boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_n)$.

Linearization of the convected balance of linear momentum. For the one-parameter family of motions $\varphi_{t,\epsilon}$, one has

$$\operatorname{div}_{\mathbf{C}^\flat_\epsilon} \boldsymbol{\Sigma}_\epsilon + \varrho_\epsilon \mathscr{B}_\epsilon = \varrho_\epsilon \mathscr{A}_\epsilon. \tag{7.63}$$

Therefore, the linearized balance of linear momentum is written as

$$\frac{d}{d\epsilon}\left[\operatorname{div}_{\mathbf{C}^\flat_\epsilon} \boldsymbol{\Sigma}_\epsilon\right]\Big|_{\epsilon=0} + \frac{d}{d\epsilon}\left[\varrho_\epsilon \mathscr{B}_\epsilon\right]\Big|_{\epsilon=0} = \frac{d}{d\epsilon}\left[\varrho_\epsilon \mathscr{A}_\epsilon\right]\Big|_{\epsilon=0}. \tag{7.64}$$

Note that for $X \in \mathcal{B}$, all the terms in (7.63) lie in the same tangent space $T_X\mathcal{B}$, and hence, linearizing the balance of linear momentum is straightforward when using the convected stress. It is a simple calculation to show that the linearized convected stress reads

$$\delta\boldsymbol{\Sigma} = \frac{4}{\mathring{J}} \frac{\partial^2 \hat{W}}{\partial \mathring{\mathbf{C}}^\flat \partial \mathring{\mathbf{C}}^\flat} : \mathring{\varphi}_t^* \boldsymbol{\epsilon} - (\boldsymbol{\epsilon} : \mathbf{g}^\sharp)\mathring{\boldsymbol{\Sigma}}. \tag{7.65}$$

Note that $\delta\boldsymbol{\sigma} = \mathring{\varphi}_{t*}\delta\boldsymbol{\Sigma}$, and hence

$$\delta\boldsymbol{\sigma} = \frac{4}{\mathring{J}} \frac{\partial^2 W}{\partial \mathbf{g} \partial \mathbf{g}} : \boldsymbol{\epsilon} - (\boldsymbol{\epsilon} : \mathbf{g}^\sharp)\mathring{\boldsymbol{\sigma}}. \tag{7.66}$$

We define the following fourth-order elasticity tensor

$$\mathbb{C} := \frac{4}{\mathring{J}} \frac{\partial^2 W}{\partial \mathbf{g} \partial \mathbf{g}}, \tag{7.67}$$

which in components reads $\mathbb{C}^{abcd} = \frac{4}{\mathring{J}} \frac{\partial^2 W}{\partial g_{ab} \partial g_{cd}}$. Note that $\mathbb{C} = \mathring{\varphi}_* \mathsf{C}/\mathring{J}$. We now expand the linearization of each term in (7.64). For the body force term one can write

$$\begin{aligned} \frac{d}{d\epsilon}\left[\varrho_\epsilon \mathscr{B}_\epsilon\right]\Big|_{\epsilon=0} &= \frac{d}{d\epsilon}\left[(\rho \circ \varphi_{t,\epsilon})\varphi^*_{t,\epsilon}\mathbf{b}\right]\Big|_{\epsilon=0} = \delta\rho \circ \mathring{\varphi}_t \, \mathring{\mathscr{B}} + \mathring{\varrho} \frac{d}{d\epsilon}\left[(\varphi^*_{t,\epsilon}\mathbf{b})\right]\Big|_{\epsilon=0} \\ &= -(\mathbf{g}^\sharp : \boldsymbol{\epsilon}) \circ \mathring{\varphi}_t \, \mathring{\varrho} \mathring{\mathscr{B}} + \mathring{\varrho} \, \mathring{\varphi}_t^* \Big(\mathbf{L}_{\mathbf{u}} \mathring{\mathbf{b}}\Big). \end{aligned} \tag{7.68}$$

Note that $\partial\mathring{\mathbf{b}}/\partial\epsilon = \mathbf{0}$, and hence, $\mathbf{L}_{\mathbf{u}}\mathring{\mathbf{b}} = \mathfrak{L}_{\mathbf{u}}\mathring{\mathbf{b}}$. Therefore, $\delta(\varrho\mathscr{B}) = -(\mathbf{g}^\sharp\!:\!\boldsymbol{\epsilon})\circ\mathring{\varphi}_t\;\mathring{\varrho}\mathring{\mathscr{B}} + \mathring{\varrho}\;\mathring{\varphi}_t^*\Big(\mathfrak{L}_{\mathbf{u}}\mathring{\mathbf{b}}\Big)$. Also note that $\mathfrak{L}_{\mathbf{u}}\mathring{\mathbf{b}} = \nabla^{\mathbf{g}}_{\mathbf{u}}\mathring{\mathbf{b}} - \nabla^{\mathbf{g}}_{\mathring{\mathbf{b}}}\mathbf{u}$. Hence

$$\delta(\varrho\mathscr{B}) = -(\mathbf{g}^\sharp\!:\!\boldsymbol{\epsilon})\circ\mathring{\varphi}_t\;\mathring{\varrho}\mathring{\mathscr{B}} + \mathring{\varrho}\;\mathring{\varphi}_t^*\Big(\nabla^{\mathbf{g}}_{\mathbf{u}}\mathring{\mathbf{b}} - \nabla^{\mathbf{g}}_{\mathring{\mathbf{b}}}\mathbf{u}\Big)\,. \tag{7.69}$$

The convected acceleration is linearized as follows

$$\begin{aligned}\frac{d}{d\epsilon}[\mathscr{A}_\epsilon]\Big|_{\epsilon=0} =&\varphi_{t,\epsilon}^*(\mathbf{L}_{\mathbf{u}}\mathbf{a}_\epsilon)\Big|_{\epsilon=0} = \varphi_{t,\epsilon}^*(\mathfrak{L}_{\mathbf{u}}\mathbf{a}_\epsilon)\Big|_{\epsilon=0} = \varphi_{t,\epsilon}^*\left[\nabla^{\mathbf{g}}_{\mathbf{u}}\mathbf{a}_\epsilon - \nabla^{\mathbf{g}}_{\mathbf{a}_\epsilon}\mathbf{u}\right]\Big|_{\epsilon=0}\\ =&T\varphi_{t,\epsilon}^{-1}\circ\left[\nabla_{\mathbf{U}}\mathbf{A}_\epsilon - \nabla_{\mathbf{A}_\epsilon}\mathbf{U}\right]\Big|_{\epsilon=0} = T\varphi_{t,\epsilon}^{-1}\circ\left[D_\epsilon\mathbf{A}_\epsilon - \nabla_{\mathbf{A}_\epsilon}\mathbf{U}\right]\Big|_{\epsilon=0}\\ =&T\varphi_{t,\epsilon}^{-1}\circ\left[D_\epsilon D_t\mathbf{V}_\epsilon - \nabla_{\mathbf{A}_\epsilon}\mathbf{U}\right]\Big|_{\epsilon=0}\\ =&T\varphi_{t,\epsilon}^{-1}\circ\left[D_tD_\epsilon\frac{\partial\varphi_{t,\epsilon}}{\partial t} + \nabla_{[\mathbf{U},\mathbf{V}_\epsilon]}\mathbf{V}_\epsilon + \boldsymbol{\mathcal{R}}_{\mathbf{g}}(\mathbf{U},\mathbf{V}_\epsilon,\mathbf{V}_\epsilon) - \nabla_{\mathbf{A}_\epsilon}\mathbf{U}\right]\Bigg|_{\epsilon=0}\\ =&\mathring{\mathbf{F}}_t^{-1}\cdot\Big[D_tD_t\mathbf{U} + \nabla_{[\mathbf{U},\mathring{\mathbf{V}}]}\mathring{\mathbf{V}} + \boldsymbol{\mathcal{R}}_{\mathbf{g}}(\mathbf{U},\mathring{\mathbf{V}},\mathring{\mathbf{V}}) - \nabla_{\mathring{\mathbf{A}}}\mathbf{U}\Big]\,.\end{aligned} \tag{7.70}$$

Therefore

$$\delta(\varrho\mathscr{A}) = -(\mathbf{g}^\sharp\!:\!\boldsymbol{\epsilon})\circ\mathring{\varphi}_t\;\mathring{\varrho}\mathring{\mathscr{A}} + \mathring{\varrho}\mathring{\mathbf{F}}_t^{-1}\cdot\Big[\ddot{\mathbf{U}} + \nabla_{[\mathbf{U},\mathring{\mathbf{V}}]}\mathring{\mathbf{V}} + \boldsymbol{\mathcal{R}}_{\mathbf{g}}(\mathbf{U},\mathring{\mathbf{V}},\mathring{\mathbf{V}}) - \nabla_{\mathring{\mathbf{A}}}\mathbf{U}\Big]\,. \tag{7.71}$$

To calculate $\delta\left(\mathrm{div}_{\mathbf{C}^\flat}\boldsymbol{\Sigma}\right)$, Sadik and Yavari [23] expanded $\mathrm{div}_{\mathbf{C}^\flat_\epsilon}\boldsymbol{\Sigma}_\epsilon$ in a coordinate chart and then took derivatives with respect to ϵ and finally evaluated at $\epsilon = 0$. They also assumed a flat ambient space. Here, we do not assume a flat ambient space. Note that

$$\delta(\mathrm{div}_{\mathbf{C}^\flat}\boldsymbol{\Sigma}) = \frac{d}{d\epsilon}\left[\mathrm{div}_{\mathbf{C}^\flat_\epsilon}\boldsymbol{\Sigma}_\epsilon\right]\Big|_{\epsilon=0} = \frac{d}{d\epsilon}\varphi_{t,\epsilon}^*\left[\mathrm{div}_{\mathbf{g}}\boldsymbol{\sigma}_\epsilon\right]\Big|_{\epsilon=0} = \varphi_{t,\epsilon}^*\mathbf{L}_{\mathbf{u}}\left[\mathrm{div}_{\mathbf{g}}\boldsymbol{\sigma}_\epsilon\right]\Big|_{\epsilon=0} = \varphi_{t,\epsilon}^*\mathfrak{L}_{\mathbf{u}}\left[\mathrm{div}_{\mathbf{g}}\boldsymbol{\sigma}_\epsilon\right]\Big|_{\epsilon=0}\,. \tag{7.72}$$

We know that [218, 43]

$$\mathfrak{L}_{\mathbf{u}}\nabla^{\mathbf{g}} = \nabla^{\mathbf{g}}\nabla^{\mathbf{g}}\mathbf{u} + \mathbf{u}\cdot\boldsymbol{\mathcal{R}}_{\mathbf{g}}\,, \tag{7.73}$$

where in components, $(\mathfrak{L}_{\mathbf{u}}\nabla^{\mathbf{g}})^a{}_{bc} = u^a{}_{|bc} + \mathcal{R}^a{}_{ebc}u^e$. We also know that [218]

$$
\begin{aligned}
\mathfrak{L}_{\mathbf{u}}\sigma^{ab}{}_{|b} &= (\mathfrak{L}_{\mathbf{u}}\sigma^{ab})_{|b} + (\mathfrak{L}_{\mathbf{u}}\gamma^a{}_{bd})\sigma^{bd} + (\mathfrak{L}_{\mathbf{u}}\gamma^b{}_{bd})\sigma^{ad} \\
&= (\mathfrak{L}_{\mathbf{u}}\sigma^{ab})_{|b} + (u^a{}_{|bd} + \mathcal{R}^a{}_{cbd}u^c)\sigma^{bd} + (u^b{}_{|bd} + \mathcal{R}^b{}_{cbd}u^c)\sigma^{ad}\,.
\end{aligned}
\tag{7.74}
$$

Note that $\mathcal{R}^a{}_{cbd} = -\mathcal{R}^a{}_{cdb}$, and hence, $\mathcal{R}^a{}_{cbd}\sigma^{bd} = 0$. Also, $\mathcal{R}^b{}_{cbd} = \mathsf{Ric}_{cd}$ is the Ricci curvature. Thus

$$
\mathfrak{L}_{\mathbf{u}}\sigma^{ab}{}_{|b} = (\mathfrak{L}_{\mathbf{u}}\sigma^{ab})_{|b} + u^a{}_{|bd}\sigma^{bd} + \left[(u^b{}_{|b})_{,d} + u^c\mathsf{Ric}_{cd}\right]\sigma^{ad}\,. \tag{7.75}
$$

Therefore

$$
\mathfrak{L}_{\mathbf{u}}\left(\operatorname{div}_{\mathbf{g}}\mathring{\boldsymbol{\sigma}}\right) = \operatorname{div}_{\mathbf{g}}\left(\mathfrak{L}_{\mathbf{u}}\mathring{\boldsymbol{\sigma}}\right) + \nabla^{\mathbf{g}}\nabla^{\mathbf{g}}\mathbf{u} : \mathring{\boldsymbol{\sigma}} + \mathbf{d}(\boldsymbol{\epsilon}\!:\!\mathbf{g}^\sharp)\cdot\mathring{\boldsymbol{\sigma}} + \mathring{\boldsymbol{\sigma}}\cdot\mathsf{\mathbf{Ric}}_{\mathbf{g}}\cdot\mathbf{u}\,. \tag{7.76}
$$

Note that

$$
\mathfrak{L}_{\mathbf{u}}\mathring{\boldsymbol{\sigma}} = \mathfrak{L}_{\mathbf{u}}\left(\frac{2}{\mathring{J}}\frac{\partial W}{\partial \mathbf{g}}\right) = -\frac{2}{\mathring{J}^2}\frac{\partial W}{\partial \mathbf{g}}\mathfrak{L}_{\mathbf{u}}\mathring{J} + \frac{2}{\mathring{J}}\left(\frac{\partial^2 W}{\partial \mathbf{g}\partial \mathbf{g}}\!:\!\mathfrak{L}_{\mathbf{u}}\mathbf{g} + \frac{\partial^2 W}{\partial \mathbf{F}\partial \mathbf{g}}\!:\!\mathfrak{L}_{\mathbf{u}}\mathbf{F}\right). \tag{7.77}
$$

However, $\mathfrak{L}_{\mathbf{u}}\mathbf{F} = \mathbf{0}$, because for any vector $\mathbf{W}(X) \in T_X\mathcal{B}$, one can write

$$
\begin{aligned}
\mathfrak{L}_{\mathbf{u}}\mathbf{F}\!\cdot\!\mathbf{W} &= \mathfrak{L}_{\mathbf{u}}\left(\mathbf{F}\!\cdot\!\mathbf{W}\right) = \varphi_{t*}\frac{d}{d\epsilon}\left[\varphi^*_{t,\epsilon}\left(\mathbf{F}_{t,\epsilon}\!\cdot\!\mathbf{W}\right)\right]\Big|_{\epsilon=0} \\
&= \varphi_{t*}\frac{d}{d\epsilon}\left[\varphi^*_{t,\epsilon}\left(\varphi_{t,\epsilon*}\circ\mathbf{W}\right)\right]\Big|_{\epsilon=0} = \varphi_{t*}\frac{d}{d\epsilon}\left[\mathbf{W}\right]\Big|_{\epsilon=0} = \mathbf{0}.
\end{aligned}
\tag{7.78}
$$

Hence

$$
\mathfrak{L}_{\mathbf{u}}\mathring{\boldsymbol{\sigma}} = \frac{4}{\mathring{J}}\frac{\partial^2 W}{\partial \mathbf{g}\partial \mathbf{g}}\!:\!\boldsymbol{\epsilon} - (\boldsymbol{\epsilon}\!:\!\mathbf{g}^\sharp)\mathring{\boldsymbol{\sigma}}\,. \tag{7.79}
$$

Therefore

$$
\delta(\operatorname{div}_{\mathbf{C}^\flat}\boldsymbol{\Sigma}) = \mathring{\varphi}^*_t\left[\nabla^{\mathbf{g}}\nabla^{\mathbf{g}}\mathbf{u} : \mathring{\boldsymbol{\sigma}} + \mathbf{d}(\boldsymbol{\epsilon}\!:\!\mathbf{g}^\sharp)\cdot\mathring{\boldsymbol{\sigma}} + \mathring{\boldsymbol{\sigma}}\cdot\mathsf{\mathbf{Ric}}_{\mathbf{g}}\cdot\mathbf{u} + \operatorname{div}_{\mathbf{g}}\left(\frac{4}{\mathring{J}}\frac{\partial^2 W}{\partial \mathbf{g}\partial \mathbf{g}}\!:\!\boldsymbol{\epsilon} - (\boldsymbol{\epsilon}\!:\!\mathbf{g}^\sharp)\mathring{\boldsymbol{\sigma}}\right)\right]. \tag{7.80}
$$

Or

$$\delta(\operatorname{div}_{\mathbf{C}^\flat}\boldsymbol{\Sigma}) = \mathring{\varphi}_t^*\left[\nabla^{\mathbf{g}}\nabla^{\mathbf{g}}\mathbf{u} : \mathring{\boldsymbol{\sigma}} - (\boldsymbol{\epsilon}\!:\!\mathbf{g}^\sharp)\operatorname{div}_{\mathbf{g}}\mathring{\boldsymbol{\sigma}} + \mathring{\boldsymbol{\sigma}}\cdot\mathsf{Ric}_{\mathbf{g}}\cdot\mathbf{u} + \operatorname{div}_{\mathbf{g}}(\mathbb{C}\!:\!\boldsymbol{\epsilon})\right] . \tag{7.81}$$

Now the push-forward of the balance of linear momentum by $\mathring{\varphi}_t$ using (7.69), (7.71), and (7.81) reads

$$\begin{aligned}
&\nabla^{\mathbf{g}}\nabla^{\mathbf{g}}\mathbf{u} : \mathring{\boldsymbol{\sigma}} - (\boldsymbol{\epsilon}\!:\!\mathbf{g}^\sharp)\operatorname{div}_{\mathbf{g}}\mathring{\boldsymbol{\sigma}} + \mathring{\boldsymbol{\sigma}}\cdot\mathsf{Ric}_{\mathbf{g}}\cdot\mathbf{u} + \operatorname{div}_{\mathbf{g}}(\mathbb{C}\!:\!\boldsymbol{\epsilon}) - \mathbf{g}^\sharp\!:\!\boldsymbol{\epsilon}\,\mathring{\rho}\mathring{\mathbf{b}} + \mathring{\rho}\left(\nabla^{\mathbf{g}}_{\mathbf{u}}\mathring{\mathbf{b}} - \nabla^{\mathbf{g}}_{\mathring{\mathbf{b}}}\mathbf{u}\right)\\
&= -\mathbf{g}^\sharp\!:\!\boldsymbol{\epsilon}\,\mathring{\rho}\mathring{\mathbf{a}} + \mathring{\rho}\left[\ddot{\mathbf{U}}\circ\mathring{\varphi}_t^{-1} + \nabla^{\mathbf{g}}_{[\mathbf{u},\mathring{\mathbf{v}}]}\mathring{\mathbf{v}} + \boldsymbol{\mathcal{R}}_{\mathbf{g}}(\mathbf{u},\mathring{\mathbf{v}},\mathring{\mathbf{v}}) - \nabla^{\mathbf{g}}_{\mathring{\mathbf{a}}}\mathbf{u}\right] .
\end{aligned} \tag{7.82}$$

We assume that the reference motion is "physical," i.e., it satisfies all the balance laws. In particular, $\operatorname{div}_{\mathbf{g}}\mathring{\boldsymbol{\sigma}} + \mathring{\rho}\mathring{\mathbf{b}} = \mathring{\rho}\mathring{\mathbf{a}}$. Thus, we have the following two identities

$$\begin{aligned}
&-\mathbf{g}^\sharp\!:\!\boldsymbol{\epsilon}\,\mathring{\rho}\mathring{\mathbf{b}} + \mathbf{g}^\sharp\!:\!\boldsymbol{\epsilon}\,\mathring{\rho}\mathring{\mathbf{a}} = (\mathbf{g}^\sharp\!:\!\boldsymbol{\epsilon})\operatorname{div}_{\mathbf{g}}\mathring{\boldsymbol{\sigma}}\\
&-\mathring{\rho}\nabla^{\mathbf{g}}_{\mathring{\mathbf{b}}}\mathbf{u} + \mathring{\rho}\nabla^{\mathbf{g}}_{\mathring{\mathbf{a}}}\mathbf{u} = \nabla^{\mathbf{g}}_{-\mathring{\rho}(\mathring{\mathbf{b}}-\mathring{\mathbf{a}})}\mathbf{u} = \nabla^{\mathbf{g}}_{\operatorname{div}_{\mathbf{g}}\mathring{\boldsymbol{\sigma}}}\mathbf{u} = \nabla^{\mathbf{g}}\mathbf{u}\cdot\operatorname{div}_{\mathbf{g}}\mathring{\boldsymbol{\sigma}} .
\end{aligned} \tag{7.83}$$

Therefore, the linearized balance of linear momentum is simplified to read

$$\nabla^{\mathbf{g}}\nabla^{\mathbf{g}}\mathbf{u} : \mathring{\boldsymbol{\sigma}} + \nabla^{\mathbf{g}}\mathbf{u}\cdot\operatorname{div}_{\mathbf{g}}\mathring{\boldsymbol{\sigma}} + \mathring{\boldsymbol{\sigma}}\cdot\mathsf{Ric}_{\mathbf{g}}\cdot\mathbf{u} + \operatorname{div}_{\mathbf{g}}(\mathbb{C}\!:\!\boldsymbol{\epsilon}) + \mathring{\rho}\nabla^{\mathbf{g}}_{\mathbf{u}}\mathring{\mathbf{b}} = \mathring{\rho}\left[\ddot{\mathbf{U}}\circ\mathring{\varphi}_t^{-1} + \nabla^{\mathbf{g}}_{[\mathbf{u},\mathring{\mathbf{v}}]}\mathring{\mathbf{v}} + \boldsymbol{\mathcal{R}}_{\mathbf{g}}(\mathbf{u},\mathring{\mathbf{v}},\mathring{\mathbf{v}})\right] . \tag{7.84}$$

Note that $\nabla^{\mathbf{g}}\nabla^{\mathbf{g}}\mathbf{u} : \mathring{\boldsymbol{\sigma}} + \nabla^{\mathbf{g}}\mathbf{u}\cdot\operatorname{div}_{\mathbf{g}}\mathring{\boldsymbol{\sigma}} = \operatorname{div}_{\mathbf{g}}(\nabla^{\mathbf{g}}\mathbf{u}\cdot\mathring{\boldsymbol{\sigma}})$. Hence, the linearized balance of linear momentum is written as[16]

$$\operatorname{div}_{\mathbf{g}}(\mathbb{C}\!:\!\boldsymbol{\epsilon}) + \operatorname{div}_{\mathbf{g}}(\nabla^{\mathbf{g}}\mathbf{u}\cdot\mathring{\boldsymbol{\sigma}}) + \mathring{\boldsymbol{\sigma}}\cdot\mathsf{Ric}_{\mathbf{g}}\cdot\mathbf{u} + \mathring{\rho}\nabla^{\mathbf{g}}_{\mathbf{u}}\mathring{\mathbf{b}} = \mathring{\rho}\left[\ddot{\mathbf{U}}\circ\mathring{\varphi}_t^{-1} + \nabla^{\mathbf{g}}_{[\mathbf{u},\mathring{\mathbf{v}}]}\mathring{\mathbf{v}} + \boldsymbol{\mathcal{R}}_{\mathbf{g}}(\mathbf{u},\mathring{\mathbf{v}},\mathring{\mathbf{v}})\right] . \tag{7.85}$$

Linearization of the spatial balance of linear momentum. For the motion $\varphi_{t,\epsilon}$, the balance of linear momentum reads $\operatorname{div}_{\mathbf{g}}\boldsymbol{\sigma}_\epsilon + \rho_\epsilon\mathbf{b}_\epsilon = \rho_\epsilon\mathbf{a}_\epsilon$. Linearization of the body force term is calculated as

[16] For a flat ambient space this is identical to the corresponding equation in [23]. However, note that even for a flat ambient space this is not identical to what Marsden and Hughes [43] obtained; they do not have the term $\operatorname{div}_{\mathbf{g}}(\nabla^{\mathbf{g}}\mathbf{u}\cdot\mathring{\boldsymbol{\sigma}})$.

$\delta(\rho\mathbf{b}) = \delta\rho\mathring{\mathbf{b}} + \mathring{\rho}\nabla^{\mathbf{g}}_{\frac{\partial}{\partial\epsilon}}\mathbf{b}_\epsilon\Big|_{\epsilon=0} = -(\mathbf{g}^\sharp\!:\!\boldsymbol{\epsilon})\mathring{\rho}\mathring{\mathbf{b}} + \mathring{\rho}\nabla^{\mathbf{g}}_{\mathbf{u}}\mathring{\mathbf{b}}$. Linearization of the inertial force term reads

$$\begin{aligned}\delta(\rho\mathbf{a}) &= \delta\rho\mathring{\mathbf{a}} + \mathring{\rho}\nabla^{\mathbf{g}}_{\frac{\partial}{\partial\epsilon}}\mathbf{a}_\epsilon\Big|_{\epsilon=0} = -(\mathbf{g}^\sharp\!:\!\boldsymbol{\epsilon})\mathring{\rho}\mathring{\mathbf{a}} + \mathring{\rho}\nabla^{\mathbf{g}}_{\frac{\partial}{\partial\epsilon}}\mathbf{a}_\epsilon\Big|_{\epsilon=0} \\ &-(\mathbf{g}^\sharp\!:\!\boldsymbol{\epsilon})\mathring{\rho}\mathring{\mathbf{a}} + \mathring{\rho}\left[\ddot{\mathbf{U}}\circ\mathring{\varphi}_t^{-1} + \nabla^{\mathbf{g}}_{[\mathbf{u},\mathring{\mathbf{v}}]}\mathring{\mathbf{v}} + \boldsymbol{\mathcal{R}}_{\mathbf{g}}\left(\mathbf{u},\mathring{\mathbf{v}},\mathring{\mathbf{v}}\right)\right].\end{aligned} \tag{7.86}$$

Note that $\delta(\operatorname{div}_{\mathbf{g}}\boldsymbol{\sigma}) = \nabla^{\mathbf{g}}_{\frac{\partial}{\partial\epsilon}}\left(\operatorname{div}_{\mathbf{g}}\boldsymbol{\sigma}_\epsilon\right)\Big|_{\epsilon=0} = \nabla^{\mathbf{g}}_{\mathbf{u}}\left(\operatorname{div}_{\mathbf{g}}\mathring{\boldsymbol{\sigma}}\right)$. Knowing that the Levi-Civita connection is torsion-free, one can write $\nabla^{\mathbf{g}}_{\mathbf{u}}\left(\operatorname{div}_{\mathbf{g}}\mathring{\boldsymbol{\sigma}}\right) = \mathfrak{L}_{\mathbf{u}}\left(\operatorname{div}_{\mathbf{g}}\mathring{\boldsymbol{\sigma}}\right) + \nabla^{\mathbf{g}}_{\operatorname{div}_{\mathbf{g}}\mathring{\boldsymbol{\sigma}}}\mathbf{u} = \mathfrak{L}_{\mathbf{u}}\left(\operatorname{div}_{\mathbf{g}}\mathring{\boldsymbol{\sigma}}\right) + \nabla^{\mathbf{g}}\mathbf{u}\cdot\operatorname{div}_{\mathbf{g}}\mathring{\boldsymbol{\sigma}}$. Hence, from (7.76), one has $\delta(\operatorname{div}_{\mathbf{g}}\boldsymbol{\sigma}) = \operatorname{div}_{\mathbf{g}}\left(\mathfrak{L}_{\mathbf{u}}\mathring{\boldsymbol{\sigma}}\right) + \operatorname{div}_{\mathbf{g}}\left(\nabla^{\mathbf{g}}\mathbf{u}\cdot\mathring{\boldsymbol{\sigma}}\right) + \mathbf{d}(\boldsymbol{\epsilon}\!:\!\mathbf{g}^\sharp)\cdot\mathring{\boldsymbol{\sigma}} + \mathring{\boldsymbol{\sigma}}\cdot\mathsf{Ric}_{\mathbf{g}}\cdot\mathbf{u}$. Using (7.79), one obtains

$$\delta(\operatorname{div}_{\mathbf{g}}\boldsymbol{\sigma}) = \operatorname{div}_{\mathbf{g}}\left(\nabla^{\mathbf{g}}\mathbf{u}\cdot\mathring{\boldsymbol{\sigma}}\right) + \mathring{\boldsymbol{\sigma}}\cdot\mathsf{Ric}_{\mathbf{g}}\cdot\mathbf{u} + \operatorname{div}_{\mathbf{g}}\left(\mathbb{C}\!:\!\boldsymbol{\epsilon}\right) - (\boldsymbol{\epsilon}\!:\!\mathbf{g}^\sharp)\operatorname{div}_{\mathbf{g}}\mathring{\boldsymbol{\sigma}}\,. \tag{7.87}$$

Remark 7.3.1. Note that linearizing a convected vector and pushing it forward to the ambient space is not the same as linearizing the corresponding spatial vector using a covariant derivative. In particular, we note that

$$\begin{aligned}\delta(\operatorname{div}_{\mathbf{g}}\boldsymbol{\sigma}) &= \mathring{\varphi}_{t*}\delta(\operatorname{div}_{\mathbf{C}^\flat}\boldsymbol{\Sigma}) + \nabla^{\mathbf{g}}\mathbf{u}\cdot\operatorname{div}_{\mathbf{g}}\mathring{\boldsymbol{\sigma}}, \\ \delta(\rho\mathbf{b}) &= \mathring{\varphi}_{t*}\delta(\varrho\mathscr{B}) + \mathring{\rho}\nabla^{\mathbf{g}}_{\mathring{\mathbf{b}}}\mathbf{u}, \\ \delta(\rho\mathbf{a}) &= \mathring{\varphi}_{t*}\delta(\varrho\mathscr{A}) + \mathring{\rho}\nabla^{\mathbf{g}}_{\mathring{\mathbf{a}}}\mathbf{u}.\end{aligned} \tag{7.88}$$

However, note that $\nabla^{\mathbf{g}}\mathbf{u}\cdot\operatorname{div}_{\mathbf{g}}\mathring{\boldsymbol{\sigma}} + \mathring{\rho}\nabla^{\mathbf{g}}_{\mathring{\mathbf{b}}}\mathbf{u} - \mathring{\rho}\nabla^{\mathbf{g}}_{\mathring{\mathbf{a}}}\mathbf{u} = \nabla^{\mathbf{g}}_{\operatorname{div}_{\mathbf{g}}\mathring{\boldsymbol{\sigma}}+\mathring{\rho}\mathring{\mathbf{b}}-\mathring{\rho}\mathring{\mathbf{a}}}\mathbf{u} = \mathbf{0}$, i.e., the two approaches give the same linearized balance of linear momentum (7.85).

Linearization of the material balance of linear momentum. The balance of linear momentum in terms of the first Piola-Kirchhoff stress is linearized as follows. Linearized body and inertial forces are (note that $\rho_0 = \rho_0(X)$ has a vanishing variation) $\delta(\rho_0\mathbf{B}) = \rho_0\nabla^{\mathbf{g}}_{\mathbf{U}}\mathring{\mathbf{B}}$, and $\delta(\rho_0\mathbf{A}) = \rho_0\left[\ddot{\mathbf{U}} + \nabla^{\mathbf{g}}_{[\mathbf{U},\mathring{\mathbf{V}}]}\mathring{\mathbf{V}} + \boldsymbol{\mathcal{R}}_{\mathbf{g}}\left(\mathbf{U},\mathring{\mathbf{V}},\mathring{\mathbf{V}}\right)\right]$. Note that $\delta\left(\operatorname{Div}\mathbf{P}\right) = \delta(J\operatorname{div}_{\mathbf{g}}\boldsymbol{\sigma}) = \delta J\operatorname{div}_{\mathbf{g}}\mathring{\boldsymbol{\sigma}} + \mathring{J}\delta(\operatorname{div}_{\mathbf{g}}\boldsymbol{\sigma}) = (\boldsymbol{\epsilon}\!:\!\mathbf{g}^\sharp)\operatorname{Div}\mathring{\mathbf{P}} + \mathring{J}\delta(\operatorname{div}_{\mathbf{g}}\boldsymbol{\sigma})$. From (7.87), we know that

$$\mathring{J}\delta(\operatorname{div}_{\mathbf{g}}\boldsymbol{\sigma}) = \mathring{J}\operatorname{div}_{\mathbf{g}}\left(\nabla^{\mathbf{g}}\mathbf{u}\cdot\mathring{\boldsymbol{\sigma}}\right) + \mathring{J}\mathring{\boldsymbol{\sigma}}\cdot\mathsf{Ric}_{\mathbf{g}}\cdot\mathbf{u} + \mathring{J}\operatorname{div}_{\mathbf{g}}\left(\mathbb{C}\!:\!\boldsymbol{\epsilon}\right) - (\boldsymbol{\epsilon}\!:\!\mathbf{g}^\sharp)\mathring{J}\operatorname{div}_{\mathbf{g}}\mathring{\boldsymbol{\sigma}}\,. \tag{7.89}$$

We next use the Piola identity and rewrite the divergence terms with respect to the reference configuration. Note that

$$\begin{aligned}\mathring{J}\,\mathrm{div}_{\mathbf{g}}\left(\nabla^{\mathbf{g}}\mathbf{u}\cdot\mathring{\boldsymbol{\sigma}}\right) &= \mathring{J}\left(u^a{}_{|b}\mathring{\sigma}^{bc}\right)_{|c}\frac{\partial}{\partial x^a} = \left[u^a{}_{|b}\mathring{J}(\mathring{F}^{-1})^B{}_c\mathring{\sigma}^{bc}\right]_{|B}\frac{\partial}{\partial x^a} \\ &= \left(u^a{}_{|b}\mathring{P}^{bB}\right)_{|B}\frac{\partial}{\partial x^a} = \mathrm{Div}(\nabla^{\mathbf{g}}\mathbf{U}\cdot\mathring{\mathbf{P}})\,.\end{aligned} \tag{7.90}$$

The last term in (7.89) is simplified to read $-(\boldsymbol{\epsilon}:\mathbf{g}^\sharp)\,\mathrm{Div}\,\mathring{\mathbf{P}}$. The second term on the right-hand side of (7.89) is simplified as $\mathring{\mathbf{F}}\mathring{\mathbf{P}}^\star\cdot\mathbf{Ric}_{\mathbf{g}}\circ\mathring{\varphi}\cdot\mathbf{U}$. The second divergence term is simplified using the Piola identity as $\mathring{J}\,\mathrm{div}_{\mathbf{g}}\left(\mathbb{C}:\boldsymbol{\epsilon}\right) = \mathring{J}\left(\mathbb{C}^{abcd}u_{c|d}\right)_{|b}\frac{\partial}{\partial x^a} = \left[\mathring{J}(\mathring{F}^{-1})^B{}_b\mathbb{C}^{abcd}u_{c|d}\right]_{|B}\frac{\partial}{\partial x^a}$. From (7.60), we know that $\mathring{J}(\mathring{F}^{-1})^B{}_b\mathbb{C}^{abcd} = F^a{}_MF^c{}_PF^d{}_Q\mathsf{C}^{BMPQ} = \mathsf{A}^{aBcQ}F^d{}_Q - \mathring{P}^{dB}g^{ac}$. Thus

$$\left[\mathring{J}(\mathring{F}^{-1})^B{}_b\mathbb{C}^{abcd}u_{c|d}\right]_{|B} = \left(\mathsf{A}^{aBcQ}F^d{}_Qu_{c|d}\right)_{|B} - \left(\mathring{P}^{dB}g^{ac}u_{c|d}\right)_{|B} = \left(\mathsf{A}^{aBcQ}u_{c|Q}\right)_{|B} - \left(\mathring{P}^{dB}u^a{}_{|d}\right)_{|B}\,. \tag{7.91}$$

Hence, $\mathring{J}\,\mathrm{div}_{\mathbf{g}}\left(\mathbb{C}:\boldsymbol{\epsilon}\right) = \mathrm{Div}(\mathbf{A}:\nabla\mathbf{U}) - \mathrm{Div}(\nabla\mathbf{U}\cdot\mathring{\mathbf{P}})$. Therefore, $\delta(\mathrm{Div}\,\mathbf{P}) = \mathrm{Div}(\mathbf{A}:\nabla\mathbf{U}) + \mathring{\mathbf{F}}\mathring{\mathbf{P}}^\star\cdot\mathbf{Ric}_{\mathbf{g}}\circ\mathring{\varphi}\cdot\mathbf{U}$. In summary, the linearized material balance of linear momentum reads

$$\mathrm{Div}(\mathbf{A}:\nabla\mathbf{U}) + \mathring{\mathbf{F}}\mathring{\mathbf{P}}^\star\cdot\mathbf{Ric}_{\mathbf{g}}\circ\mathring{\varphi}\cdot\mathbf{U} + \rho_0\nabla_{\mathbf{U}}\mathring{\mathbf{B}} = \rho_0\left[\ddot{\mathbf{U}} + \nabla_{[\mathbf{U},\mathring{\mathbf{V}}]}\mathring{\mathbf{V}} + \boldsymbol{\mathcal{R}}_{\mathbf{g}}\left(\mathbf{U},\dot{\mathbf{U}},\mathring{\mathbf{V}}\right)\right]. \tag{7.92}$$

Linearization of the balance of angular momentum. In terms of the first Piola-Kirchhoff stress, the balance of angular momentum in coordinates reads $P^{aA}F^b{}_A = P^{bA}F^a{}_A$. Its linearization reads $\delta P^{aA}\mathring{F}^b{}_A + \mathring{P}^{aA}U^b{}_{|A} = \delta P^{bA}\mathring{F}^a{}_A + \mathring{P}^{bA}U^a{}_{|A}$. Or

$$\delta\mathbf{P}\cdot\mathring{\mathbf{F}}^\star + \mathring{\mathbf{P}}\cdot(\nabla\mathbf{U})^\star = \mathring{\mathbf{F}}\cdot\delta\mathbf{P}^\star + \nabla\mathbf{U}\cdot\mathring{\mathbf{P}}^\star. \tag{7.93}$$

In terms of the second Piola-Kirchhoff stress, $\delta\mathbf{S} = \mathbf{C}:\mathring{\varphi}^*\boldsymbol{\epsilon}$, the balance of angular momentum is equivalent to $\delta\mathbf{S}^\star = \delta\mathbf{S}$, or $\mathsf{C}^{ABCD} = \mathsf{C}^{BACD}$. In terms of the first elasticity tensor, the balance of angular momentum reads

$$\mathsf{A}^{[aA}{}_m{}^MU^m{}_{|M}\mathring{F}^{b]}{}_A + \mathring{P}^{[aA}U^{b]}{}_{|A} = 0. \tag{7.94}$$

We next discuss the invariance of the governing equations of linear elasticity under both time-dependent spatial and time-independent referential changes of coordinates.

7.3.1 Spatial Covariance of Linearized Elasticity

Consider a spatial change of frame (or a coordinate transformation in the current configuration) $\xi_t : \mathcal{S} \to \mathcal{S}$. Under this change of frame, the one-parameter family of deformations $\varphi_{t,\epsilon}$ is transformed to $\varphi'_{t,\epsilon} = \xi_t \circ \varphi_{t,\epsilon} : \mathcal{B} \to \mathcal{S}$. We next study the effect of this change of frame on all the fields and governing equations of linear elasticity. The transformed variation field is defined as

$$\delta\varphi'_t(X) = \mathbf{U}'(X,t) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \xi_t \circ \varphi_{t,\epsilon}(X) = T\xi_t \cdot \delta\varphi_t = \overset{\xi}{\mathbf{F}} \cdot \mathbf{U}(X,t). \tag{7.95}$$

Transformation of the linearized velocity, acceleration, and the deformation gradient. Linearization of the material velocity with respect to the new spatial frame is calculated as $\delta\mathbf{V}'(X,t) = \nabla^{\mathbf{g}'}_{\xi_* \frac{\partial}{\partial\epsilon}} \frac{\partial\varphi'_{t,\epsilon}(X)}{\partial t}\Big|_{\epsilon=0}$, where $\mathbf{g}' = \xi_*\mathbf{g}$. Note that

$$\frac{\partial\varphi'_{t,\epsilon}(X)}{\partial t} = \frac{\partial}{\partial t}\xi_t \circ \varphi_{t,\epsilon}(X) = T\xi_t \cdot \frac{\partial\varphi_{t,\epsilon}(X)}{\partial t} + \frac{\partial\xi_t}{\partial t} \circ \varphi_{t,\epsilon}(X) = \xi_* \frac{\partial\varphi_{t,\epsilon}(X)}{\partial t} + \mathbf{w}_t \circ \varphi_{t,\epsilon}(X), \tag{7.96}$$

where $\mathbf{w}_t$ is the velocity of the change of frame. Thus

$$\begin{aligned} \delta\mathbf{V}'(X,t) &= \left(\nabla_{\xi_* \frac{\partial}{\partial\epsilon}} \xi_* \frac{\partial\varphi_{t,\epsilon}(X)}{\partial t} + \nabla_{\xi_* \frac{\partial}{\partial\epsilon}} \mathbf{w}_t \circ \varphi_{t,\epsilon}(X)\right)\Big|_{\epsilon=0} \\ &= \xi_* \left(\nabla_{\frac{\partial}{\partial\epsilon}} \frac{\partial\varphi_{t,\epsilon}(X)}{\partial t} + \nabla_{\mathbf{U}}\mathbf{W}_t \circ \varphi_{t,\epsilon}(X)\right)\Big|_{\epsilon=0}, \end{aligned} \tag{7.97}$$

where $\mathbf{W}_t = \xi_t^*\mathbf{w}_t$. Therefore, using (7.47)

$$\begin{aligned} \delta\mathbf{V}'(X,t) &= \xi_* \left(\delta\mathbf{V}(X,t) + \nabla_{\mathbf{U}}\mathbf{W}_t \circ \mathring{\varphi}_t(X)\right) \\ &= \xi_* \left(D_t\mathbf{U}(X,t) + \nabla_{\mathbf{U}}\mathbf{W}_t \circ \mathring{\varphi}_t(X)\right) \\ &= \xi_* \left(\dot{\mathbf{U}}(X,t) + \nabla_{\mathbf{U}}\mathbf{W}_t \circ \mathring{\varphi}_t(X)\right). \end{aligned} \tag{7.98}$$

The transformed linearized acceleration is calculated as

$$\begin{aligned}\delta\mathbf{A}'(X,t) &= \nabla_{\mathring{\mathbf{V}}'}\mathring{\mathbf{V}}' = \nabla_{\xi_*\mathring{\mathbf{V}}+\mathbf{w}\circ\varphi_t}(\xi_*\mathring{\mathbf{V}}+\mathbf{w}\circ\varphi_t)\\ &= \xi_*\left\{\nabla_{\mathring{\mathbf{V}}+\mathbf{W}\circ\varphi_t}(\mathring{\mathbf{V}}+\mathbf{W}\circ\varphi_t)\right\}\\ &= \xi_*\delta\mathbf{A}(X,t)+\xi_*\left(\nabla_{\mathring{\mathbf{V}}}\mathbf{W}\circ\varphi_t+\nabla_{\mathbf{W}\circ\varphi_t}\mathring{\mathbf{V}}+\nabla_{\mathbf{W}\circ\varphi_t}\mathbf{W}\circ\varphi_t\right).\end{aligned} \tag{7.99}$$

We assume that body force is transformed such that $\mathbf{A}'(X,t)-\mathbf{B}'(X,t)=\xi_{t*}(\mathbf{A}(X,t)-\mathbf{B}(X,t))$ [43]. Therefore, $\delta\mathbf{A}(X,t)-\delta\mathbf{B}(X,t)=\xi_{t*}(\delta\mathbf{A}(X,t)-\delta\mathbf{B}(X,t))$. Note that $\mathbf{F}'_{t,\epsilon}=T\varphi'_{t,\epsilon}=T(\xi_t\circ\varphi_{t,\epsilon})=T\xi_t\cdot T\varphi_{t,\epsilon}=\xi_{t*}T\varphi_{t,\epsilon}$. Therefore

$$\delta\mathbf{F}'=\nabla_{\xi_*\frac{\partial}{\partial\epsilon}}\xi_{t*}T\varphi_{t,\epsilon}\Big|_{\epsilon=0}=\xi_*\nabla_{\frac{\partial}{\partial\epsilon}}T\varphi_{t,\epsilon}\Big|_{\epsilon=0}=\xi_*\delta\mathbf{F}=\xi_*\nabla\mathbf{U}=\overset{\xi}{\mathbf{F}}\cdot\nabla\mathbf{U}. \tag{7.100}$$

Transformation of the linearized first and second Piola-Kirchhoff, and Cauchy stresses. The linearized first and second Piola-Kirchhoff stresses are written as $\delta\mathbf{P}=\mathsf{A}:\nabla^{\mathbf{g}}\mathbf{U}$, and $\delta\mathbf{S}=\mathsf{C}:\mathring{\varphi}^*\boldsymbol{\epsilon}$. We know that both A and C are tensors and under $\xi_t:\mathcal{S}\to\mathcal{S}$ are transformed as

$$\mathsf{A}'=\xi_{t*}\mathsf{A},\quad \mathsf{A}'^{a'A}{}_{b'}{}^{B}=\overset{\xi}{F}{}^{a'}{}_{a}(\overset{\xi}{F}{}^{-1})^{b}{}_{b'}\mathsf{A}^{aA}{}_{bB},\quad \mathsf{C}'=\mathsf{C}. \tag{7.101}$$

Therefore

$$\delta\mathbf{P}'=\xi_{t*}\mathsf{A}:\nabla^{\xi_{t*}\mathbf{g}}\xi_{t*}\mathbf{U}=\xi_{t*}\left(\mathsf{A}:\nabla\mathbf{U}\right)=\xi_{t*}\mathbf{P},\quad \delta\mathbf{S}'=\delta\mathbf{S}. \tag{7.102}$$

In the case of Cauchy stress, $\delta\boldsymbol{\sigma}=\mathbb{C}:\boldsymbol{\epsilon}-(\mathrm{div}_{\mathbf{g}}\,\mathbf{u})\mathring{\boldsymbol{\sigma}}$, and hence

$$\delta\boldsymbol{\sigma}'=\mathbb{C}':\boldsymbol{\epsilon}'-(\mathrm{div}_{\mathbf{g}'}\,\mathbf{u}')\mathring{\boldsymbol{\sigma}}'=\xi_{t*}\mathbb{C}:\xi_{t*}\boldsymbol{\epsilon}-(\mathrm{div}_{\xi_{t*}\mathbf{g}}\,\xi_{t*}\mathbf{u})\xi_{t*}\mathring{\boldsymbol{\sigma}}=\xi_{t*}\left[\mathbb{C}:\boldsymbol{\epsilon}-(\mathrm{div}_{\mathbf{g}}\,\mathbf{u})\mathring{\boldsymbol{\sigma}}\right]=\xi_{t*}\delta\boldsymbol{\sigma}. \tag{7.103}$$

Note that the elasticity tensor $\mathbb{C}$ is transformed as $\mathbb{C}'^{a'b'c'd'}=\overset{\xi}{F}{}^{a'}{}_{a}\overset{\xi}{F}{}^{b'}{}_{b}\overset{\xi}{F}{}^{c'}{}_{c}\overset{\xi}{F}{}^{d'}{}_{d}\,\mathbb{C}^{abcd}$.

Transformation of the linearized balance of linear momentum. The linearized spatial balance of linear momentum reads $\mathrm{div}_{\mathbf{g}}\left(\mathbb{C}:\boldsymbol{\epsilon}\right)+\mathrm{div}_{\mathbf{g}}\left(\nabla^{\mathbf{g}}\mathbf{u}\cdot\mathring{\boldsymbol{\sigma}}\right)+\mathring{\boldsymbol{\sigma}}\cdot\mathsf{Ric}_{\mathbf{g}}\cdot\mathbf{u}+\mathring{\rho}(\delta\mathbf{b}-\delta\mathbf{a})=\mathbf{0}$. Therefore, in the new spatial frame it reads

$$
\begin{aligned}
&\mathrm{div}_{\mathbf{g}'}\left(\mathbb{C}'\!:\!\boldsymbol{\epsilon}'\right)+\mathrm{div}_{\mathbf{g}'}\left(\nabla^{\mathbf{g}'}\mathbf{u}'\cdot\mathring{\boldsymbol{\sigma}}'\right)+\mathring{\boldsymbol{\sigma}}'\cdot\mathsf{Ric}'_{\mathbf{g}'}\cdot\mathbf{u}'+\mathring{\rho}'(\delta\mathbf{b}'-\delta\mathbf{a}')\\
&=\mathrm{div}_{\mathbf{g}'}\left(\mathbb{C}'\!:\!\boldsymbol{\epsilon}'\right)+\mathrm{div}_{\xi_*\mathbf{g}}\left(\nabla^{\xi_*\mathbf{g}}\xi_*\mathbf{u}\cdot\xi_*\mathring{\boldsymbol{\sigma}}\right)+\xi_*\mathring{\boldsymbol{\sigma}}\cdot\xi_*\mathsf{Ric}_{\mathbf{g}}\cdot\xi_*\mathbf{u}+(\mathring{\rho}\circ\xi)\xi_*(\delta\mathbf{b}-\delta\mathbf{a})\\
&=\xi_*\left[\mathrm{div}_{\mathbf{g}}\left(\mathbb{C}\!:\!\boldsymbol{\epsilon}\right)+\mathrm{div}_{\mathbf{g}}\left(\nabla^{\mathbf{g}}\mathbf{u}\cdot\mathring{\boldsymbol{\sigma}}\right)+\mathring{\boldsymbol{\sigma}}\cdot\mathsf{Ric}_{\mathbf{g}}\cdot\mathbf{u}+\mathring{\rho}(\delta\mathbf{b}-\delta\mathbf{a})\right]=\mathbf{0},
\end{aligned}
\tag{7.104}
$$

i.e., the linearized balance of linear momentum is spatially covariant.

The linearized material balance of linear momentum reads $\mathrm{Div}(\mathsf{A}\!:\!\nabla_0^{\mathbf{g}}\mathbf{U})+\mathring{\mathbf{F}}\mathring{\mathbf{P}}^{\star}\cdot\mathsf{Ric}_{\mathbf{g}}\circ\mathring{\varphi}\cdot\mathbf{U}+\rho_0(\delta\mathbf{B}-\delta\mathbf{A})=\mathbf{0}$. In the new spatial frame this reads

$$
\begin{aligned}
&\mathrm{Div}'(\mathsf{A}'\!:\!\nabla\mathbf{U}')+\mathring{\mathbf{F}}'\mathring{\mathbf{P}}'^{\star}\cdot\mathsf{Ric}'_{\mathbf{g}'}\circ\mathring{\varphi}'\cdot\mathbf{U}'+\rho_0'(\delta\mathbf{B}'-\delta\mathbf{A}')\\
&=\xi_*\left[\mathrm{Div}(\mathsf{A}\!:\!\nabla\mathbf{U})+\mathring{\mathbf{F}}\mathring{\mathbf{P}}^{\star}\cdot\mathsf{Ric}_{\mathbf{g}}\circ\mathring{\varphi}\cdot\mathbf{U}+\rho_0(\delta\mathbf{B}-\delta\mathbf{A})\right]=\mathbf{0}.
\end{aligned}
\tag{7.105}
$$

The proofs of the spatial covariance of the balance of angular momentum and conservation of mass are straightforward.

7.3.2 Material Covariance of Linearized Elasticity

In this section, we find the transformations of the fields and governing equations of linearized elasticity under an arbitrary time-independent material diffeomorphism (change of coordinates in the reference configuration). Consider a referential change of frame (or a coordinate transformation in the reference configuration) $\Xi:\mathcal{B}\to\tilde{\mathcal{B}}$. Under this change of frame, the one-parameter family of deformations $\varphi_{t,\epsilon}$ is transformed to $\tilde{\varphi}_{t,\epsilon}=\varphi_{t,\epsilon}\circ\Xi^{-1}:\mathcal{B}\to\mathcal{S}$. The transformed variation field is defined as

$$
\delta\tilde{\varphi}_t(\tilde{X})=\tilde{\mathbf{U}}(\tilde{X},t)=\frac{d}{d\epsilon}\Big|_{\epsilon=0}\varphi_{t,\epsilon}(\Xi^{-1}(\tilde{X}))=\delta\varphi_t\circ\Xi^{-1}(\tilde{X})=\mathbf{U}(\Xi^{-1}(\tilde{X}),t).
\tag{7.106}
$$

Hence, $\delta\tilde{\varphi}_t=\delta\varphi_t\circ\Xi^{-1}=\mathbf{U}\circ\Xi^{-1}$.

Transformation of the linearized velocity, acceleration, and deformation gradient. Linearization of the material velocity with respect to the new reference configuration is calculated as

$$\delta\tilde{\mathbf{V}}(\tilde{X},t)=\nabla_{\frac{\partial}{\partial\epsilon}}\frac{\partial\tilde{\varphi}_{t,\epsilon}(\tilde{X})}{\partial t}\Big|_{\epsilon=0}=\nabla_{\frac{\partial}{\partial\epsilon}}\frac{\partial\varphi_{t,\epsilon}(\Xi^{-1}(\tilde{X}))}{\partial t}\Big|_{\epsilon=0}=\delta\mathbf{V}(\Xi^{-1}(\tilde{X}),t). \tag{7.107}$$

Thus, $\delta\tilde{\mathbf{V}}=\delta\mathbf{V}\circ\Xi^{-1}=\dot{\mathbf{U}}\circ\Xi^{-1}$.

Linearization of the material acceleration with respect to the new reference configuration is calculated as

$$\begin{aligned}\delta\tilde{\mathbf{A}}(\tilde{X},t)&=\nabla_{\frac{\partial}{\partial\epsilon}}\nabla_{\frac{\partial}{\partial t}}\frac{\partial\tilde{\varphi}_{t,\epsilon}(\tilde{X})}{\partial t}\Big|_{\epsilon=0}\\&=\nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial\epsilon}}\frac{\partial\varphi_{t,\epsilon}(\Xi^{-1}(\tilde{X}))}{\partial t}\Big|_{\epsilon=0}+\nabla_{[\frac{\partial}{\partial\epsilon},\frac{\partial}{\partial t}]}\frac{\partial\varphi_{t,\epsilon}(\Xi^{-1}(\tilde{X}))}{\partial t}\Big|_{\epsilon=0}+\boldsymbol{\mathcal{R}}_{\mathbf{g}}\left(\frac{\partial}{\partial\epsilon},\frac{\partial}{\partial t}\right)\frac{\partial\varphi_{t,\epsilon}(\Xi^{-1}(\tilde{X}))}{\partial t}\Big|_{\epsilon=0}\\&=\nabla_{\frac{\partial}{\partial t}}\nabla_{\frac{\partial}{\partial t}}\frac{\partial\varphi_{t,\epsilon}(\Xi^{-1}(\tilde{X}))}{\partial\epsilon}\Big|_{\epsilon=0}+\nabla_{[\tilde{\mathbf{U}},\tilde{\mathring{\mathbf{V}}}]}\tilde{\mathring{\mathbf{V}}}+\boldsymbol{\mathcal{R}}_{\mathbf{g}}\left(\tilde{\mathbf{U}},\tilde{\mathring{\mathbf{V}}},\tilde{\mathring{\mathbf{V}}}\right)\\&=\delta\mathbf{A}(\Xi^{-1}(\tilde{X}),t),\end{aligned} \tag{7.108}$$

i.e., $\delta\tilde{\mathbf{A}}=\delta\mathbf{A}\circ\Xi^{-1}$.

Deformation gradient for the motion $\tilde{\varphi}_{t,\epsilon}$ is written as $\tilde{\mathbf{F}}_{t,\epsilon}=T\tilde{\varphi}_{t,\epsilon}=T(\varphi_{t,\epsilon}\circ\Xi^{-1})=T\varphi_{t,\epsilon}\circ(T\Xi)^{-1}=\Xi_{t*}T\varphi_{t,\epsilon}$. Therefore

$$\delta\tilde{\mathbf{F}}=\nabla_{\Xi_*\frac{\partial}{\partial\epsilon}}\Xi_{t*}T\varphi_{t,\epsilon}\Big|_{\epsilon=0}=\Xi_*\nabla_{\frac{\partial}{\partial\epsilon}}T\varphi_{t,\epsilon}\Big|_{\epsilon=0}=\Xi_*\delta\mathbf{F}=\Xi_*\nabla\mathbf{U}. \tag{7.109}$$

In components, $(\delta\tilde{F})^a{}_{\tilde{A}}=(\overset{\Xi}{F}{}^{-1})^A{}_{\tilde{A}}U^a{}_{|A}$.

Transformation of the linearized first Piola-Kirchhoff stress and the first elasticity tensor. With respect to the new reference configuration (cf. (7.19))

$$\tilde{W}_\epsilon=W(\tilde{X},\tilde{\mathbf{F}}_\epsilon,\tilde{\mathbf{G}},\mathbf{g}\circ\tilde{\varphi}_\epsilon,\tilde{\boldsymbol{\zeta}}_1,\ldots,\tilde{\boldsymbol{\zeta}}_n)=W(X,\Xi_*\mathbf{F}_\epsilon,\Xi_*\mathbf{G},\mathbf{g}\circ\varphi_\epsilon\circ\Xi^{-1},\Xi_*\boldsymbol{\zeta}_1,\ldots,\Xi_*\boldsymbol{\zeta}_n)=\Xi_*W_\epsilon. \tag{7.110}$$

The above relation, in particular, implies that $\delta\tilde{W} = \Xi_*\delta W$. Note that $\delta W = \frac{\partial W}{\partial \mathbf{F}} \cdot \delta\mathbf{F}$. Similarly

$$\delta\tilde{W} = \frac{\partial\tilde{W}}{\partial\tilde{\mathbf{F}}} \cdot \delta\tilde{\mathbf{F}} = \frac{\partial\tilde{W}}{\partial\tilde{\mathbf{F}}} \cdot \Xi_*\delta\mathbf{F} = \Xi_*\left(\Xi^*\frac{\partial\tilde{W}}{\partial\tilde{\mathbf{F}}} \cdot \delta\mathbf{F}\right). \tag{7.111}$$

Hence

$$\Xi^*\frac{\partial\tilde{W}}{\partial\tilde{\mathbf{F}}} \cdot \nabla\mathbf{U} = \frac{\partial W}{\partial\mathbf{F}} \cdot \nabla\mathbf{U}, \quad \forall\mathbf{U}. \tag{7.112}$$

This implies that

$$\frac{\partial\tilde{W}}{\partial\tilde{\mathbf{F}}} = \Xi_*\frac{\partial W}{\partial\mathbf{F}}. \tag{7.113}$$

Therefore

$$\delta\tilde{\mathbf{P}} = \mathbf{g}^\sharp\nabla_{\frac{\partial}{\partial\epsilon}}\frac{\partial\tilde{W}_\epsilon}{\partial\tilde{\mathbf{F}}_\epsilon}\bigg|_{\epsilon=0} = \mathbf{g}^\sharp\nabla_{\frac{\partial}{\partial\epsilon}}\Xi_*\frac{\partial W_\epsilon}{\partial\mathbf{F}_\epsilon}\bigg|_{\epsilon=0} = \Xi_*\left(\mathbf{g}^\sharp\frac{\partial^2 W}{\partial\mathring{\mathbf{F}}\partial\mathring{\mathbf{F}}} \cdot \nabla\mathbf{U}\right). \tag{7.114}$$

Hence, $\delta\tilde{\mathbf{P}} = \tilde{\mathsf{A}} \cdot \nabla\tilde{\mathrm{U}} = \Xi_*\mathsf{A} \cdot \Xi_*(\nabla\mathrm{U})$. In particular, the first elasticity tensor is transformed as

$$\tilde{\mathsf{A}}^{a\tilde{A}}{}_{b}{}^{\tilde{B}} = \overset{\Xi}{\bar{F}}{}^{\tilde{A}}{}_{A}\overset{\Xi}{\bar{F}}{}^{\tilde{B}}{}_{B}\,\mathsf{A}^{aA}{}_{b}{}^{B}. \tag{7.115}$$

Transformation of the second Piola-Kirchhoff stress and the second elasticity tensor. Note that the second elasticity tensor for hyperelastic solids (for which an energy function exists) is defined in components as $\mathsf{C}^{ABCD} = 4\frac{\partial^2 W}{\partial C_{AB}\partial C_{CD}}$. Under a referential change of coordinates C is transformed to $\tilde{\mathsf{C}} = \Xi_*\mathsf{C}$, which in components reads

$$\tilde{\mathsf{C}}^{\tilde{A}\tilde{B}\tilde{C}\tilde{D}} = \overset{\Xi}{\bar{F}}{}^{\tilde{A}}{}_{A}\overset{\Xi}{\bar{F}}{}^{\tilde{B}}{}_{B}\overset{\Xi}{\bar{F}}{}^{\tilde{C}}{}_{C}\overset{\Xi}{\bar{F}}{}^{\tilde{D}}{}_{D}\,\mathsf{C}^{ABCD}\circ\Xi^{-1}. \tag{7.116}$$

It then immediately follows that if C possesses the minor ($\mathsf{C}^{ABCD} = \mathsf{C}^{BACD} = \mathsf{C}^{ABDC}$) and major ($\mathsf{C}^{ABCD} = \mathsf{C}^{CDAB}$) symmetries, so does $\tilde{\mathsf{C}}$, i.e., the major and minor symmetries of the elasticity tensor are preserved under a material diffeomorphism. The symmetry group $\mathrm{Sym}_{\mathbf{G}}(\mathbf{T})$ of an m-order tensor field $\mathbf{T}$ is the subgroup of $\mathbf{G}$-orthogonal transformations defined as

$$\langle\mathbf{Q}\rangle_m\,\mathbf{T} = \mathbf{T}, \quad \forall\,\mathbf{Q}\in\mathrm{Sym}_{\mathbf{G}}(\mathbf{T}) \leqslant \mathrm{Orth}\,(\mathbf{G}). \tag{7.117}$$

The space of elasticity tensors $\mathbb{E}\mathrm{la}$ (consisting of all those fourth-order tensors satisfying the major and minor symmetries) may be endowed with an equivalence relation such that

$$\mathsf{C}_1 \sim \mathsf{C}_2 \iff \exists \mathbf{Q} \in \mathrm{Orth}\,(\mathbf{G}): \quad \langle \mathbf{Q} \rangle_4 \,\mathsf{C}_2 = \mathsf{C}_1\,. \tag{7.118}$$

Equivalently[17]

$$\mathsf{C}_1 \sim \mathsf{C}_2 \iff \exists \mathbf{Q} \in \mathrm{Orth}\,(\mathbf{G}): \quad \mathrm{Sym}_{\mathbf{G}}(\mathsf{C}_1) = \mathbf{Q}\, \mathrm{Sym}_{\mathbf{G}}(\mathsf{C}_2)\, \mathbf{Q}^{\mathsf{T}}\,. \tag{7.119}$$

That is, two elasticity tensors are equivalent if and only if their symmetry groups are conjugate subgroups of $SO(3)$. Thus, by an abuse of notation, one may write $\mathsf{C}_1 \sim \mathsf{C}_2$ if and only if $\mathrm{Sym}_{\mathbf{G}}(\mathsf{C}_1) \sim \mathrm{Sym}_{\mathbf{G}}(\mathsf{C}_2)$, where use was made of the fact that conjugacy is also an equivalence relation. Under this equivalence relation, $\mathbb{E}\mathrm{la}$ is divided into eight equivalence classes, known as *symmetry classes*, namely triclinic, monoclinic, trigonal, orthotropic, tetragonal, cubic, transversely isotropic, and isotropic (for proofs,[18] see [220, 221]). We note that under the material diffeomorphsim Ξ the elasticity tensor is transformed such that $\widetilde{\mathring{\mathsf{C}}} = \Xi_* \mathring{\mathsf{C}}$, where $\mathring{\mathsf{C}}$ is the linearized elasticity tensor with respect to the reference motion $\mathring{\varphi}_t$. Therefore, from (7.116), it follows that $\widetilde{\mathring{\mathsf{C}}} = \langle \bar{\bar{\bar{\mathbf{F}}}} \rangle_4 (\mathring{\mathsf{C}} \circ \Xi^{-1})$. It is straightforward to verify that [53]

$$\mathbf{Q} \in \mathrm{Orth}(\mathbf{G}) \iff \Xi_* \mathbf{Q} = \bar{\bar{\bar{\mathbf{F}}}} \mathbf{Q} \bar{\bar{\bar{\mathbf{F}}}}^{-1} \in \mathrm{Orth}(\tilde{\mathbf{G}})\,, \tag{7.120}$$

where $\tilde{\mathbf{G}} = \Xi_* \mathbf{G}$. Using the properties of the Kronecker product (see, e.g., [222]), one concludes that

$$\mathrm{Sym}_{\tilde{\mathbf{G}}}(\widetilde{\mathring{\mathsf{C}}}) = \bar{\bar{\bar{\mathbf{F}}}}\; \mathrm{Sym}_{\mathbf{G}}(\mathring{\mathsf{C}})\, \bar{\bar{\bar{\mathbf{F}}}}^{-1}\,. \tag{7.121}$$

[17] Note that (7.118) and (7.119) imply that two elasticity tensors are equivalent (represent the same type of material anisotropy) if and only if there exists an orthogonal transformation such that under its action the two elasticity tensors (or their symmetry groups) coincide.

[18] Also, see [219] for calculation of the symmetry classes of an even-order tensor space.

In other words, the symmetry groups of $\tilde{\mathring{\mathsf{C}}}$ and $\mathring{\mathsf{C}}$ are conjugate, and hence, isomorphic.[19] It then follows from (7.119) and (7.121) that

$$\begin{aligned}
\mathring{\mathsf{C}}_1 \sim \mathring{\mathsf{C}}_2 \iff & \exists \mathbf{Q} \in \mathrm{Orth}(\mathbf{G}) : \mathrm{Sym}_{\mathbf{G}}(\mathring{\mathsf{C}}_1) = \mathbf{Q}\, \mathrm{Sym}_{\mathbf{G}}(\mathring{\mathsf{C}}_2)\, \mathbf{Q}^{\mathsf{T}} \\
\iff & \exists \tilde{\mathbf{Q}} \in \mathrm{Orth}(\tilde{\mathbf{G}}) : \mathrm{Sym}_{\tilde{\mathbf{G}}}(\tilde{\mathring{\mathsf{C}}}_1) = \tilde{\mathbf{Q}}\, \mathrm{Sym}_{\tilde{\mathbf{G}}}(\tilde{\mathring{\mathsf{C}}}_2)\, \tilde{\mathbf{Q}}^{\mathsf{T}} \\
\iff & \tilde{\mathring{\mathsf{C}}}_1 \sim \tilde{\mathring{\mathsf{C}}}_2 \,,
\end{aligned} \tag{7.122}$$

where $\tilde{\mathbf{Q}} = \Xi_* \mathbf{Q}$. Therefore, $\mathring{\mathsf{C}}_1 \sim \mathring{\mathsf{C}}_2 \iff \tilde{\mathring{\mathsf{C}}}_1 \sim \tilde{\mathring{\mathsf{C}}}_2$, i.e., there is a one-to-one correspondence between the symmetry classes of the elasticity tensors in the initial and transformed reference configurations. It is important to notice that the symmetry of a tensor explicitly depends on the metric. In particular, a tensor, e.g., the elasticity tensor, may belong to a different type of symmetry class with respect to a different metric under consideration. Note that if $\overset{\equiv}{\mathbf{F}}{}^{-\star}\mathbf{G}\overset{\equiv}{\mathbf{F}}{}^{-1} = \mathbf{G}$, i.e., if $\overset{\equiv}{\mathbf{F}}$ is $\mathbf{G}$-orthogonal ($\in \mathrm{Orth}\,(\mathbf{G})$), then the symmetry class is trivially unaffected under a change of reference configuration. Mazzucato and Rachele [132] characterized the orbits of different symmetry classes of the elasticity tensor (i.e., $\psi^*\mathsf{C}$, where ψ is a diffeomorphism in the material manifold) with respect to the Euclidean metric under a change of reference configuration that fixes the boundary to the first-order. In particular, they found that the orbit of isotropic materials consists of some orthotropic and some transversely isotropic materials, in addition to isotropic materials. As they describe the material symmetry groups of C and $\tilde{\mathsf{C}}$ with respect to the Euclidean metric for both configurations, they are implicitly characterizing the symmetry group of the transformed elasticity tensor $\tilde{\mathsf{C}}$ with respect to the untransformed metric $\mathbf{G}$, i.e., $\mathrm{Sym}_{\mathbf{G}}(\tilde{\mathsf{C}})$.

Next, using (7.121) and (7.122), we show that symmetry classes are preserved under a change of reference configuration. In other words, the elasticity tensor $\mathring{\mathsf{C}}$ belongs to a given symmetry class if and only if $\tilde{\mathring{\mathsf{C}}}$ belongs to the same symmetry class in the transformed reference configuration. For instance, let $\mathring{\mathsf{C}}$ belong to the transversely isotropic symmetry class in the initial reference configuration

[19]The isomorphism is trivially given by the conjugacy relations as follows. Let $\phi : \mathrm{Sym}_{\mathbf{G}}(\mathring{\mathsf{C}}) \to \mathrm{Sym}_{\tilde{\mathbf{G}}}(\tilde{\mathring{\mathsf{C}}})$ and let $\mathbf{H} \in \mathrm{Sym}_{\mathbf{G}}(\mathring{\mathsf{C}})$. Then, $\phi(\mathbf{H}) = \overset{\equiv}{\mathbf{F}}\mathbf{H}\overset{\equiv}{\mathbf{F}}{}^{-1} \in \mathrm{Sym}_{\tilde{\mathbf{G}}}(\tilde{\mathring{\mathsf{C}}})$. It is straightforward to see that $\phi(\mathbf{H}_1\mathbf{H}_2) = \phi(\mathbf{H}_1)\phi(\mathbf{H}_2)$ for $\mathbf{H}_1, \mathbf{H}_2 \in \mathrm{Sym}_{\mathbf{G}}(\mathring{\mathsf{C}})$. Also, it is straightforward to see that ϕ is one-to-one, and thus, an isomorphism.

at a material point X with the unit vector $\mathbf{N} \in T_X\mathcal{B}$ identifying the material preferred direction. It follows that the symmetry group of $\mathring{\mathsf{C}}$ is given by

$$\mathrm{Sym}_{\mathbf{G}}(\mathring{\mathsf{C}}) = \Big\{ \mathbf{Q} \in \mathrm{Orth}\,(\mathbf{G}) : \mathbf{Q}\mathbf{N} = \pm\mathbf{N} \Big\}. \tag{7.123}$$

From (7.121)

$$\mathrm{Sym}_{\tilde{\mathbf{G}}}(\tilde{\mathring{\mathsf{C}}}) = \Big\{ \tilde{\mathbf{Q}} \in \mathrm{Orth}(\tilde{\mathbf{G}}) : \tilde{\mathbf{Q}} = \bar{\bar{\mathbf{F}}}\mathbf{Q}\bar{\bar{\mathbf{F}}}^{-1},\ \mathbf{Q} \in \mathrm{Sym}_{\mathbf{G}}(\mathring{\mathsf{C}}) \Big\}. \tag{7.124}$$

The material preferred direction is transformed to $\tilde{\mathbf{N}} = \bar{\bar{\mathbf{F}}}\mathbf{N}$. Clearly, employing (7.123) and (7.124), $\forall \tilde{\mathbf{Q}} \in \mathrm{Sym}_{\tilde{\mathbf{G}}}(\tilde{\mathring{\mathsf{C}}})$ one has $\tilde{\mathbf{Q}}\tilde{\mathbf{N}} = \pm\tilde{\mathbf{N}}$. Moreover, if $\tilde{\mathbf{Q}} \in \mathrm{Orth}(\tilde{\mathbf{G}})$ such that $\tilde{\mathbf{Q}}\tilde{\mathbf{N}} = \pm\tilde{\mathbf{N}}$, then from (7.120), there exists $\mathbf{Q} = \bar{\bar{\mathbf{F}}}^{-1}\tilde{\mathbf{Q}}\bar{\bar{\mathbf{F}}} \in \mathrm{Orth}(\tilde{\mathbf{G}})$, for which one concludes that $\mathbf{Q}\mathbf{N} = \pm\mathbf{N}$, i.e., $\mathbf{Q} \in \mathrm{Sym}_{\mathbf{G}}(\mathring{\mathsf{C}})$. Thus

$$\mathrm{Sym}_{\tilde{\mathbf{G}}}(\tilde{\mathring{\mathsf{C}}}) = \Big\{ \tilde{\mathbf{Q}} \in \mathrm{Orth}(\tilde{\mathbf{G}}) : \tilde{\mathbf{Q}}\tilde{\mathbf{N}} = \pm\tilde{\mathbf{N}} \Big\}, \tag{7.125}$$

and hence, $\tilde{\mathring{\mathsf{C}}}$ belongs to the transversely isotropic symmetry class in the transformed reference configuration. The generalization of this proof to the orthotropic case is immediate. One only needs to consider three $\mathbf{G}$-orthonormal vectors $\mathbf{N}_1$, $\mathbf{N}_2$, and $\mathbf{N}_3$ specifying the orthotropic axes and define the symmetry group in the orthotropic case as $\mathbf{Q} \in \mathrm{Orth}\,(\mathbf{G})$ such that $\mathbf{Q}\mathbf{N}_i = \pm\mathbf{N}_i$, $i = 1, 2, 3$. One can similarly prove that other symmetry classes are preserved under material diffeomorphisms as well. For the isotropic case, it immediately follows from (7.120) and (7.121) that $\mathrm{Sym}_{\mathbf{G}}(\mathring{\mathsf{C}}) = \mathrm{Orth}\,(\mathbf{G}) \iff \mathrm{Sym}_{\tilde{\mathbf{G}}}(\tilde{\mathring{\mathsf{C}}}) = \mathrm{Orth}(\tilde{\mathbf{G}})$. Therefore, we have proved the following result.

Proposition 7.3.2. *The elasticity tensor $\mathring{\mathsf{C}}$ in the reference configuration $(\mathcal{B}, \mathbf{G})$ with respect to the reference motion $\mathring{\varphi} : \mathcal{B} \to \mathcal{S}$ belongs to a given symmetry class if and only if $\tilde{\mathring{\mathsf{C}}}$, which is the elasticity tensor in the transformed reference configuration $(\tilde{\mathcal{B}}, \Xi_*\mathbf{G})$ with respect to the reference motion $\tilde{\mathring{\varphi}} : \tilde{\mathcal{B}} \to \mathcal{S}$, belongs to the same symmetry class.*

Using (7.57), one can write

$$\delta\tilde{\mathbf{S}} = 2\frac{\partial^2 \tilde{W}_\epsilon}{\partial \tilde{\mathbf{C}}^\flat_\epsilon \partial \tilde{\mathbf{C}}^\flat_\epsilon} : \frac{d}{d\epsilon}\tilde{\mathbf{C}}^\flat_\epsilon\bigg|_{\epsilon=0} = \Xi_* \left(4\frac{\partial^2 \hat{W}}{\partial \mathring{\mathbf{C}}^\flat \partial \mathring{\mathbf{C}}^\flat} : \mathring{\varphi}^* \boldsymbol{\epsilon} \right) = \tilde{\mathsf{C}} : \widetilde{\mathring{\varphi}}^* \boldsymbol{\epsilon}\,, \tag{7.126}$$

where $\tilde{\mathsf{C}} = \Xi_* \mathsf{C}$ and $\widetilde{\mathring{\varphi}}^* \boldsymbol{\epsilon} = \Xi_* \circ \mathring{\varphi}^* \boldsymbol{\epsilon} = \Xi_*(\mathring{\varphi}^* \boldsymbol{\epsilon})$, and thus, $\delta\tilde{\mathbf{S}} = \Xi_* \mathsf{C} : \Xi_*(\mathring{\varphi}^* \boldsymbol{\epsilon})$.

Transformation of the linearized balances of linear and angular momenta. The linearized spatial balance of linear momentum with respect to the new reference configuration reads

$$\operatorname{div}_{\tilde{\mathbf{g}}}(\tilde{\mathbb{C}} : \tilde{\boldsymbol{\epsilon}}) + \operatorname{div}_{\tilde{\mathbf{g}}}(\nabla^{\mathbf{g}} \tilde{\mathbf{u}} \cdot \tilde{\mathring{\boldsymbol{\sigma}}}) + \tilde{\mathring{\boldsymbol{\sigma}}} \cdot \widetilde{\mathsf{Ric}}_{\mathbf{g}} \cdot \tilde{\mathbf{u}} + (\mathring{\rho} \nabla^{\mathbf{g}}_{\mathbf{u}} \mathring{\mathbf{b}}) \circ \Xi^{-1} = \mathring{\rho} \left[\ddot{\mathbf{U}} \circ \mathring{\varphi}_t{}^{-1} + \nabla^{\mathbf{g}}_{[\mathbf{u}, \mathring{\mathbf{v}}]} \mathring{\mathbf{v}} + \boldsymbol{\mathcal{R}}_{\mathbf{g}}(\mathbf{u}, \mathring{\mathbf{v}}, \mathring{\mathbf{v}}) \right] \circ \Xi^{-1}\,. \tag{7.127}$$

Note that

$$\tilde{\mathbb{C}} = \frac{1}{\tilde{\mathring{J}}} \tilde{\mathring{\varphi}}_* \tilde{\mathsf{C}} = \frac{1}{\mathring{J}} (\mathring{\varphi} \circ \Xi^{-1})_* \Xi_* \mathsf{C} = \frac{1}{\mathring{J}} \mathring{\varphi}_* \mathsf{C} = \mathbb{C}\,. \tag{7.128}$$

Knowing that all the spatial tensors transform like a scalar, i.e., $\tilde{\boldsymbol{\epsilon}} = \boldsymbol{\epsilon} \circ \Xi^{-1}$, $\tilde{\mathbf{g}} = \mathbf{g} \circ \Xi^{-1}$, $\widetilde{\mathsf{Ric}}_{\mathbf{g}} = \mathsf{Ric}_{\mathbf{g}} \circ \Xi^{-1}$, $\tilde{\mathbf{u}} = \mathbf{u} \circ \Xi^{-1}$, and $\tilde{\mathring{\boldsymbol{\sigma}}} = \mathring{\boldsymbol{\sigma}} \circ \Xi^{-1}$, one concludes that the linearized spatial balance of linear momentum is materially covariant.

Similarly, the linearized material balance of linear momentum transforms like a scalar, i.e.,

$$\begin{aligned} &\widetilde{\operatorname{Div}}(\tilde{\mathsf{A}} : \nabla \tilde{\mathbf{U}}) + \tilde{\mathring{\mathbf{F}}} \tilde{\mathring{\mathbf{P}}}^\star \cdot \widetilde{\mathsf{Ric}}_{\mathbf{g}} \circ \tilde{\mathring{\varphi}} \cdot \tilde{\mathbf{U}} + \tilde{\rho}_0 \nabla_{\tilde{\mathbf{U}}} \tilde{\mathring{\mathbf{B}}} - \tilde{\rho}_0 \left[\ddot{\tilde{\mathbf{U}}} + \nabla_{[\tilde{\mathbf{U}}, \tilde{\mathring{\mathbf{V}}}]} \tilde{\mathring{\mathbf{V}}} + \tilde{\boldsymbol{\mathcal{R}}}_{\mathbf{g}}(\tilde{\mathbf{U}}, \tilde{\mathring{\mathbf{U}}}, \tilde{\mathring{\mathbf{V}}}) \right] \\ &= \left\{ \operatorname{Div}(\mathsf{A} : \nabla \mathbf{U}) + \mathring{\mathbf{F}} \mathring{\mathbf{P}}^\star \cdot \mathsf{Ric}_{\mathbf{g}} \circ \mathring{\varphi} \cdot \mathbf{U} + \rho_0 \nabla_{\mathbf{U}} \mathring{\mathbf{B}} - \rho_0 \left[\ddot{\mathbf{U}} + \nabla_{[\mathbf{U}, \mathring{\mathbf{V}}]} \mathring{\mathbf{V}} + \boldsymbol{\mathcal{R}}_{\mathbf{g}}(\mathbf{U}, \mathring{\mathbf{U}}, \mathring{\mathbf{V}}) \right] \right\} \circ \Xi^{-1} = \mathbf{0}. \end{aligned} \tag{7.129}$$

The balance of angular momentum transforms like a scalar as well.

Wave equation and its spatial and material covariance. Let us assume that $\mathbf{u}(x,t) = \bar{\mathbf{u}}(x) e^{-\mathrm{i}\omega t}$, where ω is the frequency. For a Euclidean ambient space, the wave equation is written as $\operatorname{div}_{\mathbf{g}}(\mathbb{C} : \bar{\boldsymbol{\epsilon}}) + \operatorname{div}_{\mathbf{g}}(\nabla^{\mathbf{g}} \bar{\mathbf{u}} \cdot \mathring{\boldsymbol{\sigma}}) + \mathring{\rho} \nabla^{\mathbf{g}}_{\bar{\mathbf{u}}} \mathring{\mathbf{b}} = \mathring{\rho} \left[-\omega^2 \bar{\mathbf{u}} + \nabla^{\mathbf{g}}_{[\bar{\mathbf{u}}, \mathring{\mathbf{v}}]} \mathring{\mathbf{v}} \right]$, where $2\bar{\boldsymbol{\epsilon}} = \nabla^{\mathbf{g}} \bar{\mathbf{u}}^\flat + (\nabla^{\mathbf{g}} \bar{\mathbf{u}}^\flat)^\star$. Similarly, in material form $\mathbf{U}(X,t) = \bar{\mathbf{U}}(X)e^{-\mathrm{i}\omega t}$, and the wave equation in material form reads

$$\mathrm{Div}(\mathbf{A}\!:\!\nabla\bar{\mathbf{U}}) + \rho_0 \nabla_{\bar{\mathbf{U}}}\mathring{\mathbf{B}} = \rho_0 \left[-\omega^2\bar{\mathbf{U}} + \nabla_{[\bar{\mathbf{U}},\mathring{\mathbf{V}}]}\mathring{\mathbf{V}}\right]. \tag{7.130}$$

If the reference motion is static, i.e., $\mathring{\mathbf{V}} = \mathbf{0}$ and body force is ignored, the spatial and material wave equations are simplified to read

$$\begin{aligned} &\mathrm{div}_{\mathbf{g}}(\mathbb{C}\!:\!\nabla^{\mathbf{g}}\bar{\mathbf{u}}^{\flat}) + \mathrm{div}_{\mathbf{g}}(\nabla^{\mathbf{g}}\bar{\mathbf{u}}\cdot\mathring{\boldsymbol{\sigma}}) + \mathring{\rho}\omega^2\bar{\mathbf{u}} = \mathbf{0}, \\ &\mathrm{Div}(\mathbf{A}\!:\!\nabla\bar{\mathbf{U}}) + \rho_0\omega^2\bar{\mathbf{U}} = \mathbf{0}. \end{aligned} \tag{7.131}$$

Note that as a consequence of the spatial and material covariance of the balance of linear momentum and its linearization, the wave equation is form-invariant under arbitrary spatial coordinate transformations and arbitrary time-independent referential coordinate transformations.

7.4 A Mathematical Formulation of the Problem of Cloaking a Cavity in Nonlinear and Linearized Elastodynamics

Suppose an object is to be hidden from elastic waves. This object lies in a cavity inside an elastic body. The hole needs to be reinforced by a cloak, which is a layer of specially designed inhomogeneous and anisotropic material that will deflect away any incoming elastic waves from the cloaked object. For an observer away from the hole the elastic waves are passing through the body as if there was no hole. Let us assume that the body is homogeneous and is made of an isotropic material. The idea of elastodynamic transformation cloaking is to first map the stress-free body in its reference configuration to a corresponding homogeneous and isotropic body in its stress-free reference configuration. We assume that the homogeneous and isotropic transformed body (virtual body) has a very small hole of radius ϵ ($\epsilon \to 0$). Consider a map that shrinks the hole to a very small hole and is the identity outside the cloak. Note that there are many such mappings. Next, the important requirement is that the physical body with the hole and the homogeneous and isotropic virtual body must have identical current configurations outside the cloak. In other words, to an elastic wave the two bodies are identical outside the cloaking region. Inside the cloak we will impose certain requirements. The last step is to check if these requirements are enough to specify the elastic properties of the cloak, its mass density, and the external loads and the boundary conditions in the virtual body, or if they result in an overdetermined system with no solution. One should also check if all the balance laws are satisfied in both the virtual and physical bodies. To formulate this problem the understanding of the transformation properties of the governing equations of nonlinear and linearized elasticity that we established in the previous two sections is crucial.

7.4.1 Nonlinear Elastodynamic Transformation Cloaking

Let us consider a body $\mathcal{B}$ with a hole $\mathcal{H}$ (see Fig.7.3). An object is placed inside $\mathcal{H}$ and needs to be hidden from elastic waves. The hole is reinforced by a cloak $\mathcal{C}$, which we can assume is an annulus. The elastic properties and mass density of $\mathcal{C}$ are, in general, inhomogeneous and anisotropic. For the sake of simplicity and without loss of generality, let us assume that in $\mathcal{B} \setminus \mathcal{C}$ the body is homogeneous and isotropic. This means that the mass density ρ_0 is a constant and the body has an energy function $W = W(I_1, I_2, I_3)$, where I_i $(i = 1, 2, 3)$ are the principal invariants of the left (or right) Cauchy-Green strain. The motion of $\mathcal{B}$ is represented by a map $\varphi_t : \mathcal{B} \to \mathcal{S}$ in Fig.7.3. A spatial diffeomorphism leaves the governing equations form invariant. However, spatial diffeomorphisms (spatial coordinate transformations) would not be useful in designing a cloak because a change of spatial coordinates corresponds to the same body as seen through a warped lens — but the material is the same. The cloaking transformation is assumed to be a time-independent map $\Xi : \mathcal{B} \to \tilde{\mathcal{B}}$ such that the annulus $\mathcal{C}$ is transformed to a disk with a very small hole (or a spherical ball with a very small hole in 3D). The mapping Ξ is assumed to be the identity in $\mathcal{B} \setminus \mathcal{C}$. It is also assumed that the virtual body has the uniform and isotropic mechanical properties of $\mathcal{B} \setminus \mathcal{C}$.

One may be tempted to think that the cloak can be designed using a referential change of frame (coordinate transformation). We have shown that the governing equations of both nonlinear and linearized elasticity are spatially and referentially covariant. In other words, the governing equations of the same body are invariant under arbitrary time-dependent coordinate transformations in the current

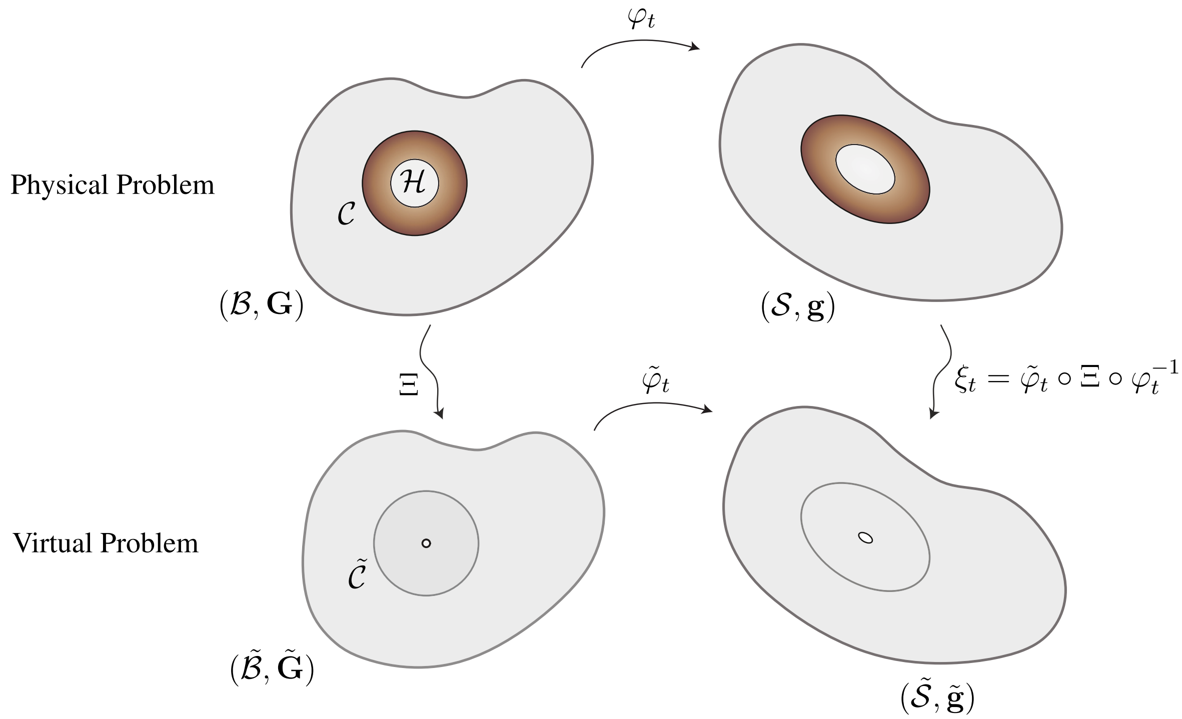


Figure 7.3: *A cloaking transformation Ξ transforms a body with a hole $\mathcal{H}$ to another body with an infinitesimal hole that is homogeneous and isotropic. The cloaking transformation is defined to be the identity map outside the cloak $\mathcal{C}$. Note that Ξ is not a referential change of coordinates and ξ is not a spatial change of coordinates.*

configuration and arbitrary time-independent coordinate transformations in the reference configuration. This, however, is not useful for cloaking applications, because a change of the material frame does not lead to a new elastic body. In other words, the two bodies $(\mathcal{B}, \mathbf{G})$ and $(\Xi(\mathcal{B}), \Xi_* \mathbf{G})$ are isometric and essentially the same elastic body with the same mechanical response. One instead needs two different elastic bodies that cannot be distinguished by an elastic wave in their current configurations outside the cloak. Motion of the virtual (homogeneous and isotropic) body is represented by a map $\tilde{\varphi}_t$ as is schematically shown in Fig.7.3. The spatial configurations of the two bodies with their corresponding deformations φ_t and $\tilde{\varphi}_t$ are required to be identical outside the cloaking region, i.e., in $\mathcal{B} \setminus \mathcal{C}$. In particular, this implies that any elastic measurements made in the current configurations of the two bodies outside the cloak are identical, and hence, the two bodies cannot be distinguished by an observer located in $\mathcal{B} \setminus \mathcal{C}$. Moreover, in the virtual body $\tilde{\mathcal{B}}$, the influence of the hole is negligible as $\mathcal{H}$ in $\mathcal{B}$ is mapped to an infinitesimal hole in $\tilde{\mathcal{B}}$. It is important to notice that our approach does not impose any restrictions on the size of the hole, and thus, that of the concealment as illustrated in

Fig.7.3.

Let us consider a body (physical body) that has a material manifold $(\mathcal{B}, \mathbf{G})$ and is in a time-dependent current configuration $\varphi_t(\mathcal{B})$. Suppose that the stress-free reference configuration of the body is in a one-to-one correspondence with that of another body $\tilde{\mathcal{B}}$ (virtual body) in its reference configuration $(\tilde{\mathcal{B}}, \tilde{\mathbf{G}})$ (see Fig.7.3). Let us denote the bijection between the two bodies by $\Xi : \mathcal{B} \to \tilde{\mathcal{B}}$, which is a diffeomorphism. We assume that the two stress-free bodies are embedded in the Euclidean space, and hence, $\mathbf{G}$ and $\tilde{\mathbf{G}}$ are their corresponding induced Euclidean metrics. This immediately implies that Ξ is not a simple change of material frame (referential coordinate transformation) as $\tilde{\mathbf{G}} \neq \Xi_* \mathbf{G}$, in general. The boundary-value problems related to the motions $\varphi_t : \mathcal{B} \to \mathcal{S}$ and $\tilde{\varphi}_t : \tilde{\mathcal{B}} \to \tilde{\mathcal{S}}$ are called the physical and virtual problems, respectively. The deformation gradients corresponding to the physical and virtual problems are denoted by $\mathbf{F} = T\varphi_t$ and $\tilde{\mathbf{F}} = T\tilde{\varphi}_t$, respectively. The energy function of the physical problem in $\mathcal{B} \setminus \mathcal{C}$ is known, while it is not known a priori in $\mathcal{C}$. The energy function of the virtual body is known everywhere in $\tilde{\mathcal{B}}$. The applied loads (body force and traction boundary conditions) and the essential boundary conditions are given for the physical problem. They are, however, not known a priori for the virtual problem.

Shifters in Euclidean ambient space. We assume that the reference configurations of both the physical and virtual bodies are embedded in the Euclidean space. To relate vector fields in the physical problem to those in the virtual problem we would need to use shifters. We first consider the physical body $\mathcal{B} \subset \mathcal{S} = \mathbb{R}^n$ ($n = 2$ or 3). The map $\mathsf{s} : T\mathcal{S} \to T\mathcal{S}$, $\mathsf{s}(x, \mathbf{w}) = (\tilde{x}, \mathbf{w})$ is called the shifter map (see Fig.7.4). The restriction of s to $x \in \mathcal{S}$ is denoted by $\mathsf{s}_x = \mathsf{s}(x) : T_x\mathcal{S} \to T_{\tilde{x}}\mathcal{S}$, and shifts $\mathbf{w}$ based at $x \in \mathcal{S}$ to $\mathbf{w}$ based at $\tilde{x} \in \mathcal{S}$ (shifter map in the reference configuration is defined similarly). For $\mathcal{S}$ we choose two global colinear Cartesian coordinates $\{\tilde{z}^{\tilde{i}}\}$ and $\{z^i\}$ for the virtual and physical deformed configurations, respectively. We also use curvilinear coordinates $\{\tilde{x}^{\tilde{a}}\}$ and $\{x^a\}$ for these configurations. Note that $\mathsf{s}^{\tilde{i}}{}_i = \delta^{\tilde{i}}_i$. One can show that [43]

$$\mathsf{s}^{\tilde{a}}{}_a(x) = \frac{\partial \tilde{x}^{\tilde{a}}}{\partial \tilde{z}^{\tilde{i}}}(\tilde{x}) \frac{\partial z^i}{\partial x^a}(x) \delta^{\tilde{i}}_i \,. \tag{7.132}$$

Note that s preserves inner products, and hence, $\mathsf{s}^{\mathsf{T}} = \mathsf{s}^{-1}$. In components, $(\mathsf{s}^{\mathsf{T}})^a{}_{\tilde{a}} = g^{ab}\mathsf{s}^{\tilde{b}}{}_b \tilde{g}_{\tilde{a}\tilde{b}}$. Note also that

$$\mathsf{s}^{\tilde{a}}{}_{a|\tilde{b}} = \frac{\partial \mathsf{s}^{\tilde{a}}{}_a}{\partial \tilde{x}^{\tilde{b}}} + \tilde{\gamma}^{\tilde{a}}{}_{\tilde{b}\tilde{c}}\mathsf{s}^{\tilde{c}}{}_a - \frac{\partial x^b}{\partial \tilde{x}^{\tilde{b}}}\gamma^c{}_{ab}\mathsf{s}^{\tilde{a}}{}_c\,, \tag{7.133}$$

where $\gamma^a{}_{bc} = \frac{\partial x^a}{\partial z^k}\frac{\partial^2 z^k}{\partial x^b \partial x^c}$ and $\tilde{\gamma}^{\tilde{a}}{}_{\tilde{b}\tilde{c}} = \frac{\partial \tilde{x}^{\tilde{a}}}{\partial \tilde{z}^{\tilde{k}}}\frac{\partial^2 \tilde{z}^{\tilde{k}}}{\partial \tilde{x}^{\tilde{b}} \partial \tilde{x}^{\tilde{c}}}$. It is easily shown that $\mathsf{s}^{\tilde{a}}{}_{a|\tilde{b}} = 0$, i.e., the shifter is covariantly constant.

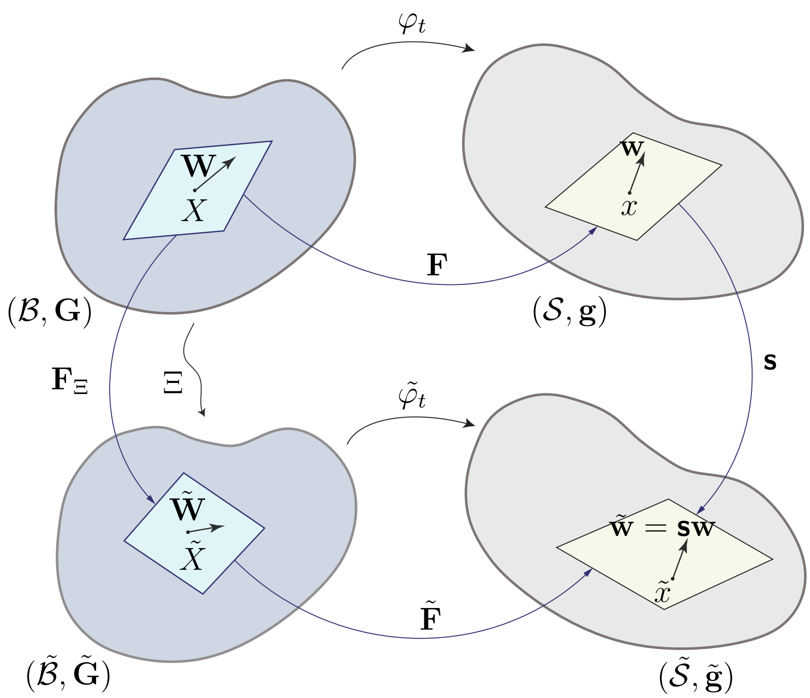


Figure 7.4: *The mapping s is the shifter map, which parallel transports vectors in $T_x\mathcal{S}$ to vectors in $T_{\tilde{x}}\tilde{\mathcal{S}}$.*

Example 7.4.1. Consider cylindrical coordinates (r, θ, z) and $(\tilde{r}, \tilde{\theta}, \tilde{z})$ at $x \in \mathbb{R}^3$ and $\tilde{x} \in \mathbb{R}^3$, respectively. The shifter map has the following matrix representation with respect to these coordinates

$$\mathsf{s} = \begin{bmatrix} \cos(\tilde{\theta} - \theta) & r\sin(\tilde{\theta} - \theta) & 0 \\ -\sin(\tilde{\theta} - \theta)/\tilde{r} & r\cos(\tilde{\theta} - \theta)/\tilde{r} & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{7.134}$$

Example 7.4.2. Consider spherical coordinates (r, θ, ϕ) and $(\tilde{r}, \tilde{\theta}, \tilde{\phi})$ at $x \in \mathbb{R}^3$ and $\tilde{x} \in \mathbb{R}^3$, respectively. The shifter map has the following matrix representation with respect to these coordinates

$$\mathsf{s} = \begin{bmatrix} \cos(\tilde{\phi}-\phi)\sin\tilde{\theta}\sin\theta+\cos\tilde{\theta}\cos\theta & r[\cos(\tilde{\phi}-\phi)\sin\tilde{\theta}\cos\theta-\cos\tilde{\theta}\sin\theta] & r\sin(\tilde{\phi}-\phi)\sin\tilde{\theta}\sin\theta \\ [\cos(\tilde{\phi}-\phi)\cos\tilde{\theta}\sin\theta-\sin\tilde{\theta}\cos\theta]/\tilde{r} & r[\cos(\tilde{\phi}-\phi)\cos\tilde{\theta}\cos\theta+\sin\tilde{\theta}\sin\theta]/\tilde{r} & r\sin(\tilde{\phi}-\phi)\cos\tilde{\theta}\sin\theta/\tilde{r} \\ -\sin(\tilde{\phi}-\phi)\sin\theta/(\tilde{r}\sin\tilde{\theta}) & -r\sin(\tilde{\phi}-\phi)\cos\theta/(\tilde{r}\sin\tilde{\theta}) & r\cos(\tilde{\phi}-\phi)\sin\theta/(\tilde{r}\sin\tilde{\theta}) \end{bmatrix}. \tag{7.135}$$

Balance of linear momentum in the physical and virtual bodies. The balance of linear momentum for the physical body reads: $\operatorname{Div}\mathbf{P} + \rho_0\mathbf{B} = \rho_0\mathbf{A}$. We use the Piola identity and write the divergence term with respect to the reference configuration of the virtual body. The Piola identity [43] tells us that for a vector field $\mathbf{W}$ on $\mathcal{B}$, one can write (see the appendix)

$$\operatorname{Div}\mathbf{W} = J_\Xi\,\widetilde{\operatorname{Div}}\tilde{\mathbf{W}}, \quad \tilde{\mathbf{W}} = J_\Xi^{-1}\overset{\Xi}{\mathbf{F}}\mathbf{W}\,. \tag{7.136}$$

In components, this reads

$$W^A{}_{|A} = J_\Xi\left[J_\Xi^{-1}\overset{\Xi}{F}{}^{\tilde{A}}{}_A W^A\right]_{|\tilde{A}}\,. \tag{7.137}$$

$\tilde{\mathbf{W}}$ is called the Piola transform of $\mathbf{W}$. We use the local coordinate charts $\{X^A\}$ and $\{\tilde{X}^{\tilde{A}}\}$ for $\mathcal{B}$ and $\tilde{\mathcal{B}}$, respectively. Let us start with $\operatorname{Div}\mathbf{P}$ and rewrite it with respect to the reference configuration of the virtual body. In coordinates and using the Piola identity one can write[20]

$$P^{aA}{}_{|A} = J_\Xi\left(J_\Xi^{-1}\overset{\Xi}{F}{}^{\tilde{A}}{}_A P^{aA}\right)_{|\tilde{A}} = J_\Xi(\tilde{P}^{a\tilde{A}}{}_{|\tilde{A}})\circ\Xi\,, \tag{7.139}$$

where

$$\tilde{P}^{a\tilde{A}}\circ\Xi = J_\Xi^{-1}\overset{\Xi}{F}{}^{\tilde{A}}{}_A P^{aA}, \tag{7.140}$$

[20]Note that the Piola identity can be written as

$$\left[J_\Xi^{-1}\overset{\Xi}{F}{}^{\tilde{A}}{}_A\right]_{|\tilde{A}} = 0\,. \tag{7.138}$$

Thus

$$\left[J_\Xi^{-1}\overset{\Xi}{F}{}^{\tilde{A}}{}_A P^{aA}\right]_{|\tilde{A}} = \left[J_\Xi^{-1}\overset{\Xi}{F}{}^{\tilde{A}}{}_A\right]_{|\tilde{A}} P^{aA} + J_\Xi^{-1}\overset{\Xi}{F}{}^{\tilde{A}}{}_A P^{aA}{}_{|\tilde{A}}\,.$$

Therefore, using (7.138) one can write

$$J_\Xi\left[J_\Xi^{-1}\overset{\Xi}{F}{}^{\tilde{A}}{}_A P^{aA}\right]_{|\tilde{A}} = \overset{\Xi}{F}{}^{\tilde{A}}{}_A P^{aA}{}_{|\tilde{A}} = P^{aA}{}_{|A}\,.$$

and J_Ξ is the Jacobian of the *cloaking transformation* Ξ, which is written as

$$J_\Xi = \sqrt{\frac{\det \tilde{\mathbf{G}} \circ \Xi}{\det \mathbf{G}}} \det \bar{\bar{\mathbf{F}}}^{\Xi} . \tag{7.141}$$

However, this is a vector in $T_x\mathcal{S}$. The corresponding vector in $T_{\tilde{x}}\mathcal{S}$ is defined using the shifter $\mathsf{s}(x)$ as

$$\mathsf{s}^{\tilde{a}}{}_a \tilde{P}^{a\tilde{A}} \circ \Xi = J_\Xi^{-1} \mathsf{s}^{\tilde{a}}{}_a P^{aA} \bar{\bar{F}}^{\Xi}{}^{\tilde{A}}{}_A. \tag{7.142}$$

Note that $\mathsf{s}^{\tilde{a}}{}_a P^{aA}{}_{|A} = (\mathsf{s}^{\tilde{a}}{}_a P^{aA})_{|A}$. Therefore

$$\mathsf{s} \circ \varphi \operatorname{Div} \mathbf{P} = J_\Xi (\widetilde{\operatorname{Div}} \tilde{\mathbf{P}}) \circ \Xi \, , \tag{7.143}$$

where

$$\tilde{\mathbf{P}} = J_\Xi^{-1} \mathsf{s} \circ \varphi \mathbf{P} \bar{\bar{\mathbf{F}}}^{\Xi\star}, \text{ or } \quad \tilde{P}^{\tilde{a}\tilde{A}} \circ \Xi = J_\Xi^{-1} \bar{\bar{F}}^{\Xi}{}^{\tilde{A}}{}_A \mathsf{s}^{\tilde{a}}{}_a P^{aA}. \tag{7.144}$$

Equivalently, the transformed second Piola-Kirchhoff stress reads

$$\tilde{\mathbf{S}} \circ \Xi = J_\Xi^{-1} \tilde{\mathbf{F}}^{-1} \circ \tilde{\varphi} \, \mathsf{s} \circ \varphi \mathbf{F} \mathbf{S} \bar{\bar{\mathbf{F}}}^{\Xi\star}. \tag{7.145}$$

Note that force on an infinitesimal area dA is calculated as

$$t^a dA = P^{aA} N_A dA = P^{aA} J_\Xi^{-1} \bar{\bar{F}}^{\Xi}{}^{\tilde{A}}{}_A \tilde{N}_{\tilde{A}} d\tilde{A} = \tilde{P}^{a\tilde{A}} \tilde{N}_{\tilde{A}} d\tilde{A} = \tilde{t}^a d\tilde{A} \, , \tag{7.146}$$

and hence $\mathsf{s}^{\tilde{a}}{}_a t^a dA = \tilde{t}^{\tilde{a}} d\tilde{A}$, i.e., this force is the same in the physical and the virtual bodies if one assumes that the first Piola-Kirchhoff stress in the virtual body is given by (7.144). Equivalently, if one assumes that forces on the corresponding infinitesimal areas in the two bodies are equal, i.e., are related by the shifter map, then the relation between the stresses is given in (7.144).

Remark 7.4.3. Instead of (7.142), i.e., assuming that $\tilde{P}^{\tilde{a}\tilde{A}} = \mathsf{s}^{\tilde{a}}{}_a \tilde{P}^{a\tilde{A}} \circ \Xi$ one may assume that

$$\tilde{P}^{\tilde{a}\tilde{A}} = \mathsf{w}^{\tilde{a}}{}_a \tilde{P}^{a\tilde{A}} \circ \Xi, \tag{7.147}$$

where $\mathsf{w}^{\tilde{a}}{}_a$ are the components of some two-point tensor $\mathbf{w}$. Hence

$$\begin{aligned} \tilde{P}^{\tilde{a}\tilde{A}}{}_{|\tilde{A}} &= \left(J_\Xi^{-1} \mathsf{w}^{\tilde{a}}{}_a P^{aA} \overset{\Xi}{\bar{F}}{}^{\tilde{A}}{}_A \right)_{|\tilde{A}} = J_\Xi^{-1} \left(\mathsf{w}^{\tilde{a}}{}_a P^{aA} \right)_{|\tilde{A}} \overset{\Xi}{\bar{F}}{}^{\tilde{A}}{}_A + \mathsf{w}^{\tilde{a}}{}_a P^{aA} \left(J_\Xi^{-1} \overset{\Xi}{\bar{F}}{}^{\tilde{A}}{}_A \right)_{|\tilde{A}} \\ &= J_\Xi^{-1} \left(\mathsf{w}^{\tilde{a}}{}_a P^{aA} \right)_{|A} = J_\Xi^{-1} \mathsf{w}^{\tilde{a}}{}_a P^{aA}{}_{|A} + J_\Xi^{-1} P^{aA} \mathsf{w}^{\tilde{a}}{}_{a|A}\,, \end{aligned} \tag{7.148}$$

where use was made of the identity $(J_\Xi^{-1} \overset{\Xi}{\bar{F}}{}^{\tilde{A}}{}_A)_{|\tilde{A}} = 0$. In order to preserve the form of the balance of linear momentum under a cloaking map one needs to impose $J_\Xi^{-1} P^{aA} \mathsf{w}^{\tilde{a}}{}_{a|A} = J_\Xi^{-1} P^{aA} \mathsf{w}^{\tilde{a}}{}_{a|b} F^b{}_A = 0$, which implies that $\mathsf{w}^{\tilde{a}}{}_{a|b} = 0$, i.e., $\mathbf{w}$ must be a covariantly constant two-point tensor defined on the Euclidean ambient spaces $\mathcal{S}$ and $\tilde{\mathcal{S}}$. Therefore, when using Cartesian coordinates $\{z^i\}$ and $\{\tilde{z}^{\tilde{i}}\}$ for $\mathcal{S}$ and $\tilde{\mathcal{S}}$, respectively, $\mathbf{w}$ would have constant components $\mathsf{w}^{\tilde{i}}{}_i$. However, knowing that on the outer boundary of the cloak (and also outside the cloak) forces on the corresponding infinitesimal areas are equal one concludes that $\mathbf{w}$ is the identity, i.e., $\mathsf{w}^{\tilde{i}}{}_i = \delta^{\tilde{i}}_i$. Hence, $\mathbf{w}$ is the shifter when general coordinate charts are used for $\mathcal{S}$ and $\tilde{\mathcal{S}}$.

Now in the virtual body $\tilde{\mathbf{P}} = J_\Xi^{-1} \mathbf{s} \circ \varphi \mathbf{P} \overset{\Xi}{\bar{\mathbf{F}}}{}^\star$ is the shifted Piola transform of $\mathbf{P}$ and

$$\mathbf{P} = \mathbf{g}^\sharp \frac{\partial W}{\partial \mathbf{F}}, \qquad \tilde{\mathbf{P}} = \mathbf{g}^\sharp \frac{\partial \tilde{W}}{\partial \tilde{\mathbf{F}}}, \tag{7.149}$$

where $\tilde{W} = \tilde{W}(\tilde{\mathbf{G}}, \mathbf{g} \circ \tilde{\varphi}, \tilde{\mathbf{F}})$ as we assume that the virtual body is isotropic and homogeneous. To ensure that the relation $\tilde{\mathbf{P}} = J_\Xi^{-1} \mathbf{s} \circ \varphi \mathbf{P} \overset{\Xi}{\bar{\mathbf{F}}}{}^\star$ holds one needs to define W in $\mathcal{B} \setminus \mathcal{C}$ as $W|_{\mathcal{B}\setminus\mathcal{C}} = \tilde{W} \circ \Xi$, $\forall X \in \mathcal{B} \setminus \mathcal{C}$. This is because Ξ is defined to be the identity map in $\mathcal{B} \setminus \mathcal{C}$, and hence, the material outside the cloak will be isotropic and identical to that of the virtual body. The cloaking region $\mathcal{C}$, nevertheless, will be anisotropic, in general, i.e., $W_\mathcal{C} = W_\mathcal{C}(X, \mathbf{G}, \mathbf{g} \circ \varphi, \mathbf{F}, \boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_m)$, $\forall X \in \mathcal{C}$, where $\boldsymbol{\zeta}_1, \ldots, \boldsymbol{\zeta}_m$ are the structural tensors that characterize the symmetry group of the material in the cloaking region.[21] Note that $W_{\mathcal{C}}$ must satisfy the following relation

$$(\mathbf{g}^\sharp \circ \varphi_t)\frac{\partial W_{\mathcal{C}}}{\partial \mathbf{F}} = J_\Xi(\mathbf{s}^{-1}\circ\tilde{\varphi})\,(\mathbf{g}^\sharp\circ\tilde{\varphi}_t\circ\Xi)\left(\frac{\partial \tilde{W}}{\partial \tilde{\mathbf{F}}}\circ\Xi\right)\bar{\bar{\mathbf{F}}}^{-\star}\,. \tag{7.150}$$

One can now rewrite the balance of linear momentum $\mathrm{Div}\,\mathbf{P}+\rho_0\mathbf{B}=\rho_0\mathbf{A}$ as

$$\widetilde{\mathrm{Div}}\tilde{\mathbf{P}}+\tilde{\rho}_0\tilde{\mathbf{B}}=\tilde{\rho}_0\tilde{\mathbf{A}}\,, \tag{7.151}$$

where $\tilde{\rho}_0=J_\Xi^{-1}\rho_0\circ\Xi^{-1}$, $\tilde{\mathbf{B}}=(\mathbf{s}\circ\varphi)\mathbf{B}\circ\Xi^{-1}$, and $\tilde{\mathbf{A}}=(\mathbf{s}\circ\varphi)\mathbf{A}\circ\Xi^{-1}$. In other words, assuming that in the virtual body the first Piola-Kirchhoff stress is given by (7.144), the virtual body has the mass density $\tilde{\rho}_0$, is under the body force $\tilde{\mathbf{B}}$, and has the material acceleration $\tilde{\mathbf{A}}$, the balance of linear momentum is satisfied. Conversely, if the balance of linear momentum is satisfied for the virtual body with the above transformed fields, it is satisfied for the physical body as well. We refer to this construction as *transformation elasticity*.

Balance of angular momentum in the physical and virtual bodies. In the physical body the balance of angular momentum is equivalent to $S^{AB}=S^{BA}$, or $\mathbf{S}^\star=\mathbf{S}$. The balance of angular momentum must hold in the virtual body as well, i.e., $\tilde{\mathbf{S}}^\star=\tilde{\mathbf{S}}$, which holds if and only if

$$(\mathbf{RS})^\star=\mathbf{RS}\,, \tag{7.152}$$

where $\mathbf{R}=(\bar{\bar{\mathbf{F}}}^{-1}\circ\Xi)(\tilde{\mathbf{F}}^{-1}\circ\tilde{\varphi}\circ\Xi)(\mathbf{s}\circ\varphi)\mathbf{F}$. In terms of $\mathbf{R}$, the second Piola-Kirchhoff stress in the physical and virtual bodies are related as

$$\tilde{\mathbf{S}}=J_\Xi^{-1}\bar{\bar{\mathbf{F}}}\mathbf{RS}\bar{\bar{\mathbf{F}}}^\star,\quad\text{or}\quad \mathbf{S}=J_\Xi\mathbf{R}^{-1}\bar{\bar{\mathbf{F}}}^{-1}\tilde{\mathbf{S}}\bar{\bar{\mathbf{F}}}^{-\star}\,. \tag{7.153}$$

Note that assuming that the balance of angular momentum is satisfied in one configuration it may or may not be satisfied in the other configuration. We will see in §8.4.2 and §7.4.4 that the balance of

[21] Note that we do not know a priori how many structural tensors are needed and what they are.

angular momentum is the obstruction to transformation cloaking in both classical linear elastodynamics and the small-on-large theory. It happens to be the obstruction to transformation cloaking in linear gradient elasticity and linear generalized Cosserat elasticity as well as we will show in §7.5.

Remark 7.4.4. Note that although we use the same metric for the ambient space for the two bodies, after deformation, a physical point and its corresponding virtual point will be mapped to two different points in the ambient space, in general. For example, in cylindrical coordinates, $r(R, \Theta, \Phi) \neq \tilde{r}(\tilde{R}, \tilde{\Theta}, \tilde{\Phi})$, in general, where $\Xi(R, \Theta, \Phi) = (\tilde{R}, \tilde{\Theta}, \tilde{\Phi})$. Therefore, when calculating J and $\tilde{J}$, $\det \mathbf{g}$ at two different points is used.

Remark 7.4.5. We showed that if $\tilde{\mathbf{G}} = \Xi_* \mathbf{G}$, then the symmetry class of the material is preserved under Ξ (cf. (7.34) in the nonlinear case and Prop. 7.3.2 in the linear case). However, for general Ξ and $\tilde{\mathbf{G}}$, the type of the symmetry class of the body $\tilde{\mathcal{B}}$ with respect to the metric $\tilde{\mathbf{G}}$ can be quite non-trivial and different from that of the body $\mathcal{B}$ with respect to the metric $\mathbf{G}$. This is a geometric interpretation of the expected anisotropy of an elastic cloak.

Remark 7.4.6. The relation $\tilde{\rho}_0 = J_{\Xi}^{-1} \rho_0 \circ \Xi^{-1}$ implies that the total mass of the cloak is equal to the mass of the corresponding transformed region in the virtual structure with uniform density ρ_0, i.e., $m(\mathcal{C}) = \rho_0 \operatorname{vol}_{\tilde{\mathbf{G}}}(\tilde{\mathcal{C}})$. For example, the mass of a cylindrical cloak, which is an annulus with inner radius R_i and outer radius R_o in the physical body is equal to the mass of the annulus in the virtual body (filled with a homogeneous solid) with inner radius $f(R_i) = \epsilon$ and outer radius $f(R_o) = R_o$. Therefore, as the virtual and physical bodies are identical outside the cloak, one concludes that the two bodies have the same total mass.

Even if the balance of angular momentum is satisfied in both configurations, it turns out that hiding a hole in a nonlinear elastic medium from arbitrary finite-amplitude excitations is not possible.

Proposition 7.4.7. *Nonlinear elastodynamic transformation cloaking is not possible. In other words, it is not possible to design a cloak that would hide a cavity of any shape from any (finite) time-dependent elastic disturbance (or wave).*

Proof. The idea of nonlinear elastodynamic transformation cloaking can be summarized as follows. Starting from the balance of linear momentum one assumes the relation (7.144) between the first Piola-Kirchhoff stresses in the virtual and physical bodies. The energy function of the virtual body is given while that of the cloak in the physical body is not known. This implies that to be able to find the energy function of the cloak one would need to relate the kinematics of the physical and virtual bodies. This kinematic constraint is the relation between accelerations: $\tilde{\mathbf{A}} \circ \Xi = (\mathbf{s} \circ \varphi)\mathbf{A}$.[22] All these will have to be consistent with the balance of angular momentum (7.152). We start from the relation $\tilde{\mathbf{A}} \circ \Xi = (\mathbf{s} \circ \varphi)\mathbf{A}$ and write it in Cartesian coordinates: $\tilde{A}^{\tilde{i}} = \mathsf{s}^{\tilde{i}}{}_{i} A^{i} = \delta^{\tilde{i}}_{i} A^{i}$. Thus, $\tilde{V}^{\tilde{i}}(\Xi(X), t) = \delta^{\tilde{i}}_{i} V^{i}(X, t) + a_1$, where a_1 is a constant. Assuming that $\tilde{V}^{\tilde{i}}(\Xi(X), 0) = 0$ and $V^{i}(X, 0) = 0$, $a_1 = 0$, and hence, $\tilde{\varphi}^{\tilde{i}}(\Xi(X), t) = \delta^{\tilde{i}}_{i} \varphi^{i}(X, t) + a_0$, where a_0 is a constant. Knowing that $\tilde{\varphi}$ and φ satisfy the same essential boundary conditions on $\partial_d \tilde{\mathcal{B}} = \partial_d \mathcal{B}$, $a_0 = 0$ (boundary conditions will be discussed in detail in §7.5.2). Therefore, we have concluded that

$$\tilde{\varphi}^{\tilde{i}}(\Xi(X), t) = \delta^{\tilde{i}}_{i} \varphi^{i}(X, t) \,. \tag{7.154}$$

In the virtual body there is a circular (spherical) hole of radius $\epsilon \to 0$. (7.154) is telling us that the deformed configuration of $\partial\mathcal{H}$ is identical to that of the infinitesimal cavity in the virtual body. This is true for any loading of the physical body, and even in the limit of vanishing loads. This implies that the hole surrounded by a cloak in the physical body would collapse to a cavity of radius $\epsilon \to 0$ (anti-cavitation), i.e., the physical body is unstable. Therefore, nonlinear elastodynamic transformation cloaking is not possible. $\square$

However, nonlinear elastostatic transformation cloaking may be possible as discussed next.

[22]The kinematic relationship in linear elastodynamic cloaking turns out to be $\tilde{\mathbf{U}} \circ \Xi = \mathbf{U}$.

7.4.2 Nonlinear Elastostatic Cloaking

We ignore inertial forces and explore the possibility of static cloaking in nonlinear elasticity. Note that the virtual body is assumed to be isotropic, and therefore, its constitutive equation is given by

$$\tilde{\mathbf{S}} = 2\frac{\partial \tilde{W}}{\partial \tilde{\mathbf{C}}^\flat} = 2\tilde{W}_{\tilde{I}_1}\tilde{\mathbf{G}}^\sharp + 2\tilde{W}_{\tilde{I}_2}(\tilde{I}_2\tilde{\mathbf{C}}^{-1} - \tilde{I}_3\tilde{\mathbf{C}}^{-2}) + 2\tilde{W}_{\tilde{I}_3}\tilde{I}_3\tilde{\mathbf{C}}^{-1}\,, \tag{7.155}$$

where $\tilde{\mathbf{S}}$ is the second Piola-Kirchhoff stress and

$$\tilde{I}_1 = \operatorname{tr}\tilde{\mathbf{C}}\,,\quad \tilde{I}_2 = \det\tilde{\mathbf{C}}\operatorname{tr}\tilde{\mathbf{C}}^{-1}\,,\quad \tilde{I}_3 = \det\tilde{\mathbf{C}}\,,\quad \text{and}\quad \tilde{W}_{\tilde{I}_n} := \frac{\partial \tilde{W}}{\partial \tilde{I}_n}\,,\quad n = 1,2,3. \tag{7.156}$$

Note that

$$\tilde{\mathbf{C}}^{-1} = \tilde{\mathbf{F}}^{-1}\tilde{\mathbf{g}}^\sharp\tilde{\mathbf{F}}^{-\star},\quad \tilde{\mathbf{C}}^{-2} = \tilde{\mathbf{C}}^{-1}\tilde{\mathbf{G}}\tilde{\mathbf{C}}^{-1},\quad \boldsymbol{s}^{-1}\tilde{\mathbf{g}}^\sharp = \mathbf{g}^\sharp\boldsymbol{s}^\star,\quad \mathbf{C}^{-1} = \mathbf{F}^{-1}\mathbf{g}^\sharp\mathbf{F}^{-\star}. \tag{7.157}$$

Hence, from (7.153), (7.155), and (7.157) one can write

$$J_\Xi^{-1}\mathbf{S}_\mathcal{C} = \alpha\mathbf{R}^{-1}\Xi^*\tilde{\mathbf{G}}^\sharp + (\tilde{I}_2\beta + \tilde{I}_3\gamma)\mathbf{C}^{-1}\mathbf{R}^\star - \tilde{I}_3\beta\mathbf{C}^{-1}\mathbf{R}^\star(\Xi^*\tilde{\mathbf{G}})\mathbf{R}\mathbf{C}^{-1}\mathbf{R}^\star\,, \tag{7.158}$$

where $\alpha = 2\tilde{W}_{\tilde{I}_1}$, $\beta = 2\tilde{W}_{\tilde{I}_2}$, and $\gamma = 2\tilde{W}_{\tilde{I}_3}$. In components

$$\begin{aligned} J_\Xi^{-1}S_\mathcal{C}^{AB} &= \alpha(R^{-1})^A{}_M(\Xi^*\tilde{G})^{MB} + (\tilde{I}_2\beta + \tilde{I}_3\gamma)(C^{-1})^{AM}R^B{}_M \\ &\quad - \tilde{I}_3\beta(C^{-1})^{AM}R^K{}_M(\Xi^*\tilde{G})_{KL}R^L{}_P(C^{-1})^{PQ}R^B{}_Q\,, \end{aligned} \tag{7.159}$$

where $(\Xi^*\tilde{G})^{MB} = (\bar{\bar{\bar{F}}}^{-1})^M{}_{\tilde{M}}(\bar{\bar{\bar{F}}}^{-1})^B{}_{\tilde{N}}\tilde{G}^{\tilde{M}\tilde{N}}$, and $(\Xi^*\tilde{G})_{KL} = \bar{\bar{\bar{F}}}^{\tilde{K}}{}_K\bar{\bar{\bar{F}}}^{\tilde{L}}{}_L\tilde{G}_{\tilde{K}\tilde{L}}$.

Next we consider an infinitely-long cylindrical bar (or an infinite body) with a cylindrical hole and a spherical ball (or an infinite body) with a spherical cavity. In both examples we assume radial deformations. Under this assumption we will find the constitutive equations of the corresponding cloaks. It should be emphasized that these static cloaks are the nonlinear analogues of Mansfield

[172]'s neutral holes.

Example 1: A nonlinear static cylindrical cloak. Let us consider radial deformations of an infinitely-long solid cylinder with a cylindrical hole of radius R_i. This hole is covered by a cloak with the outer radius $R_o > R_i$. For the physical body in the cylindrical coordinates (R, Θ, Z) and (r, θ, z) for the reference configuration and the ambient space, respectively, we consider deformations of the form $(r, \theta, z) = (r(R), \Theta, Z)$. Similarly, for the virtual body in the cylindrical coordinates $(\tilde{R}, \tilde{\Theta}, \tilde{Z})$ and $(\tilde{r}, \tilde{\theta}, \tilde{z})$ we have $(\tilde{r}, \tilde{\theta}, \tilde{z}) = (\tilde{r}(\tilde{R}), \tilde{\Theta}, \tilde{Z})$. The cloaking map is assumed radial, i.e., $(\tilde{R}, \tilde{\Theta}, \tilde{Z}) = \Xi(R, \Theta, Z) = (f(R), \Theta, Z)$. We will design a nonlinear elastic cloak of inner and outer radii R_i and R_o in the physical body such that the static response of the body with a hole in radial deformations outside the cloak is identical to that of an isotropic and homogeneous elastic body with an infinitesimal hole. We assume that $f(R_i) = \epsilon$, and $f(R_o) = R_o$. As we will see in §5.2, the function f must satisfy the extra condition $f'(R_o) = 1$. We also assume that $f(R) = R$, for $R \geq R_o$. Let us assume the following relation between the kinematics of deformations in the physical and virtual bodies[23]

$$\frac{\tilde{r}(\tilde{R})}{\tilde{R}} = \frac{r(R)}{R}. \tag{7.160}$$

Therefore

$$(\tilde{r}, \tilde{\theta}, \tilde{z}) = (\tilde{r}(\tilde{R}), \tilde{\Theta}, \tilde{Z}) = \left(\frac{f(R)}{R} r(R), \Theta, Z \right). \tag{7.161}$$

Hence

$$\tilde{r}'(\tilde{R}) = \frac{1}{f'(R)} \frac{d}{dR} \left[\frac{f(R)}{R} r(R) \right] = \frac{f(R)}{R f'(R)} r'(R) + \left[1 - \frac{f(R)}{R f'(R)} \right] \frac{r(R)}{R}. \tag{7.162}$$

[23] In the case of elastodynamic cloaking the kinematic relation between the physical and virtual problems was the equality (up to a shift) of acceleration vectors. In elastostatics there is no such constraint and one has freedom in choosing a kinematic relation between the two problems. Note that (7.160) is just one choice and one may assume

$$\frac{\tilde{r}(\tilde{R})}{\tilde{R}} = h \left(\frac{r(R)}{R} \right),$$

for any positive and strictly increasing function h such that $h(1) = 1$. Note also that (7.160) is a nonlinear analogue of Olsson and Wall [197]'s kinematic assumption.

In particular, $\tilde{r}(\tilde{R}_o) = r(R_o)$, and $\tilde{r}'(\tilde{R}_o) = r'(R_o)$. Note that the spatial shifter map has the following coordinate representation

$$\mathsf{s} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r(R)/\tilde{r}(\tilde{R}) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R/f(R) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{7.163}$$

Therefore

$$\mathbf{R} = \begin{bmatrix} r'(R)/[f'(R)\tilde{r}'(\tilde{R})] & 0 & 0 \\ 0 & R/f(R) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{C}^{-1} = \begin{bmatrix} 1/r'(R)^2 & 0 & 0 \\ 0 & 1/r(R)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{7.164}$$

Using (7.158) the constitutive equations of the cloak read

$$\begin{aligned} \mathbf{S} =& \alpha \begin{bmatrix} f\tilde{r}'/(Rr') & 0 & 0 \\ 0 & f'/R^2 & 0 \\ 0 & 0 & ff'/R \end{bmatrix} + (\tilde{I}_2\beta + \tilde{I}_3\gamma) \begin{bmatrix} f/(Rr'\tilde{r}') & 0 & 0 \\ 0 & f'/r^2 & 0 \\ 0 & 0 & ff'/R \end{bmatrix} \\ & - \tilde{I}_3\beta \begin{bmatrix} f/(Rr\tilde{r}'^3) & 0 & 0 \\ 0 & fR^2/r^4 & 0 \\ 0 & 0 & ff'/R \end{bmatrix}, \end{aligned} \tag{7.165}$$

where

$$\tilde{I}_1 = 1 + \tilde{r}'^2(f(R)) + \frac{r^2(R)}{R^2}, \quad \tilde{I}_2 = \frac{1}{2}\tilde{r}'^2(1 - \tilde{r}'^2) + \frac{r^2(R)}{2R^2}\left[1 - \frac{r^2(R)}{R^2}\right], \quad \tilde{I}_3 = \tilde{J}^2 = \left[\frac{r(R)}{R}\tilde{r}'(f(R))\right]^2. \tag{7.166}$$

It is seen that $\mathbf{S}$ is symmetric, and hence, the balance of angular momentum is satisfied. The surface of the cavity is traction free, and hence, we have the boundary condition $S^{RR}(R_i) = 0$. The boundary conditions in the physical and virtual bodies will be discussed in detail in Section 7.5.2. In particular, it will be shown that the boundary of the hole is traction-free in the virtual body if and only if the boundary of the physical hole is traction-free. As an example we can assume that the physical and virtual bodies have finite outer radii both equal to R_e on which a radial traction is specified, i.e., $P^{rR}(R_e) = T_e$, and $P^{\theta R}(R_e) = 0$.

Note that radial deformations cannot be maintained in an arbitrary compressible isotropic solid only by applying boundary tractions. This is a result of Ericksen's theorem, which states that the only universal deformations in compressible isotropic solids are homogeneous deformations [223] (see [224] for an extension of this result to compressible solids with finite eigenstrains). Note, however, that if the radial stretch is uniform, i.e., $\tilde{r}(\tilde{R})/\tilde{R} = r(R)/R = \lambda$, then the deformation will be homogeneous, and hence, universal, i.e., it can be maintained solely by applying boundary tractions in any compressible isotropic solid.

Example 2: A nonlinear static spherical cloak. Let us consider radial deformations of a finite solid sphere with a spherical cavity of radius R_i. For the physical body in the spherical coordinates (R, Θ, Φ) and (r, θ, ϕ) for the reference configuration and the ambient space, respectively, we consider deformations of the form $(r, \theta, \phi) = (r(R), \Theta, \Phi)$. Similarly, for the virtual body in the cylindrical coordinates $(\tilde{R}, \tilde{\Theta}, \tilde{\Phi})$ and $(\tilde{r}, \tilde{\theta}, \tilde{\phi})$ we have $(\tilde{r}, \tilde{\theta}, \tilde{\phi}) = (\tilde{r}(\tilde{R}), \tilde{\Theta}, \tilde{\Phi})$. Notice that $(\tilde{R}, \tilde{\Theta}, \tilde{\Phi}) = \Xi(R, \Theta, Z) = (f(R), \Theta, \Phi)$. We again assume the following relation between the kinematics of deformations in the physical and virtual bodies $\tilde{r}(\tilde{R})/\tilde{R} = r(R)/R$. Therefore

$$(\tilde{r}, \tilde{\theta}, \tilde{\phi}) = (\tilde{r}(\tilde{R}), \tilde{\Theta}, \tilde{\Phi}) = \left(\frac{f(R)}{R} r(R), \Theta, \Phi \right). \tag{7.167}$$

Note that $\tilde{r}'(\tilde{R})$ is given in (7.162). We will design a nonlinear elastic cloak of inner and outer radii R_i and R_o in the physical body such that the static response of the body with a cavity in radial deformations outside the cloak be identical to that of an isotropic and homogeneous elastic body with an infinitesimal cavity. The function f satisfies the following conditions: $f(R_i) = \epsilon$, $f(R_o) = R_o$, and $f'(R_o) = 1$. Note that the spatial shifter map has the following coordinate representation

$$\mathsf{s} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r(R)/\tilde{r}(\tilde{R}) & 0 \\ 0 & 0 & r(R)/\tilde{r}(\tilde{R}) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R/f(R) & 0 \\ 0 & 0 & R/f(R) \end{bmatrix}. \tag{7.168}$$

Therefore

$$\mathbf{R} = \begin{bmatrix} r'(R)/(f'(R)\tilde{r}'(\tilde{R})) & 0 & 0 \\ 0 & R/f(R) & 0 \\ 0 & 0 & R/f(R) \end{bmatrix}, \quad \mathbf{C}^{-1} = \begin{bmatrix} 1/r'(R)^2 & 0 & 0 \\ 0 & 1/r(R)^2 & 0 \\ 0 & 0 & 1/(r(R)^2 \sin^2\Theta) \end{bmatrix}. \tag{7.169}$$

The constitutive equations of the cloak read

$$\begin{aligned}\mathbf{S} =& \alpha \begin{bmatrix} f^2\tilde{r}'/(R^2 r') & 0 & 0 \\ 0 & ff'/R^3 & 0 \\ 0 & 0 & ff'/(R^3\sin^2\Theta) \end{bmatrix} + (\tilde{I}_2\beta + \tilde{I}_3\gamma) \begin{bmatrix} f^2/(R^2 r'\tilde{r}') & 0 & 0 \\ 0 & ff'/(Rr^2) & 0 \\ 0 & 0 & ff'/(Rr^2\sin^2\Theta) \end{bmatrix} \\ & - \tilde{I}_3\beta \begin{bmatrix} f^2/(R^2 r'\tilde{r}'^3) & 0 & 0 \\ 0 & Rff'/r^4 & 0 \\ 0 & 0 & Rff'/(r^4\sin^2\Theta) \end{bmatrix},\end{aligned} \tag{7.170}$$

where

$$\tilde{I}_1 = \tilde{r}'^2(f(R)) + \frac{2r^2(R)}{R^2}, \quad \tilde{I}_2 = \frac{1}{2}\tilde{r}'^2(1-\tilde{r}'^2) + \frac{r^2(R)}{R^2}\left[1 - \frac{r^2(R)}{R^2}\right], \quad \tilde{I}_3 = \tilde{J}^2 = \left[\frac{r(R)}{R}\tilde{r}'(f(R))\right]^4. \tag{7.171}$$

It is seen that $\mathbf{S}$ is symmetric, and hence, the balance of angular momentum is satisfied. The surface of the cavity is traction free, and hence, we have the boundary condition $S^{RR}(R_i) = 0$.

7.4.3 Linear Elastodynamic Transformation Cloaking

In this section we explore the possibility of transformation cloaking in classical linear elasticity. Let us assume that the reference motion is static, i.e., $\mathring{\mathbf{V}}(X,t) = \mathbf{0}$, and the ambient space is Euclidean, that is, $\mathsf{\mathbf{Ric}}_{\mathbf{g}} = \mathbf{0}$. In this special case, the linearized material balance of linear momentum (7.92) is simplified to read

$$\mathrm{Div}(\mathsf{\mathbf{A}}\!:\!\nabla_0^{\mathbf{g}}\mathbf{U}) + \rho_0 \nabla_{\mathbf{U}}^{\mathbf{g}}\mathring{\mathbf{B}} = \rho_0 \ddot{\mathbf{U}}\,. \tag{7.172}$$

Now suppose that the reference configuration of the physical body $\mathcal{B}$ is mapped to the reference configuration of the virtual body $\tilde{\mathcal{B}}$, where $\mathcal{B}$ and $\tilde{\mathcal{B}}$ are endowed with the Euclidean metrics $\mathbf{G}(X)$ and $\tilde{\mathbf{G}}(\tilde{X})$, respectively. Let us denote the map between the two stress-free reference configurations by $\Xi : \mathcal{B} \to \tilde{\mathcal{B}}$. The Jacobian of this cloaking map is calculated as

$$J_\Xi = \sqrt{\frac{\det \tilde{\mathbf{G}}(\tilde{X}(X))}{\det \mathbf{G}(X)}}\,\det \overset{\Xi}{\mathbf{F}}\,. \tag{7.173}$$

Linearized balance of linear momentum in the physical and virtual bodies. Using the Piola identity, the divergence term in (7.172) can be rewritten as

$$\left(\mathsf{A}^{aA}{}_b{}^B U^b{}_{|B}\right)_{|A} = J_\Xi \left[J_\Xi^{-1} \overset{\Xi}{F}{}^{\tilde{A}}{}_A \mathsf{A}^{aA}{}_b{}^B U^b{}_{|B}\right]_{|\tilde{A}} = J_\Xi \left(\tilde{\mathsf{A}}^{a\tilde{A}}{}_b{}^{\tilde{B}} U^b{}_{|\tilde{B}}\right)_{|\tilde{A}}\,, \tag{7.174}$$

where

$$U^b{}_{|\tilde{B}} = (\overset{\Xi}{F}{}^{-1})^B{}_{\tilde{B}} U^b{}_{|B}\,, \quad (\tilde{\mathsf{A}}^{a\tilde{A}}{}_b{}^{\tilde{B}}) \circ \Xi = J_\Xi^{-1} \overset{\Xi}{F}{}^{\tilde{A}}{}_A \overset{\Xi}{F}{}^{\tilde{B}}{}_B \mathsf{A}^{aA}{}_b{}^B\,. \tag{7.175}$$

Hence, using the shifter map and knowing that it is covariantly constant one can write

$$\mathsf{s}^{\tilde{a}}{}_a \left(\mathsf{A}^{aA}{}_b{}^B U^b{}_{|B}\right)_{|A} = J_\Xi \left(\tilde{\mathsf{A}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}} \tilde{U}^{\tilde{b}}{}_{|\tilde{B}}\right)_{|\tilde{A}}\,, \tag{7.176}$$

where[24]

$$\tilde{\mathsf{A}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}} \circ \Xi = J_{\Xi}^{-1} \bar{\bar{\bar{F}}}{}^{\tilde{A}}{}_{A} \bar{\bar{\bar{F}}}{}^{\tilde{B}}{}_{B}\, \mathsf{s}^{\tilde{a}}{}_{a} (\mathsf{s}^{-1})^{b}{}_{\tilde{b}}\, \mathsf{A}^{aA}{}_{b}{}^{B}, \quad \tilde{U}^{\tilde{a}} \circ \Xi = \mathsf{s}^{\tilde{a}}{}_{a} \circ \varphi U^{a}\,. \tag{7.177}$$

Therefore

$$\boldsymbol{\mathsf{s}} \circ \varphi \,\mathrm{Div}(\boldsymbol{\mathsf{A}} \!:\! \nabla_0^{\mathbf{g}} \mathbf{U}) = J_{\Xi}\, \widetilde{\mathrm{Div}}(\tilde{\boldsymbol{\mathsf{A}}} \!:\! \tilde{\nabla}_0^{\mathbf{g}} \tilde{\mathbf{U}}) \circ \Xi\,, \tag{7.178}$$

where $\tilde{\mathbf{U}} \circ \Xi = \boldsymbol{\mathsf{s}} \circ \varphi \mathbf{U}$, i.e., one assumes that the displacements in the two bodies are equal at the corresponding points. Again, knowing that the shifter is covariantly constant, one obtains $\dot{\tilde{\mathbf{U}}} = (\boldsymbol{\mathsf{s}} \circ \varphi)\dot{\mathbf{U}} \circ \Xi^{-1}$. Similarly, $\ddot{\tilde{\mathbf{U}}} = (\boldsymbol{\mathsf{s}} \circ \mathring{\varphi})\ddot{\mathbf{U}} \circ \Xi^{-1}$. Substituting (7.178) into $(7.172)_1$, one can write the linearized balance of linear momentum in the virtual body as

$$\widetilde{\mathrm{Div}}(\tilde{\boldsymbol{\mathsf{A}}} \!:\! \tilde{\nabla}_0^{\mathbf{g}} \tilde{\mathbf{U}}) + \tilde{\rho}_0 \tilde{\nabla}^{\mathbf{g}}_{\tilde{\mathbf{U}}} \mathring{\tilde{\mathbf{B}}} = \tilde{\rho}_0 \ddot{\tilde{\mathbf{U}}}\,, \tag{7.179}$$

where $\tilde{\rho}_0 = J_{\Xi}^{-1} \rho_0 \circ \Xi^{-1}$ and $\mathring{\tilde{\mathbf{B}}} = \boldsymbol{\mathsf{s}} \circ \mathring{\varphi} \mathring{\mathbf{B}} \circ \Xi^{-1}$. In summary, the linearized balance of linear momentum is form-invariant under the following *cloaking transformations*:

$$\begin{aligned} &\tilde{X} = \Xi(X), \quad \tilde{\mathbf{U}} = \boldsymbol{\mathsf{s}} \circ \mathring{\varphi}\, \mathbf{U} \circ \Xi^{-1}, \;\; \mathring{\tilde{\mathbf{B}}} = \boldsymbol{\mathsf{s}} \circ \mathring{\varphi}\, \mathring{\mathbf{B}} \circ \Xi^{-1}, \\ &\tilde{\rho}_0 = J_{\Xi}^{-1} \rho_0 \circ \Xi^{-1}, \;\; \tilde{\boldsymbol{\mathsf{A}}} = (J_{\Xi}^{-1} \boldsymbol{\mathsf{s}}^{-1} \circ \mathring{\varphi}\, \bar{\bar{\bar{\mathbf{F}}}}\, \boldsymbol{\mathsf{s}} \circ \mathring{\varphi}\, \boldsymbol{\mathsf{A}}\, \bar{\bar{\bar{\mathbf{F}}}}{}^{\star}) \circ \Xi^{-1}\,. \end{aligned} \tag{7.180}$$

Linearized balance of angular momentum in the physical and virtual bodies. Suppose that the reference motions of both the physical and virtual bodies are their corresponding undeformed configurations, i.e., both $\mathring{\mathbf{F}}$ and $\mathring{\tilde{\mathbf{F}}}$ are identity maps (there are no initial stresses). Therefore, for any $\mathbf{W} \in T_X\mathcal{B}$, one has $\mathring{\tilde{\mathbf{F}}} \boldsymbol{\mathsf{S}} \mathbf{W} = \boldsymbol{\mathsf{s}} \mathring{\mathbf{F}} \mathbf{W}$, and hence, $\boldsymbol{\mathsf{S}} = \mathring{\tilde{\mathbf{F}}}{}^{-1} \boldsymbol{\mathsf{s}} \mathring{\mathbf{F}}$ (in components, $\mathsf{S}^{\tilde{A}}{}_{A} = \delta^{\tilde{A}}_{\tilde{a}} \mathsf{s}^{\tilde{a}}{}_{a} \delta^{a}_{A}$). Let us next linearize (7.152) with respect to $\mathring{\varphi}$ and $\mathring{\tilde{\varphi}}$, which reads $(\mathring{\mathbf{R}} \delta \mathbf{S})^{\star} = \mathring{\mathbf{R}} \delta \mathbf{S}$, where $\mathring{\mathbf{R}} = (\bar{\bar{\bar{\mathbf{F}}}}{}^{-1} \circ \Xi) \boldsymbol{\mathsf{S}}$. In components

$$(\bar{\bar{\bar{F}}}{}^{-1})^{A}{}_{\tilde{A}} \mathsf{S}^{\tilde{A}}{}_{M} \mathsf{C}^{MBKL} = (\bar{\bar{\bar{F}}}{}^{-1})^{B}{}_{\tilde{A}} \mathsf{S}^{\tilde{A}}{}_{M} \mathsf{C}^{MAKL}\,. \tag{7.181}$$

[24]Note that transformation of the elastic constants under a cloaking map is different from that under a material change of coordinates (7.115).

Clearly, the balance of angular momentum may not be satisfied for a given cloaking map Ξ. In terms of the first Piola-Kirchhoff stress, the balance of angular momentum reads $P^{[aA}F^{c]}{}_{A} = 0$, where $2P^{[aA}F^{c]}{}_{A} = P^{aA}F^{c}{}_{A} - P^{cA}F^{a}{}_{A}$. Assuming that there is no initial stress, the linearized balance of angular momentum reads $\delta P^{[aA}F^{c]}{}_{A} = 0$, or $\mathsf{A}^{[aAbB}\mathring{F}^{c]}{}_{A} = 0$. Assuming that the balance of angular momentum in the virtual body is satisfied, i.e., $\tilde{\mathsf{A}}^{[\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\mathring{\tilde{F}}^{\tilde{c}]}{}_{\tilde{A}} = 0$, the balance of angular momentum in the physical body requires that

$$(\mathsf{s}^{-1})^{[a}{}_{\tilde{a}}\mathring{F}^{c]}{}_{A}(\mathsf{s}^{-1})^{b}{}_{\tilde{b}}(\overset{\Xi}{\bar{F}}{}^{-1})^{A}{}_{\tilde{A}}(\overset{\Xi}{\bar{F}}{}^{-1})^{B}{}_{\tilde{B}}\tilde{\mathsf{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} = 0\,. \tag{7.182}$$

Note that (7.182) is not necessarily satisfied.

Transformed second elasticity tensor. Using (7.60) and the relation $\mathbf{S} = \mathring{\tilde{\mathbf{F}}}{}^{-1}\mathbf{s}\mathring{\mathbf{F}}$, it is straightforward to show that $\tilde{\mathbf{C}} = J_{\Xi}^{-1}\overset{\Xi}{\bar{\mathbf{F}}}\mathbf{S}\overset{\Xi}{\bar{\mathbf{F}}}\mathbf{S}\mathbf{C} \circ \Xi$. In components one has

$$\tilde{\mathsf{C}}^{\tilde{A}\tilde{B}\tilde{C}\tilde{D}} = J_{\Xi}^{-1}\overset{\Xi}{\bar{F}}{}^{\tilde{A}}{}_{A}\,\mathsf{S}^{\tilde{B}}{}_{B}\,\overset{\Xi}{\bar{F}}{}^{\tilde{C}}{}_{C}\,\mathsf{S}^{\tilde{D}}{}_{D}\,\mathsf{C}^{ABCD}\,. \tag{7.183}$$

Or[25]

$$\mathsf{C}^{ABCD} = J_{\Xi}(\overset{\Xi}{\bar{F}}{}^{-1})^{A}{}_{\tilde{A}}\,(\mathsf{S}^{-1})^{B}{}_{\tilde{B}}\,(\overset{\Xi}{\bar{F}}{}^{-1})^{C}{}_{\tilde{C}}\,(\mathsf{S}^{-1})^{D}{}_{\tilde{D}}\,\tilde{\mathsf{C}}^{\tilde{A}\tilde{B}\tilde{C}\tilde{D}}\,. \tag{7.184}$$

We assume that the virtual structure is isotropic, and hence, $\tilde{\mathsf{C}}^{\tilde{A}\tilde{B}\tilde{C}\tilde{D}} = \lambda\tilde{G}^{\tilde{A}\tilde{B}}\tilde{G}^{\tilde{C}\tilde{D}} + \mu(\tilde{G}^{\tilde{A}\tilde{C}}\tilde{G}^{\tilde{B}\tilde{D}} + \tilde{G}^{\tilde{A}\tilde{D}}\tilde{G}^{\tilde{C}\tilde{B}})$ are given. Note that the transformed second elasticity tensor $\mathbf{C}$ possesses the major symmetries, i.e., $\mathsf{C}^{ABCD} = \mathsf{C}^{CDAB}$. However, the minor symmetries are not satisfied, in general. This is in agreement with the previous works in the literature.

Proposition 7.4.8. *Classical linear elasticity does not allow for elastodynamic transformation cloaking regardless of the shape of the hole and the cloak. In other words, elastodynamic transformation cloaking is not possible if the cloak is required to be made of a classical linear elastic solid.*

[25] When Cartesian coordinates are used in both the physical and virtual bodies one writes $\tilde{\mathsf{C}}^{\tilde{A}\tilde{B}\tilde{C}\tilde{D}} = J_{\Xi}^{-1}\overset{\Xi}{\bar{F}}{}^{\tilde{A}}{}_{A}\,\overset{\Xi}{\bar{F}}{}^{\tilde{C}}{}_{C}\,\delta^{\tilde{B}}_{B}\delta^{\tilde{D}}_{D}\,\mathsf{C}^{ABCD}$. This is identical to Norris and Parnell [201]'s Eq.(2.6). See also Al-Attar and Crawford [179]'s Eq.(129).

Proof. The balance of angular momentum in the physical body (7.182) is expanded to read

$$(\mathsf{s}^{-1})^a{}_{\tilde{a}}\mathring{F}^c{}_A(\mathsf{s}^{-1})^b{}_{\tilde{b}}(\bar{\bar{\bar{F}}}^{-1})^A{}_{\tilde{A}}(\bar{\bar{\bar{F}}}^{-1})^B{}_{\tilde{B}}\tilde{\mathsf{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} = (\mathsf{s}^{-1})^c{}_{\tilde{a}}\mathring{F}^a{}_A(\mathsf{s}^{-1})^b{}_{\tilde{b}}(\bar{\bar{\bar{F}}}^{-1})^A{}_{\tilde{A}}(\bar{\bar{\bar{F}}}^{-1})^B{}_{\tilde{B}}\tilde{\mathsf{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\,. \quad (7.185)$$

Thus, knowing that $\mathring{F}^a{}_A = \delta^a_A$, one obtains

$$\begin{aligned}&(\mathsf{s}^{-1})^a{}_{\tilde{a}}(\mathsf{s}^{-1})^b{}_{\tilde{b}}(\bar{\bar{\bar{F}}}^{-1})^c{}_{\tilde{A}}(\bar{\bar{\bar{F}}}^{-1})^B{}_{\tilde{B}}\tilde{\mathsf{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\\ &= (\mathsf{s}^{-1})^c{}_{\tilde{a}}(\mathsf{s}^{-1})^b{}_{\tilde{b}}(\bar{\bar{\bar{F}}}^{-1})^a{}_{\tilde{A}}(\bar{\bar{\bar{F}}}^{-1})^B{}_{\tilde{B}}\tilde{\mathsf{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\,, \quad \forall\, a,b,c,B \in \{1,2,3\}.\end{aligned} \quad (7.186)$$

Without loss of generality, we may represent (7.186) in Cartesian coordinates in which the shifter is the identity, i.e., $\mathsf{s}^{\tilde{a}}{}_a = \delta^{\tilde{a}}{}_a$. Therefore

$$(\bar{\bar{\bar{F}}}^{-1})^B{}_{\tilde{B}}\left[(\bar{\bar{\bar{F}}}^{-1})^c{}_{\tilde{A}}\tilde{\mathsf{A}}^{a\tilde{A}b\tilde{B}} - (\bar{\bar{\bar{F}}}^{-1})^a{}_{\tilde{A}}\tilde{\mathsf{A}}^{c\tilde{A}b\tilde{B}}\right] = 0\,, \quad \forall\, a,b,c,B \in \{1,2,3\}. \quad (7.187)$$

Knowing that in Cartesian coordinates $\tilde{\mathsf{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} = \lambda\delta^{\tilde{a}\tilde{A}}\delta^{\tilde{b}\tilde{B}} + \mu(\delta^{\tilde{a}\tilde{b}}\delta^{\tilde{A}\tilde{B}} + \delta^{\tilde{a}\tilde{B}}\delta^{\tilde{b}\tilde{A}})$, for an arbitrary cloaking transformation with components $\mathsf{F}_{ij} = (\bar{\bar{\bar{F}}}^{-1})^i{}_j$, $i,j \in \{1,2,3\}$, (7.187) is simplified for $i \neq j \neq k \in \{1,2,3\}$ to read

$$(\lambda+\mu)\mathsf{F}_{ii}(\mathsf{F}_{ij} - \mathsf{F}_{ji}) - \mu(\mathsf{F}_{ij}\mathsf{F}_{jj} + \mathsf{F}_{ik}\mathsf{F}_{jk} + \mathsf{F}_{ii}\mathsf{F}_{ji}) = 0\,, \quad (7.188)$$

$$\mu(\mathsf{F}_{ij}\mathsf{F}_{ik} - \mathsf{F}_{ii}\mathsf{F}_{kj}) + \lambda\mathsf{F}_{ij}(\mathsf{F}_{ik} - \mathsf{F}_{ki}) = 0\,, \quad (7.189)$$

$$\mu(\mathsf{F}_{ii}^2 + \mathsf{F}_{ik}^2) + (\lambda+2\mu)\mathsf{F}_{ij}^2 - \mu\mathsf{F}_{ii}\mathsf{F}_{jj} - \lambda\mathsf{F}_{ij}\mathsf{F}_{ji} = 0\,. \quad (7.190)$$

(7.188) can be rewritten as ($i \leftrightarrow j$)

$$(\lambda+\mu)\mathsf{F}_{jj}(\mathsf{F}_{ji} - \mathsf{F}_{ij}) - \mu(\mathsf{F}_{ji}\mathsf{F}_{ii} + \mathsf{F}_{jk}\mathsf{F}_{ik} + \mathsf{F}_{jj}\mathsf{F}_{ij}) = 0\,. \quad (7.191)$$

Using (7.188) and (7.191), one obtains

$$(\lambda+\mu)(\mathsf{F}_{ii}+\mathsf{F}_{jj})(\mathsf{F}_{ij}-\mathsf{F}_{ji})=0. \tag{7.192}$$

Note that $\lambda+\mu=\frac{1}{3}(3\lambda+2\mu)+\frac{1}{3}\mu>0$. Thus, either $\mathsf{F}_{ij}=\mathsf{F}_{ji}$, or $\mathsf{F}_{ii}=-\mathsf{F}_{jj}$. Note that $\mathsf{F}_{ii}=-\mathsf{F}_{jj}$ immediately implies that $\mathsf{F}_{ii}=0$, and from (7.188), $\mathsf{F}_{jk}=0$, resulting in $\bar{\mathbf{F}}^{\Xi\,-1}=\mathbf{0}$, which is not possible as the tangent map of a cloaking transformation cannot be singular. Thus, $\mathsf{F}_{ij}=\mathsf{F}_{ji}$, which simplifies (7.188)-(7.190) to read

$$\mathsf{F}_{ii}\mathsf{F}_{ij}(\mathsf{F}_{jj}+\mathsf{F}_{ii})+\mathsf{F}_{ik}\mathsf{F}_{ii}\mathsf{F}_{kj}=0\,, \tag{7.193}$$

$$\mathsf{F}_{ij}\mathsf{F}_{ik}=\mathsf{F}_{ii}\mathsf{F}_{kj}\,, \tag{7.194}$$

$$\mathsf{F}_{ii}^2+\mathsf{F}_{ik}^2+2\mathsf{F}_{ij}^2-\mathsf{F}_{ii}\mathsf{F}_{jj}=0\,. \tag{7.195}$$

Now, substituting (7.194) into (7.193), one obtains

$$\mathsf{F}_{ij}(\mathsf{F}_{ii}^2+\mathsf{F}_{ii}\mathsf{F}_{jj}+\mathsf{F}_{ik}^2)=0. \tag{7.196}$$

Using this relation and (7.195), one obtains $\mathsf{F}_{ij}(\mathsf{F}_{ij}^2-\mathsf{F}_{ii}\mathsf{F}_{jj})=0$, which implies that either $\mathsf{F}_{ij}=0$, or $\mathsf{F}_{ij}^2=\mathsf{F}_{ii}\mathsf{F}_{jj}$. However, $\mathsf{F}_{ij}^2=\mathsf{F}_{ii}\mathsf{F}_{jj}$ is not acceptable because from (7.195) one concludes that $\bar{\mathbf{F}}^{\Xi\,-1}=\mathbf{0}$. Therefore, $\mathsf{F}_{ij}=0$, for $i\neq j\in\{1,2,3\}$. Using (7.195) one concludes that $\mathsf{F}_{ii}=\mathsf{F}_{jj}$, i.e., $\bar{\mathbf{F}}^{\Xi}=\beta\mathbf{I}$, where $\mathbf{I}$ denotes the identity and β is a scalar. As the cloaking transformation Ξ must be the identity on the outer boundary of the cloak $\partial_o\mathcal{C}$ one concludes that $\beta=1$, and thus, $\Xi=id$. Therefore, transformation cloaking is not possible in classical linear elasticity. $\square$

Remark 7.4.9. We observe that the balance of angular momentum is the obstruction to cloaking, and not the acceleration term. This implies that transformation cloaking is not possible in classical linear elasticity at fixed frequency or classical linear elastostatics either.

Remark 7.4.10. Note that when $\mu = 0$ (a pentamode material [225]) transformation cloaking may be possible. When $\mu = 0$, the only constraint imposed by the balance of angular momentum for the physical body (cf. (7.188), (7.189), and (7.190)) is that $\overset{\Xi}{\mathbf{F}}$ be symmetric. One should note that $\overset{\Xi}{\mathbf{F}}$ is a two-point tensor. However, as the reference configuration of both the virtual and the physical bodies are embedded in the Euclidean space, symmetry of $\overset{\Xi}{\mathbf{F}}$ makes sense. For an arbitrary hole surrounded by a cloak (with an arbitrary shape), if the derivative of a cloaking map is symmetric, the elastic constants of the cloak are fully symmetric. This happens to be the case for cylindrical and spherical cloaks. As we will see in §5.3, for the virtual and physical boundary-value problems to be equivalent outside the cloak, $\overset{\Xi}{\mathbf{F}}$ must be the identity map on the outer boundary of the cloak, i.e., $\overset{\Xi}{\mathbf{F}}|_{\partial_o \mathcal{C}} = id$.

Next we critically examine a cloaking geometry that has been suggested in several previous works in the literature. Consider an infinitely-long hollow solid cylinder that in its stress-free reference configuration has inner and outer radii R_i and R_o, respectively. Let us transform the reference configuration to the reference configuration of another body (virtual body) that is a hollow cylinder with inner and outer radii ϵ and R_o, respectively, using a cloaking map $\Xi(R, \Theta, Z) = (f(R), \Theta, Z)$ such that $f(R_o) = R_o$. For such a map we have

$$\overset{\Xi}{\mathbf{F}} = \begin{bmatrix} f'(R) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{7.197}$$

Let us consider an infinitely extended isotropic homogeneous elastic medium with shear modulus μ, Lamé constant λ, and mass density ρ_0 containing an infinitely-long cylindrical cavity $\mathcal{H}$ with radius R_i in its stress-free reference configuration. Let (R, Θ, Z) be the cylindrical coordinates such that $R = 0$ corresponds to the centerline of the hole. The cloaking device $\mathcal{C}$ is an infinitely-long hollow solid cylinder with inner radius R_i (radius of the hole) and outer radius R_o, where $R_o < R_s$ (R_s is the distance of a line source from the origin) surrounding the hole. The elastic properties of the cloaking device are to be determined. In doing so, the reference configuration of the physical body

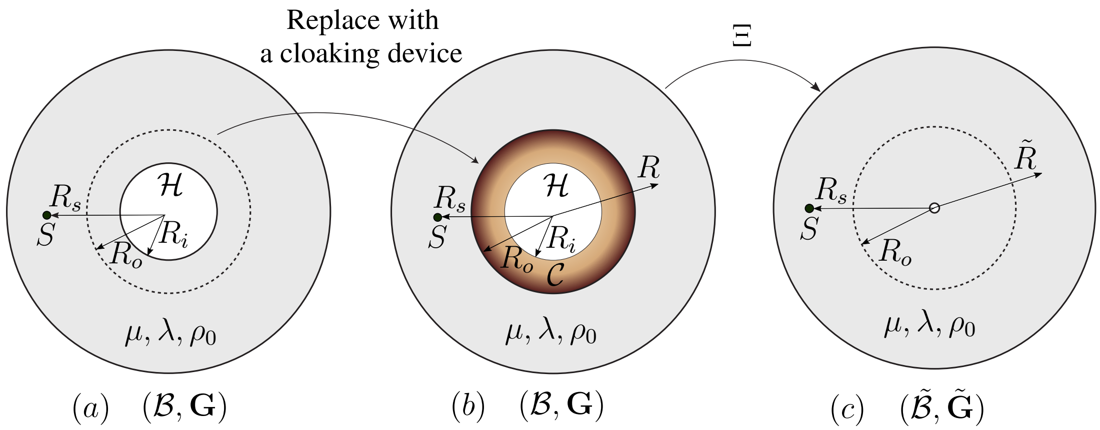


Figure 7.5: *Cloaking an object inside a hole $\mathcal{H}$ from elastic waves generated by a line source located at a distance R_s from the center of the hole. The system (a) is an isotropic and homogeneous medium containing a finite hole $\mathcal{H}$. The system (c) is isotropic and homogeneous with the same elastic properties as the medium in the reference configuration (b) outside the cloaking region $\mathcal{C}$. The configuration (b) is mapped to the new reference configuration (c) such that the hole $\mathcal{H}$ is mapped to an infinitesimal hole with negligible effects on the elastic waves. The cloaking transformation is the identity mapping in $\mathcal{B} \setminus \mathcal{C}$.*

$\mathcal{B}$ is mapped to that of the virtual body $\tilde{\mathcal{B}}$ via a mapping $\Xi : \mathcal{B} \to \tilde{\mathcal{B}}$, where for $R_i \leq R \leq R_o$, it is defined as, $(\tilde{R}, \tilde{\Theta}, \tilde{Z}) = \Xi(R, \Theta, Z) = (f(R), \Theta, Z)$ such that $f(R_i) = \epsilon$ and $f(R_o) = R_o$, and for $R \geq R_o$ it is the identity map. We assume that the transformed reference configuration is also isotropic and homogeneous with the same elastic properties as the medium in the physical reference configuration. Let the reference configurations of $\mathcal{B}$ and $\tilde{\mathcal{B}}$ be endowed with the flat metric in the cylindrical coordinates, i.e., $\mathbf{G} = \mathrm{diag}(1, R^2, 1)$ and $\tilde{\mathbf{G}} = \mathrm{diag}(1, \tilde{R}^2, 1)$ in the cylindrical coordinates (R, Θ, Z) and $(\tilde{R}, \tilde{\Theta}, \tilde{Z})$, respectively. Also, let the ambient space be endowed with the Euclidean metric $\mathbf{g} = \mathrm{diag}(1, r^2, 1)$ in the coordinates (r, θ, z). Therefore, the elasticity tensor for the material in the virtual body is written as

$$\tilde{\mathsf{C}}^{\tilde{A}\tilde{B}\tilde{C}\tilde{D}} = \lambda \tilde{G}^{\tilde{A}\tilde{B}} \tilde{G}^{\tilde{C}\tilde{D}} + \mu(\tilde{G}^{\tilde{A}\tilde{C}} \tilde{G}^{\tilde{B}\tilde{D}} + \tilde{G}^{\tilde{A}\tilde{D}} \tilde{G}^{\tilde{C}\tilde{B}})\,. \tag{7.198}$$

In Voigt representation $\tilde{\mathsf{C}}$ reads[26]

$$\tilde{\mathsf{C}} = \begin{bmatrix} \lambda+2\mu & \lambda/\tilde{R}^2 & \lambda & 0 & 0 & 0 \\ \lambda/\tilde{R}^2 & (\lambda+2\mu)/\tilde{R}^4 & \lambda/\tilde{R}^2 & 0 & 0 & 0 \\ \lambda & \lambda/\tilde{R}^2 & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu/\tilde{R}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu/\tilde{R}^2 \end{bmatrix}. \tag{7.199}$$

As Ξ is the identity mapping for $R \geq R_o$, we note that $\mathsf{C} = \tilde{\mathsf{C}}$ in $\mathcal{B} \setminus \mathcal{C}$, for which $R \geq R_o$. For $R_i \leq R \leq R_o$ (the cloaking device region $\mathcal{C}$), we have

$$\mathsf{C}_{\mathcal{C}}^{ABCD} = J_{\Xi}^{\mathcal{C}} (\bar{\bar{F}}^{-1})^A{}_{\tilde{A}} (\mathsf{S}^{-1})^B{}_{\tilde{B}} (\bar{\bar{F}}^{-1})^C{}_{\tilde{C}} (\mathsf{S}^{-1})^D{}_{\tilde{D}} \tilde{\mathsf{C}}^{\tilde{A}\tilde{B}\tilde{C}\tilde{D}} \circ \Xi \,, \tag{7.200}$$

where $\bar{\bar{\mathbf{F}}}^{-1} = \operatorname{diag}(f'(R)^{-1}, 1, 1)$, and hence

$$J_{\Xi}^{\mathcal{C}} = \sqrt{\frac{\det \tilde{\mathbf{G}}(\tilde{X}(X))}{\det \mathbf{G}(X)}} \det \bar{\bar{\mathbf{F}}} = \frac{f(R) f'(R)}{R} \,, \quad R_i \leq R \leq R_o \,. \tag{7.201}$$

Note that

$$\mathsf{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & R/f(R) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{7.202}$$

Therefore, in the cloaking device ($R_i \leq R \leq R_o$), the elasticity tensor has the following non-zero

[26]Note that in the Voigt notation one has the following bijection between indices $\{11, 22, 33, 23, 13, 12\} \rightarrow \{1, 2, 3, 4, 5, 6\}$.

physical components[27]

$$
\begin{aligned}
&\hat{\mathsf{C}}^{RRRR} = \frac{(\lambda+2\mu)f(R)}{Rf'(R)}, \quad \hat{\mathsf{C}}^{\Theta\Theta\Theta\Theta} = \frac{(\lambda+2\mu)Rf'(R)}{f(R)}, \quad \hat{\mathsf{C}}^{RR\Theta\Theta} = \hat{\mathsf{C}}^{\Theta\Theta RR} = \lambda, \\
&\hat{\mathsf{C}}^{R\Theta R\Theta} = \frac{\mu f(R)}{Rf'(R)}, \quad \hat{\mathsf{C}}^{\Theta R\Theta R} = \frac{\mu Rf'(R)}{f(R)}, \quad \hat{\mathsf{C}}^{\Theta RR\Theta} = \hat{\mathsf{C}}^{R\Theta\Theta R} = \mu\,, \\
&\hat{\mathsf{C}}^{RRZZ} = \hat{\mathsf{C}}^{ZZRR} = \frac{\lambda f(R)}{R}, \quad \hat{\mathsf{C}}^{\Theta\Theta ZZ} = \hat{\mathsf{C}}^{ZZ\Theta\Theta} = \lambda f'(R), \quad \hat{\mathsf{C}}^{ZZZZ} = \frac{(\lambda+2\mu)f(R)f'(R)}{R}\,, \\
&\hat{\mathsf{C}}^{RZRZ} = \frac{\mu f(R)}{Rf'(R)}\,, \quad \hat{\mathsf{C}}^{ZRZR} = \frac{\mu f(R)f'(R)}{R}\,, \quad \hat{\mathsf{C}}^{ZRRZ} = \hat{\mathsf{C}}^{RZZR} = \frac{\mu f(R)}{R}\,, \\
&\hat{\mathsf{C}}^{Z\Theta Z\Theta} = \frac{\mu f(R)f'(R)}{R}\,, \quad \hat{\mathsf{C}}^{\Theta Z\Theta Z} = \frac{\mu Rf'(R)}{f(R)}\,, \quad \hat{\mathsf{C}}^{\Theta ZZ\Theta} = \hat{\mathsf{C}}^{Z\Theta\Theta Z} = \mu f'(R)\,.
\end{aligned}
\tag{7.203}
$$

Remark 7.4.11. The balance of angular momentum reads $\delta S^{AB} = \delta S^{BA}$, where $\delta S^{AB} = \mathsf{C}^{ABCD}e_{CD}$, and $e_{CD} = (\mathring{\varphi}^*\epsilon)_{CD}$. From (7.203), it is seen that the balance of angular momentum is satisfied if and only if $e_{R\Theta} = e_{\Theta Z} = e_{RZ} = 0$. In other words, for the class of infinitesimal deformations with diagonal linearized strains cloaking is possible. This is the linear analogue of what was discussed for nonlinear elastostatic cloaking in Section 7.4.2.

Remark 7.4.12. When $\mu = 0$ (energy density is not positive-definite anymore), the elastic constants of the cloak can be written as

$$
\begin{aligned}
&\hat{\mathsf{C}}^{RRRR} = \frac{\lambda f(R)}{Rf'(R)}, \quad \hat{\mathsf{C}}^{\Theta\Theta\Theta\Theta} = \frac{\lambda Rf'(R)}{f(R)}, \quad \hat{\mathsf{C}}^{RR\Theta\Theta} = \hat{\mathsf{C}}^{\Theta\Theta RR} = \lambda, \\
&\hat{\mathsf{C}}^{R\Theta R\Theta} = \hat{\mathsf{C}}^{\Theta R\Theta R} = \hat{\mathsf{C}}^{\Theta RR\Theta} = \hat{\mathsf{C}}^{R\Theta\Theta R} = 0\,, \\
&\hat{\mathsf{C}}^{RRZZ} = \hat{\mathsf{C}}^{ZZRR} = \frac{\lambda f(R)}{R}, \quad \hat{\mathsf{C}}^{\Theta\Theta ZZ} = \hat{\mathsf{C}}^{ZZ\Theta\Theta} = \lambda f'(R), \quad \hat{\mathsf{C}}^{ZZZZ} = \frac{\lambda f(R)f'(R)}{R}\,, \\
&\hat{\mathsf{C}}^{RZRZ} = \hat{\mathsf{C}}^{ZRZR} = \hat{\mathsf{C}}^{ZRRZ} = \hat{\mathsf{C}}^{RZZR} = 0\,, \\
&\hat{\mathsf{C}}^{Z\Theta Z\Theta} = \hat{\mathsf{C}}^{\Theta Z\Theta Z} = \hat{\mathsf{C}}^{\Theta ZZ\Theta} = \hat{\mathsf{C}}^{Z\Theta\Theta Z} = 0\,.
\end{aligned}
\tag{7.204}
$$

It is seen that in this special case the elastic constants of the cloak are fully symmetric. One should note that this can be seen as equivalent to acoustics [159].

[27]Note that the physical components of the elasticity tensor $\hat{C}^{ABCD}$ are related to the components of the elasticity tensor as $\hat{C}^{ABCD} = \sqrt{G_{AA}}\sqrt{G_{BB}}\sqrt{G_{CC}}\sqrt{G_{DD}}C^{ABCD}$ (no summation) [67].

Note that $\rho = J_\Xi\, \tilde{\rho} \circ \Xi$, and therefore, the mass density in $\mathcal{B} \setminus \mathcal{C}$ is homogeneous and is equal to ρ_0. The mass density in the cloaking device is inhomogeneous and is given by

$$\rho_{\mathcal{C}}(R) = \frac{f(R) f'(R)}{R} \rho_0 \,, \qquad R_i \leq R \leq R_o \,. \tag{7.205}$$

In the case of anti-plane waves, the only non-trivial equilibrium equation is the one in the z-direction, which using (7.60) for the physical body in the cloaking region is written as[28]

$$\frac{1}{R} \frac{\partial}{\partial R} \left(R \hat{\mathsf{C}}^{RZRZ}(R) \frac{\partial W}{\partial R} \right) + \frac{1}{R^2} \hat{\mathsf{C}}^{\Theta Z \Theta Z}(R) \frac{\partial^2 W}{\partial \Theta^2} + \rho_{\mathcal{C}}(R) \omega^2 W = \frac{a_m}{R_s} \delta(R - R_s) \delta(\Theta - \Theta_s) \,, \tag{7.206}$$

where a_m is the amplitude of a time-harmonic line source with frequency ω located at (R_s, Θ_s) such that the induced displacement is given as $\delta\varphi_t(R, \Theta) = \mathbf{U}(R, \Theta, t) = \big(0, 0, \Re\big[W(R, \Theta) e^{-\mathrm{i}\omega t}\big]\big)$. The only non-zero stress components in the cloak read

$$\delta S^{RZ} = \mathsf{C}^{ZRRZ} W_{|R} \,, \quad \delta S^{ZR} = \mathsf{C}^{RZRZ} W_{|R} \,, \quad \delta S^{\Theta Z} = \mathsf{C}^{Z\Theta\Theta Z} W_{|\Theta} \,, \quad \delta S^{Z\Theta} = \mathsf{C}^{\Theta Z \Theta Z} W_{|\Theta} \,. \tag{7.207}$$

As $\hat{\mathsf{C}}^{ZRRZ} \neq \hat{\mathsf{C}}^{RZRZ}$ and $\hat{\mathsf{C}}^{Z\Theta\Theta Z} \neq \hat{\mathsf{C}}^{\Theta Z \Theta Z}$ (cf. (7.203)), the balance of angular momentum is not satisfied in the simple case of anti-plane waves. One should note that the imbalance of angular momentum is actually significant as $\hat{\mathsf{C}}^{Z\Theta\Theta Z} - \hat{\mathsf{C}}^{\Theta Z \Theta Z} = \mu f'(R) \left[1 - \frac{R}{f(R)}\right]$ will be unbounded at the inner boundary of the cloak ($R = R_i$) in the limit of $f(R_i) = \epsilon \to 0$. Thus, the loss of the minor

[28] Using (7.60), one obtains

$$\big(\mathsf{A}^{aA}{}_z{}^C W^z{}_{|C}\big)_{|A} = \Big(\mathsf{C}^{ABCN} \mathring{F}^a{}_B \mathring{F}^n{}_N g_{zn} W^z{}_{|C}\Big)_{|A} = \big(\mathsf{C}^{ABCZ} \delta^a{}_B \delta^z{}_Z g_{zz} W^z{}_{|C}\big)_{|A} \,.$$

Note that

$$\begin{aligned}
\big(\mathsf{C}^{ABCZ} W_{|C}\big)_{|A} &= \big(\mathsf{C}^{RBRZ} W_{|R} + \mathsf{C}^{RB\Theta Z} W_{|\Theta}\big)_{|R} + \big(\mathsf{C}^{\Theta BRZ} W_{|R} + \mathsf{C}^{\Theta B \Theta Z} W_{|\Theta}\big)_{|\Theta} \\
&= \big(\mathsf{C}^{RZRZ} W_{|R}\big)_{|R} + \big(\mathsf{C}^{\Theta Z \Theta Z} W_{|\Theta}\big)_{|\Theta} \,.
\end{aligned}$$

symmetry of the elasticity tensor cannot be ignored. For a linear mapping[29] one has

$$\hat{\mathsf{C}}^{RZRZ}(R) = \mu\frac{R-R_i}{R}\,, \quad \hat{\mathsf{C}}^{\Theta Z\Theta Z}(R) = \mu\frac{R}{R-R_i}\,, \quad \rho_{\mathcal{C}}(R) = \frac{R_o^2}{(R_o-R_i)^2}\left(1-\frac{R_i}{R}\right)\rho_0\,. \tag{7.208}$$

These are identical to Parnell [199]'s Eq.(2.5). In the next section we will show that even in the presence of pre-stress elastodynamic cloaking is not possible for anti-plane deformations. However, pre-stress will allow for elastodynamic cloaking for in-plane deformations.

[194] and many other researchers have used a linear function for $f(R)$, which has been borrowed from the similar calculations in electromagnetism. However, a linear $f(R)$ is not acceptable for elasticity as will be explained in §7.5.2. Let us ignore this condition and use a linear $f(R)$, i.e.,

$$f(R) = \frac{R_o(R-R_i)}{R_o-R_i}\,. \tag{7.209}$$

For this (inappropriate) choice of $f(R)$ the mass density in the cloak reads

$$\rho_{\mathcal{C}}(R) = \frac{R_o^2}{(R_o-R_i)^2}\left(1-\frac{R_i}{R}\right)\rho_0\,, \tag{7.210}$$

which is identical to [194]'s mass density. Our elastic constants in the cloak read

$$\begin{aligned} &\hat{\mathsf{C}}^{RRRR} = (\lambda+2\mu)\frac{R-R_i}{R}, \quad \hat{\mathsf{C}}^{\Theta\Theta\Theta\Theta} = (\lambda+2\mu)\frac{R}{R-R_i}, \quad \hat{\mathsf{C}}^{RR\Theta\Theta} = \hat{\mathsf{C}}^{\Theta\Theta RR} = \lambda, \\ &\hat{\mathsf{C}}^{\Theta R R\Theta} = \hat{\mathsf{C}}^{R\Theta\Theta R} = \mu, \qquad \hat{\mathsf{C}}^{R\Theta R\Theta} = \frac{R-R_i}{R}\mu, \qquad \hat{\mathsf{C}}^{\Theta R\Theta R} = \frac{R}{R-R_i}\mu\,, \end{aligned} \tag{7.211}$$

which are identical to [194]'s. [194] and many other authors (e.g., Norris and Shuvalov [196]) have claimed that the lack of minor symmetry in the elastic constants of the cloak implies that a cloak is made of a Cosserat solid. This claim is, unfortunately, incorrect. We will show in §7.5.2 that transformation cloaking is not possible in (generalized) Cosserat solids.

[29]This map does not satisfy the required traction continuity condition, i.e., $f'(R_o) = 1$, but nevertheless has been extensively used in the literature (see §7.5.2).

7.4.4 Small-on-Large Transformation Cloaking

In this section we formulate transformation cloaking in the small-on-large theory of Green *et al.* [195] and explore the possibility of transformation cloaking when a cloak can be pre-stressed.

Linearized balance of angular momentum with respect to a stressed initial configuration. Note that the form of the linearized balance of linear momentum (7.172) in the case of a flat ambient space is independent of the initial stress $\mathring{\mathbf{P}}$, i.e., the balance of linear momentum when the initial configuration is pre-stressed is still (7.172). Next, we explore the possibility of transformation cloaking in linearized elasticity with respect to a pre-stressed configuration. Let us assume that the initial configuration of the virtual body is its undeformed stress-free reference configuration, while the initial configuration of the physical body is stressed within the cloak with the initial second Piola-Kirchhoff stress $\mathring{\mathbf{S}}$. Let us denote the cloak in the reference configuration by $\mathring{\mathcal{C}}$ and in the stressed initial configuration by $\mathcal{C}$. We assume that the stressed initial configuration is the result of a deformation $\mathring{\varphi}:\mathcal{B}\to\mathcal{S}$ from the stress-free configuration $(\mathcal{B},\mathbf{G})$. We consider linearized elasticity with respect to the two configurations $(\mathcal{B},\mathbf{G})$ and $(\tilde{\mathcal{B}},\tilde{\mathbf{G}})$. Assuming that the physical body has an energy function W with respect to its stress-free configuration the pre-stress $\mathring{\mathbf{P}}$ is expressed as

$$\mathring{P}^{aA} = g^{ab}\frac{\partial W}{\partial F^{b}{}_{B}}\Big|_{\mathring{\mathbf{F}}}\,. \tag{7.212}$$

We still assume that the two corresponding variation fields are related through shifters. The balance of angular momentum $S^{AB}=S^{BA}$ when linearized about any (stress-free or stressed) initial configuration reads $\delta S^{AB}=\delta S^{BA}$, which is equivalent to $\mathsf{C}^{ABCD}=\mathsf{C}^{BACD}$. We assume that $\mathring{\varphi}\big|_{\mathcal{B}\setminus\mathring{\mathcal{C}}}=id$, and $\mathring{\mathbf{F}}\big|_{\partial_o\mathcal{C}}=\mathbf{I}$ (note that $\partial_o\mathcal{C}=\partial_o\mathring{\mathcal{C}}$).

Transformed second elasticity tensor. As the form of the balance of linear momentum is not affected by pre-stress, the transformation relation for the first elasticity tensor is still (7.180), i.e.,

$\tilde{\mathsf{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} \circ \Xi = J_{\Xi}^{-1} \bar{\bar{F}}^{\tilde{A}}{}_{A} \bar{\bar{F}}^{\tilde{B}}{}_{B}\, \mathsf{s}^{\tilde{a}}{}_{a} \mathsf{s}^{\tilde{b}}{}_{b}\, \mathsf{A}^{aAbB}$. From (7.60) we have the following two relations

$$\begin{aligned} \mathsf{A}^{aAbB} &= \mathsf{C}^{AMBN} \mathring{F}^{a}{}_{M} \mathring{F}^{b}{}_{N} + \mathring{S}^{AB} g^{ab}, \\ \tilde{\mathsf{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} &= \tilde{\mathsf{C}}^{\tilde{A}\tilde{M}\tilde{B}\tilde{N}} \mathring{\tilde{F}}^{\tilde{a}}{}_{\tilde{M}} \mathring{\tilde{F}}^{\tilde{b}}{}_{\tilde{N}} \,. \end{aligned} \tag{7.213}$$

When the reference configuration of the physical body is pre-stressed the cloaking map is not $\Xi : \mathcal{B} \to \tilde{\mathcal{B}}$; it is the map $\xi : \mathring{\varphi}(\mathcal{B}) \to \tilde{\mathcal{S}}$ (see Fig.7.3). Note that $\bar{\bar{\mathbf{F}}} = \mathring{\tilde{\mathbf{F}}}^{-1} \overset{\xi}{\mathbf{F}} \mathring{\mathbf{F}}$, where $\mathring{\tilde{\mathbf{F}}} = id$, i.e., $(\tilde{\mathbf{F}})^{\tilde{a}}{}_{\tilde{A}} = \delta^{\tilde{a}}_{\tilde{A}}$. Using (7.180) and (7.213), one can calculate the second elasticity constants of the physical body as

$$\begin{aligned} \mathsf{C}^{ABCD} = J_{\Xi} &\left[(\mathring{F}^{-1})^{B}{}_{k} (\mathsf{s}^{-1})^{k}{}_{\tilde{a}} \right] \left[(\mathring{F}^{-1})^{D}{}_{l} (\mathsf{s}^{-1})^{l}{}_{\tilde{b}} \right] \left[(\mathring{F}^{-1})^{A}{}_{m} (\overset{\xi}{F}{}^{-1})^{m}{}_{\tilde{m}} \delta^{\tilde{m}}_{\tilde{A}} \right] \left[(\mathring{F}^{-1})^{C}{}_{n} (\overset{\xi}{F}{}^{-1})^{n}{}_{\tilde{n}} \delta^{\tilde{n}}_{\tilde{B}} \right] \tilde{\mathsf{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} \\ &- \mathring{S}^{AC} (\mathring{\varphi}^{*} \mathbf{g}^{\sharp})^{BD} \,. \end{aligned} \tag{7.214}$$

It is observed that under a cloaking map the major symmetries are preserved. Pushing forward the elastic constants to the initial configuration, one obtains

$$(\mathring{\varphi}_{*}\mathsf{C})^{abcd} = J_{\Xi} (\mathsf{s}^{-1})^{b}{}_{\tilde{a}} (\mathsf{s}^{-1})^{d}{}_{\tilde{b}} \left[(\overset{\xi}{F}{}^{-1})^{a}{}_{\tilde{m}} \delta^{\tilde{m}}_{\tilde{A}} \right] \left[(\overset{\xi}{F}{}^{-1})^{c}{}_{\tilde{n}} \delta^{\tilde{n}}_{\tilde{B}} \right] \tilde{\mathsf{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} - (\mathring{\varphi}_{*}\mathring{S})^{ac} g^{bd} \,. \tag{7.215}$$

Note that $J_{\Xi} = J_{\xi} \mathring{J}$, and hence

$$\mathsf{c}^{abcd} = \frac{1}{\mathring{J}} (\mathring{\varphi}_{*}\mathsf{C})^{abcd} = J_{\xi} \left[(\overset{\xi}{F}{}^{-1})^{a}{}_{\tilde{m}} \delta^{\tilde{m}}_{\tilde{A}} \right] (\mathsf{s}^{-1})^{b}{}_{\tilde{a}} \left[(\overset{\xi}{F}{}^{-1})^{c}{}_{\tilde{n}} \delta^{\tilde{n}}_{\tilde{B}} \right] (\mathsf{s}^{-1})^{d}{}_{\tilde{b}}\, \tilde{\mathsf{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} - \mathring{\sigma}^{ac} g^{bd} \,. \tag{7.216}$$

The balance of angular momentum in the physical body is equivalent to $\mathsf{c}^{[ab]cd} = 0$, which is written as

$$\mathring{\sigma}^{[ac} g^{b]d} = J_{\xi} \left[(\overset{\xi}{F}{}^{-1})^{[a}{}_{\tilde{m}} \delta^{\tilde{m}}_{\tilde{A}} \right] (\mathsf{s}^{-1})^{b]}{}_{\tilde{a}} \left[(\overset{\xi}{F}{}^{-1})^{c}{}_{\tilde{n}} \delta^{\tilde{n}}_{\tilde{B}} \right] (\mathsf{s}^{-1})^{d}{}_{\tilde{b}}\, \tilde{\mathsf{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} \,. \tag{7.217}$$

Note that $\mathring{\sigma}^{ac} = \mathring{\sigma}^{ca}$ is not specified a priori.

From (7.180), the referential mass densities of the physical and virtual bodies are related as $\rho_0 = J_{\Xi} \tilde{\rho}_0 \circ \Xi$. Conservation of mass for the virtual and physical bodies imply that $\rho_0 = \rho \mathring{J}$ and $\tilde{\rho}_0 = \tilde{\rho}$. Therefore, the spatial mass density of the cloak is given by $\rho = J_{\xi} \tilde{\rho}$.

Proposition 7.4.13. *In the small-on-large theory, i.e., linearized elasticity with respect to a pre-*

stressed configuration, elastodynamic transformation cloaking is not possible regardless of the shape of the hole and the cloak.

Proof. Let us assume that the physical body is pre-stressed with the initial Cauchy stress $\mathring{\boldsymbol{\sigma}}$ and the initial body force $\mathring{\mathbf{b}}$. The balance of linear and angular momenta for the physical body in its initial configuration read

$$\mathring{\sigma}^{ab}{}_{|b} + \rho\mathring{b}^{a} = 0\,, \quad \text{and} \quad \mathring{\sigma}^{[ab]} = 0\,. \tag{7.218}$$

Note that $\mathring{\boldsymbol{\sigma}}$ and $\mathring{\mathbf{b}}$ are supported on the interior of the cloaking region, i.e., on $\mathcal{C} \setminus \{\partial\mathcal{H} \cup \partial_o\mathcal{C}\}$. Without loss of generality, we may work in Cartesian coordinates, where the shifter and the metrics have trivial representations. In Cartesian coordinates, the first elasticity tensor for the virtual body is given by $\tilde{\mathsf{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} = \left[\lambda\delta^{\tilde{A}\tilde{a}}\delta^{\tilde{B}\tilde{b}} + \mu(\delta^{\tilde{A}\tilde{B}}\delta^{\tilde{a}\tilde{b}} + \delta^{\tilde{A}\tilde{b}}\delta^{\tilde{B}\tilde{a}})\right]$. With an abuse of notation let $(\overset{\xi}{F}{}^{-1})^{i}{}_{j} = \mathsf{F}_{ij}, i, j \in \{1,2,3\}$, and thus, $J_\xi = \left[\epsilon_{lmn}\epsilon_{ijk}\mathsf{F}_{il}\mathsf{F}_{jm}\mathsf{F}_{kn}\right]^{-1}$. Expanding (7.217), one obtains the following relations for $i \neq j \neq k \in \{1,2,3\}$

$$\mathring{\sigma}^{ii} = J_\xi\left[\mathsf{F}_{ij}\left(\mathsf{F}_{ij} - \mathsf{F}_{ji}\right)\lambda + \left(\mathsf{F}_{ii}^2 + 2\mathsf{F}_{ij}^2 + \mathsf{F}_{ik}^2 - \mathsf{F}_{ii}\mathsf{F}_{jj}\right)\mu\right]\,, \tag{7.219}$$

$$\mathring{\sigma}^{ii} = J_\xi\left[\mathsf{F}_{ik}\left(\mathsf{F}_{ik} - \mathsf{F}_{ki}\right)\lambda + \left(\mathsf{F}_{ii}^2 + 2\mathsf{F}_{ik}^2 + \mathsf{F}_{ij}^2 - \mathsf{F}_{ii}\mathsf{F}_{kk}\right)\mu\right]\,, \tag{7.220}$$

$$\mathring{\sigma}^{ij} = J_\xi\left[\mathsf{F}_{ii}\left(\mathsf{F}_{ji} - \mathsf{F}_{ij}\right)\lambda + \left(\mathsf{F}_{ij}\mathsf{F}_{jj} + 2\mathsf{F}_{ii}\mathsf{F}_{ji} + \mathsf{F}_{ik}\mathsf{F}_{jk} - \mathsf{F}_{ii}\mathsf{F}_{ij}\right)\mu\right]\,, \tag{7.221}$$

$$\mathring{\sigma}^{ij} = J_\xi\left[\mathsf{F}_{jj}\left(\mathsf{F}_{ij} - \mathsf{F}_{ji}\right)\lambda + \left(\mathsf{F}_{ii}\mathsf{F}_{ji} + 2\mathsf{F}_{jj}\mathsf{F}_{ij} + \mathsf{F}_{ik}\mathsf{F}_{jk} - \mathsf{F}_{jj}\mathsf{F}_{ji}\right)\mu\right]\,, \tag{7.222}$$

$$\mathsf{F}_{ii}\left(\mathsf{F}_{kj} - \mathsf{F}_{jk}\right)\lambda + \left(\mathsf{F}_{ij}\mathsf{F}_{ki} - \mathsf{F}_{ik}\mathsf{F}_{ji}\right)\mu = 0\,, \tag{7.223}$$

$$\mathsf{F}_{ik}\left(\mathsf{F}_{ji} - \mathsf{F}_{ij}\right)\lambda + \left(\mathsf{F}_{ii}\mathsf{F}_{jk} - \mathsf{F}_{ij}\mathsf{F}_{ik}\right)\mu = 0\,. \tag{7.224}$$

Using (7.221) and (7.222), one has

$$J_\xi\,(\lambda+\mu)\,(\mathsf{F}_{ij}-\mathsf{F}_{ji})\,(\mathsf{F}_{ii}+\mathsf{F}_{jj})=0\,. \tag{7.225}$$

Noting that $\lambda+\mu=\frac{1}{3}(3\lambda+2\mu)+\frac{1}{3}\mu>0$, either $\mathsf{F}_{ij}=\mathsf{F}_{ji}$, or $\mathsf{F}_{ii}=-\mathsf{F}_{jj}$. If $\mathsf{F}_{ii}=-\mathsf{F}_{jj}$, then it follows that $\mathsf{F}_{ii}=0$, and from (7.223), $\mathsf{F}_{ij}\mathsf{F}_{ki}=\mathsf{F}_{ik}\mathsf{F}_{ji}$. Using this relation and (7.224), one obtains $\mathsf{F}_{ij}\mathsf{F}_{ki}\lambda=\mathsf{F}_{ij}\mathsf{F}_{ik}\,(\lambda+\mu)$. This relation holds if $\mathsf{F}_{ij}=0$. If $\mathsf{F}_{ij}\neq 0$, then $\mathsf{F}_{ki}\lambda=\mathsf{F}_{ik}\,(\lambda+\mu)$. This can be rewritten as $\mathsf{F}_{ik}\lambda=\mathsf{F}_{ki}\,(\lambda+\mu)$, which implies that $\mathsf{F}_{ik}=0$. Therefore, $\overset{\xi}{\mathbf{F}}{}^{-1}=\mathbf{0}$, which is not possible. Thus, $\mathsf{F}_{ij}=\mathsf{F}_{ji}$, and hence, using (7.224), $\mathsf{F}_{ii}\mathsf{F}_{jk}=\mathsf{F}_{ij}\mathsf{F}_{ik}$. Therefore, either $\mathsf{F}_{jk}=0$, or $\mathsf{F}_{ii}=\mathsf{F}_{ij}\mathsf{F}_{ik}/\mathsf{F}_{jk}$. If $\mathsf{F}_{jk}=0$, then either $\mathsf{F}_{ij}=0$ or $\mathsf{F}_{ik}=0$. Without loss of generality, let us assume that $\mathsf{F}_{ij}=0$ for some $i\neq j$ and $\mathsf{F}_{ik}\neq 0$ ($\mathsf{F}_{jk}=\mathsf{F}_{ij}=0$, $\mathsf{F}_{ik}\neq 0$). As $\mathsf{F}_{ik}\neq 0$, from $\mathsf{F}_{jj}\mathsf{F}_{ik}=\mathsf{F}_{ji}\mathsf{F}_{jk}$, we note that $\mathsf{F}_{jj}=0$. Therefore, $\overset{\xi}{\mathbf{F}}{}^{-1}$ has a zero column vector, i.e., it is singular, which is not possible. Our conclusion is that either all the off-diagonal terms of $\overset{\xi}{\mathbf{F}}{}^{-1}$ are zero or they are all non-zero. If $\mathsf{F}_{ij}=\mathsf{F}_{jk}=\mathsf{F}_{ik}=0$, then $\overset{\xi}{\mathbf{F}}=\beta\mathbf{I}$ (cf.[30] (7.226)) for some nonzero scalar β, which means that cloaking is not possible. If the off-diagonal terms are non-zero, then $\mathsf{F}_{ii}=\mathsf{F}_{ij}\mathsf{F}_{ik}/\mathsf{F}_{jk}$, $\mathsf{F}_{jj}=\mathsf{F}_{ji}\mathsf{F}_{jk}/\mathsf{F}_{ik}$, and $\mathsf{F}_{kk}=\mathsf{F}_{kj}\mathsf{F}_{ki}/\mathsf{F}_{ji}$. Substituting these relations into $J_\xi^{-1}=\epsilon_{lmn}\epsilon_{ijk}\mathsf{F}_{il}\mathsf{F}_{jm}\mathsf{F}_{kn}$, it is straightforward to see that $J_\xi^{-1}=0$, i.e., the inverse cloaking map is singular, which is not possible. $\square$

If one uses linearized elasticity with respect to a pre-stressed configuration the best one can do is to perform cloaking for a restricted class of deformations. In the following, we will find such an example in the case of a cylindrical cloak.

An example of a pre-stressed linear elastic cloak. Let us consider the cylindrical cloak example in the presence of the initial stress $\mathring{\boldsymbol{\sigma}}$. The cloaking map ξ transforms a pre-stressed cylindrical annulus in the physical body with inner and outer radii r_i and r_o, respectively, to a cylindrical annulus

[30] Knowing that $\mathsf{F}_{ij}=\mathsf{F}_{ji}$, from (7.219) and (7.220), one obtains

$$0=\mathsf{F}_{ij}^2-\mathsf{F}_{ik}^2=\mathsf{F}_{ii}\,(\mathsf{F}_{jj}-\mathsf{F}_{kk})\,. \tag{7.226}$$

in the virtual body with inner and outer radii ϵ and r_o, respectively. In cylindrical coordinates let $(\tilde{r},\tilde{\theta},\tilde{z}) = \xi(r,\theta,z) = (f(r),\theta,z)$. The tangent map of the cloaking transformation is given by

$$\overset{\xi}{\mathbf{F}} = \begin{bmatrix} f'(r) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \tag{7.227}$$

where $f(r_o) = r_o$ and $f(r_i) = \epsilon$. Thus

$$J_\xi = \sqrt{\frac{\det \tilde{\mathbf{g}}(\tilde{x}(x))}{\det \mathbf{g}(x)}} \det \overset{\xi}{\mathbf{F}} = \frac{f(r)f'(r)}{r} , \tag{7.228}$$

where $\mathbf{g} = \mathrm{diag}(1, r^2, 1)$ and $\tilde{\mathbf{g}} = \mathrm{diag}(1, \tilde{r}^2, 1)$ are, respectively, the Euclidean metrics of the physical and virtual bodies in their current configurations. Note that $\mathbf{s} = \mathrm{diag}(1, r/f(r), 1)$ and

$$\tilde{\mathsf{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} = \left[\lambda \tilde{g}^{\tilde{A}\tilde{M}}\tilde{g}^{\tilde{B}\tilde{N}} + \mu\left(\tilde{g}^{\tilde{A}\tilde{B}}\tilde{g}^{\tilde{M}\tilde{N}} + \tilde{g}^{\tilde{A}\tilde{N}}\tilde{g}^{\tilde{B}\tilde{M}}\right)\right] \mathring{\tilde{F}}^{\tilde{a}}{}_{\tilde{M}} \mathring{\tilde{F}}^{\tilde{b}}{}_{\tilde{N}} , \tag{7.229}$$

where $\mathring{\tilde{F}}^{\tilde{a}}{}_{\tilde{A}} = \delta^{\tilde{a}}{}_{\tilde{A}}$. Noting that $\rho = J_\xi \tilde{\rho}$, the mass density of the cloaking device in its current configuration is given by

$$\rho(r) = \frac{f(r)f'(r)}{r}\tilde{\rho} , \qquad r_i \le r \le r_o . \tag{7.230}$$

Using (7.216), the non-zero physical components of the elasticity tensor of the cloaking device in the current configuration are written as ($r_i \leq r \leq r_o$)

$$
\begin{aligned}
&\hat{\mathsf{c}}^{rrrr} = \frac{(\lambda+2\mu)f(r)}{rf'(r)} - \mathring{\hat{\sigma}}^{rr}\,, \quad \hat{\mathsf{c}}^{\theta\theta\theta\theta} = \frac{(\lambda+2\mu)rf'(r)}{f(r)} - \mathring{\hat{\sigma}}^{\theta\theta}\,, \quad \hat{\mathsf{c}}^{rr\theta\theta} = \hat{\mathsf{c}}^{\theta\theta rr} = \lambda\,,\\
&\hat{\mathsf{c}}^{r\theta r\theta} = \frac{\mu f(r)}{rf'(r)} - \mathring{\hat{\sigma}}^{rr}, \qquad \hat{\mathsf{c}}^{\theta r\theta r} = \frac{\mu rf'(r)}{f(r)} - \mathring{\hat{\sigma}}^{\theta\theta}, \qquad \hat{\mathsf{c}}^{\theta rr\theta} = \hat{\mathsf{c}}^{r\theta\theta r} = \mu\,,\\
&\hat{\mathsf{c}}^{rrzz} = \hat{\mathsf{c}}^{zzrr} = \frac{\lambda f(r)}{r}\,, \qquad \hat{\mathsf{c}}^{\theta\theta zz} = \hat{\mathsf{c}}^{zz\theta\theta} = \lambda f'(r)\,, \qquad \hat{\mathsf{c}}^{zzzz} = \frac{(\lambda+2\mu)f(r)f'(r)}{r} - \mathring{\hat{\sigma}}^{zz}\,,\\
&\hat{\mathsf{c}}^{rzrz} = \frac{\mu f(r)}{rf'(r)} - \mathring{\hat{\sigma}}^{rr}\,, \qquad \hat{\mathsf{c}}^{zrzr} = \frac{\mu f(r)f'(r)}{r} - \mathring{\hat{\sigma}}^{zz}\,, \quad \hat{\mathsf{c}}^{zrrz} = \hat{\mathsf{c}}^{rzzr} = \frac{\mu f(r)}{r}\,,\\
&\hat{\mathsf{c}}^{z\theta z\theta} = \frac{\mu f(r)f'(r)}{r} - \mathring{\hat{\sigma}}^{zz}\,, \quad \hat{\mathsf{c}}^{\theta z\theta z} = \frac{\mu rf'(r)}{f(r)} - \mathring{\hat{\sigma}}^{\theta\theta}\,, \qquad \hat{\mathsf{c}}^{\theta zz\theta} = \hat{\mathsf{c}}^{z\theta\theta z} = \mu f'(r)\,,\\
&\hat{\mathsf{c}}^{rr\theta r} = \hat{\mathsf{c}}^{\theta rrr} = -\mathring{\hat{\sigma}}^{r\theta}\,, \qquad \hat{\mathsf{c}}^{rrzr} = \hat{\mathsf{c}}^{zrrr} = -\mathring{\hat{\sigma}}^{rz}\,, \qquad \hat{\mathsf{c}}^{r\theta\theta\theta} = \hat{\mathsf{c}}^{\theta\theta r\theta} = -\mathring{\hat{\sigma}}^{r\theta}\,,\\
&\hat{\mathsf{c}}^{r\theta z\theta} = \hat{\mathsf{c}}^{z\theta r\theta} = -\mathring{\hat{\sigma}}^{rz}\,, \qquad \hat{\mathsf{c}}^{rz\theta z} = \hat{\mathsf{c}}^{\theta zrz} = -\mathring{\hat{\sigma}}^{r\theta}\,, \qquad \hat{\mathsf{c}}^{rzzz} = \hat{\mathsf{c}}^{zzrz} = -\mathring{\hat{\sigma}}^{rz}\,,\\
&\hat{\mathsf{c}}^{\theta rzr} = \hat{\mathsf{c}}^{zr\theta r} = -\mathring{\hat{\sigma}}^{\theta z}\,, \qquad \hat{\mathsf{c}}^{\theta\theta z\theta} = \hat{\mathsf{c}}^{z\theta\theta\theta} = -\mathring{\hat{\sigma}}^{\theta z}\,, \qquad \hat{\mathsf{c}}^{\theta zzz} = \hat{\mathsf{c}}^{zz\theta z} = -\mathring{\hat{\sigma}}^{\theta z}\,.
\end{aligned}
\tag{7.231}
$$

Let us examine the possibility of designing a cylindrical cloak using pre-stress for the special cases of in-plane and anti-plane deformations.

(i) Anti-plane deformations: In this case, the only non-zero components of the strain tensor are ϵ_{rz} and $\epsilon_{\theta z}$. Therefore, from (7.231), the balance of angular momentum, i.e., $\mathsf{c}^{[ab]rz} = \mathsf{c}^{[ab]zr} = \mathsf{c}^{[ab]\theta z} = \mathsf{c}^{[ab]z\theta} = 0$, implies that

$$
\begin{aligned}
&\mathring{\hat{\sigma}}^{rz} = \mathring{\hat{\sigma}}^{\theta z} = 0\,, \quad \text{for} \quad a=r,\ b=\theta\,,\\
&\mathring{\hat{\sigma}}^{rr} = \frac{\mu f(r)}{r}\left[\frac{1}{f'(r)} - 1\right]\,, \quad \mathring{\hat{\sigma}}^{zz} = \frac{\mu f(r)}{r}\left[f'(r) - 1\right]\,, \quad \mathring{\hat{\sigma}}^{r\theta} = 0\,, \quad \text{for} \quad a=r,\ b=z\,,\\
&\mathring{\hat{\sigma}}^{\theta\theta} = \mu f'(r)\left[\frac{r}{f(r)} - 1\right]\,, \quad \mathring{\hat{\sigma}}^{zz} = \mu f'(r)\left[\frac{f(r)}{r} - 1\right]\,, \quad \mathring{\hat{\sigma}}^{r\theta} = 0\,, \quad \text{for} \quad a=\theta,\ b=z\,.
\end{aligned}
\tag{7.232}
$$

Thus, from the two expressions for $\mathring{\hat{\sigma}}^{zz}$ it follows that $rf'(r) = f(r)$. Knowing that $f(r_o) = r_o$, one obtains $f(r) = r$, which implies that cloaking is not possible for anti-plane deformations.

(ii) In-plane deformations: For the in-plane case, the non-zero components of the linearized strain are ϵ_{rr}, $\epsilon_{\theta\theta}$, and $\epsilon_{r\theta}$. The balance of angular momentum $\mathsf{c}^{[ab]rr} = \mathsf{c}^{[ab]\theta\theta} = \mathsf{c}^{[ab]r\theta} = \mathsf{c}^{[ab]\theta r} = 0$, gives

$$
\begin{aligned}
\mathring{\hat{\sigma}}^{r\theta} = 0\,, \quad \mathring{\hat{\sigma}}^{rr} = \mu\left[\frac{f(r)}{rf'(r)} - 1\right]\,, \quad \mathring{\hat{\sigma}}^{\theta\theta} = \mu\left[\frac{rf'(r)}{f(r)} - 1\right]\,, \quad &\text{for} \quad a = r,\ b = \theta\,,\\
\mathring{\hat{\sigma}}^{rz} = \mathring{\hat{\sigma}}^{\theta z} = 0\,, \quad &\text{for} \quad a = r,\ b = z\,,\\
\mathring{\hat{\sigma}}^{rz} = \mathring{\hat{\sigma}}^{\theta z} = 0\,, \quad &\text{for} \quad a = \theta,\ b = z\,.
\end{aligned}
\tag{7.233}
$$

The traction vector in the physical and virtual bodies are given by

$$
T^a = \mathring{P}^{aA} N_A + \delta P^{aA} N_A\,, \quad \text{and} \quad \tilde{T}^{\tilde{a}} = \delta \tilde{P}^{\tilde{a}\tilde{A}} \tilde{N}_{\tilde{A}}\,,
\tag{7.234}
$$

where $\delta\mathbf{P}$ and $\delta\tilde{\mathbf{P}}$ are, respectively, the linearized first Piola-Kirchhoff stress tensors for the physical and virtual bodies related by $\delta P^{aA} = J_{\Xi}(\overset{\Xi}{\bar{F}}{}^{-1})^A{}_{\tilde{A}}(\mathsf{s}^{-1})^a{}_{\tilde{a}}\delta\tilde{P}^{\tilde{a}\tilde{A}}$, and $\mathring{\mathbf{P}}$ is the initial first Piola-Kirchhoff stress in the physical body. On the outer boundary of the cloak $\partial_o\mathcal{C}$, the traction vector in the two problems must be identical. Therefore, the initial traction in the physical body must vanish on the outer boundary of the cloak, viz., $\mathring{\mathbf{t}}|_{\partial_o\mathcal{C}} = \langle\!\langle \mathring{\boldsymbol{\sigma}}, \boldsymbol{n} \rangle\!\rangle_{\mathbf{g}}|_{\partial_o\mathcal{C}} = \mathbf{0}$, where $\mathbf{n}$ is the normal to the outer boundary of the cloak in its current configuration. Thus, $\mathring{\hat{\sigma}}^{rr}(r_o) = \mathring{\hat{\sigma}}^{r\theta}(r_o) = 0$, which holds if $f'(r_o) = 1$ (i.e., $\overset{\xi}{\mathbf{F}}\big|_{\partial_o\mathcal{C}} = \mathbf{I}$). To ensure that the hole surface is traction-free in the physical body, the initial traction needs to vanish on $\partial\mathcal{H}$ as well, i.e., $\mathring{\mathbf{t}}|_{\partial\mathcal{H}} = \mathbf{0}$.[31] Hence, it follows that $\mathring{\hat{\sigma}}^{rr}(r_i) = \mathring{\hat{\sigma}}^{r\theta}(r_i) = 0$, amounting to $f'(r_i) = \epsilon/r_i$. Moreover, the linearized tractions must also be identical on the outer boundary of the cloak in the two systems, i.e., one must have, $\overset{\Xi}{\bar{\mathbf{F}}}|_{\partial_o\mathcal{C}} = \mathbf{I}$. But note that $\overset{\Xi}{\bar{\mathbf{F}}} = \mathring{\bar{\mathbf{F}}}{}^{-1}\overset{\xi}{\mathbf{F}}\mathring{\mathbf{F}}$, and hence, $\overset{\Xi}{\bar{\mathbf{F}}}|_{\partial_o\mathcal{C}} = \overset{\xi}{\mathbf{F}}|_{\partial_o\mathcal{C}}\ \mathring{\mathbf{F}}|_{\partial_o\mathcal{C}} = \mathbf{I}$ already holds. Equilibrium equations in the initial configuration $\mathring{\sigma}^{ab}{}_{|b} + \rho\mathring{b}^a = 0$, has only one non-trivial component, which in terms of the physical components

[31] The inner boundary of the virtual and physical holes must be traction-free. This condition for the virtual hole reads $\delta\tilde{\mathbf{t}} = \delta\tilde{P}^{\tilde{a}\tilde{A}}\tilde{N}_{\tilde{A}}|_{\partial\tilde{\mathcal{H}}} = 0$. However, note that $\mathbf{s}\,\delta\mathbf{t}\,dA = \delta\tilde{\mathbf{t}}\,d\tilde{A} = \mathbf{0}$, i.e., the linearized traction in the physical body vanishes, and thus, in order for the hole to be traction-free in the physical body, the initial traction must vanish, i.e., $\mathring{P}^{aA}N_A|_{\partial\mathcal{H}} = 0$.

of the Cauchy stress reads

$$\frac{d}{dr}\mathring{\hat{\sigma}}^{rr} + \frac{1}{r}(\mathring{\hat{\sigma}}^{rr} - \mathring{\hat{\sigma}}^{\theta\theta}) + \rho(r)\mathring{b}^{r} = 0. \tag{7.235}$$

Therefore

$$\rho(r)\mathring{b}^{r}(r) = -\mu\left[\frac{f(r)}{rf'(r)} - 1\right]' + \frac{\mu}{r}\left[\frac{rf'(r)}{f(r)} - \frac{f(r)}{rf'(r)}\right] = \mu\left[\frac{f(r)f''(r)}{rf'(r)^2} + \frac{f'(r)}{f(r)} - \frac{1}{r}\right]. \tag{7.236}$$

Note that the simplest polynomial that satisfies the boundary conditions $f(r_o) = r_o$, $f(r_i) = \epsilon$, $f'(r_i) = \epsilon/r_i$, and $f'(r_o) = 1$ is

$$\begin{aligned} f(r) =& \frac{(r_i + r_o)(r_i - \epsilon)}{r_i(r_i - r_o)^3}r^3 - \frac{2\left(r_i^2 + r_i r_o + r_o^2\right)(r_i - \epsilon)}{r_i(r_i - r_o)^3}r^2 \\ &+ \frac{r_i^2\left(r_i^2 + r_i r_o + 4r_o^2\right) - r_o\epsilon\left(4r_i^2 + r_i r_o + r_o^2\right)}{r_i(r_i - r_o)^3}r + \frac{2r_i r_o^2(\epsilon - r_i)}{(r_i - r_o)^3}. \end{aligned} \tag{7.237}$$

7.5 Elastodynamic Transformation Cloaking in Solids with Microstructure

Having established that classical linear elasticity is not flexible enough to allow for transformation cloaking a possible solution would be to see if transformation cloaking may be achieved in solids with microstructure. In such continua the Cauchy stress does not need to be symmetric. This is the reason that in the literature of elastodynamic transformation cloaking it has been suggested that a cloak should be made of a Cosserat solid. The first systematic formulation of generalized continua goes back to the seminal work of Cosserat brothers [226], which remained unnoticed until the interest in continua with microstructure was revived in the 1950s, 1960s, and 1970s [227, 228, 229, 230, 231, 232] and now there is a vast literature on generalized continua.

In formulating the cloaking problem, we assume that both the virtual and physical bodies are made of solids with microstructure. We can follow two paths: i) Assume that energy depends on $\nabla\mathbf{F}$ with components $F^a{}_{A|B}$, i.e., gradient elasticity. This seems to be more natural for our purposes because first there is no ambiguity in the physical meaning of microstructure, and second there are well-established connections between strain gradient elasticity and atomistic calculations (see [233] and references therein). ii) In addition to the deformation mapping φ assume a set of director fields that have their own independent kinematics, i.e., (generalized) Cosserat elasticity. Energy depends on these extra fields as well. One should note that there are many different choices for describing microstcuture. This has been discussed in some detail in the monograph [234]. See also [208]. We discuss elastodynamic transformation cloaking in both gradient and generalized Cosserat solids. The goal of this section is to show that adding microstructure does not remedy the obstruction to elastodynamic transformation cloaking.

7.5.1 Elastodynamic Transformation Cloaking in Gradient Elastic Solids

In gradient elasticity (or strain-gradient elasticity) energy function depends on the (covariant) derivative of the deformation gradient as well, i.e., [229]

$$W = W(X, \mathbf{F}, \nabla\mathbf{F}, \mathbf{G}, \mathbf{g} \circ \varphi)\,. \tag{7.238}$$

Note that (bulk) compatibility equations are written as ${F^a}_{A|B} = {F^a}_{B|A}$ [235]. Objectivity of an energy function means invariance under (rigid-body) rotations in the ambient space. One can think of $\mathbf{F}$ as a vector-valued 1-form [207]. The 1-form part would not be affected by changes of coordinates in the ambient space. Now the question is how should W depend on $\mathbf{F}$ in order to be isotropic? Suppose $f = f(\mathbf{u})$, where $\mathbf{u}$ is a vector. We know that for f to be isotropic it should have the form $f = \hat{f}(\mathbf{u} \cdot \mathbf{u})$, where $\mathbf{u} \cdot \mathbf{u} = u^a u^b g_{ab}$ [49]. This new variable in the case of deformation gradient is ${F^a}_A {F^b}_A g_{ab} = C_{AB}$. In gradient elasticity one has an extra independent variable. Let us first consider a scalar function of two vectors $f = f(\mathbf{u}, \mathbf{v})$. For f to be isotropic, one must have $f = \hat{f}(\mathbf{u} \cdot \mathbf{u}, \mathbf{u} \cdot \mathbf{v}, \mathbf{v} \cdot \mathbf{v})$. In the case of energy function these three variables are

$${F^a}_A {F^b}_B g_{ab} = C_{AB}, \quad {F^a}_A {F^b}_{B|C} g_{ab} =: D_{ABC}, \quad {F^a}_{A|B} {F^b}_{C|D} g_{ab} =: E_{ABCD}\,. \tag{7.239}$$

Note the symmetries of the new measures of strain: $D_{ABC} = D_{ACB}$, $E_{ABCD} = E_{CDAB} = E_{BACD} = E_{ABDC} = E_{BADC}$. The strain measure $\mathbf{E}$ is, however, functionally dependent on $\mathbf{C}^\flat$ and $\mathbf{D}$ as noticed by Toupin [229]. More specifically, given $\mathbf{D}$, one can write $F^b{}_{B|C} = g^{ab}(F^{-1})^A{}_a D_{ABC}$. Hence, $E_{ABCD} = C^{-MN} D_{MAB} D_{NCD}$. Therefore, $W = \hat{W}(X, C_{AB}, D_{ABC}, G_{AB})$. Note that $C_{AB|C} = F^a{}_{A|C} F^b{}_B g_{ab} + F^a{}_A F^b{}_{B|C} g_{ab} = D_{BAC} + D_{ABC}$. Thus, $D_{ABC} = \frac{1}{2}\left(C_{AB|C} + C_{AC|B} - C_{BC|A}\right)$. Therefore, one can assume that energy density depends on $\mathbf{C}^\flat$ and $\nabla^{\mathbf{G}}\mathbf{C}^\flat$, i.e., $W = \hat{W}(X, C_{AB}, C_{AB|C}, G_{AB})$, which is exactly the way Toupin expressed the energy density.

Balance of linear momentum. We next derive the governing equations of gradient elasticity using Hamilton's principle of least action. Assuming the Lagrangian density $\mathcal{L} = W - T$, where $T = \frac{1}{2}\rho_0 \langle\!\langle \mathbf{V}, \mathbf{V} \rangle\!\rangle_{\mathbf{g}} = \frac{1}{2}\rho_0 V^a V^b g_{ab}$ is the classical kinetic energy density, the action is defined as

$$S = \int_{t_1}^{t_2} \int_{\mathcal{B}} (W - T) dV dt \,, \tag{7.240}$$

where $t_1 < t_2$ are arbitrary time instances, and dV is the Riemannian volume element of $(\mathcal{B}, \mathbf{G})$. Hamilton's principle is written as

$$\delta S = \int_{t_1}^{t_2} \int_{\mathcal{B}} (\delta W - \delta T) dV dt = \int_{\partial_t \mathcal{B}} T^a \delta\varphi^b g_{ab} dA \,, \tag{7.241}$$

where T^a is traction, and $\partial_t \mathcal{B} \subset \partial\mathcal{B}$ is part of the boundary on which tractions are specified. The only term that is different from that of classical nonlinear elasticity is δW, which is calculated as

$$\delta W = \nabla^{\varphi_\epsilon}_{\frac{\partial}{\partial\epsilon}} W \Big|_{\epsilon=0} = \frac{\partial W}{\partial \mathbf{F}} : \delta\mathbf{F} + \frac{\partial W}{\partial \nabla\mathbf{F}} : \delta\nabla^\varphi \mathbf{F} = \frac{\partial W}{\partial F^a{}_A} \delta F^a{}_A + \frac{\partial W}{\partial F^a{}_{A|B}} \delta F^a{}_{A|B} \,, \tag{7.242}$$

where we used the fact that $\nabla\mathbf{g} = \mathbf{0}$. The covariant derivative of the deformation gradient is linearized as follows.

$$\delta F^a{}_{A|B} = \nabla_{\frac{\partial}{\partial\epsilon}} \nabla_{\frac{\partial}{\partial X^B}} \frac{\partial \varphi^a_\epsilon}{\partial X^A}\Big|_{\epsilon=0} = \nabla_{\frac{\partial}{\partial X^B}} \nabla_{\frac{\partial}{\partial\epsilon}} \frac{\partial \varphi^a_\epsilon}{\partial X^A}\Big|_{\epsilon=0} = \nabla_{\frac{\partial}{\partial X^B}} \delta F^a{}_A = U^a{}_{|A|B}, \tag{7.243}$$

where use was made of the fact that ∇ is a flat connection, and hence, order of covariant differentiation can be interchanged (see the appendix). Therefore

$$\delta W = \frac{\partial W}{\partial F^a{}_A}\delta\varphi^a{}_{|A} + \frac{\partial W}{\partial F^a{}_{A|B}}\delta\varphi^a{}_{|A|B}\,. \tag{7.244}$$

Note that $\frac{\partial W}{\partial F^a{}_A}\delta\varphi^a{}_{|A} = \left(\frac{\partial W}{\partial F^a{}_A}\delta\varphi^a\right)_{|A} - \left(\frac{\partial W}{\partial F^a{}_A}\right)_{|A}\delta\varphi^a$. Thus

$$\int_{\mathcal{B}} \frac{\partial W}{\partial F^a{}_A}\delta\varphi^a{}_{|A}dV = -\int_{\mathcal{B}}\left(\frac{\partial W}{\partial F^a{}_A}\right)_{|A}\delta\varphi^a dV + \int_{\partial\mathcal{B}}\frac{\partial W}{\partial F^a{}_A}N_A\delta\varphi^a dA\,, \tag{7.245}$$

where $\mathbf{N}$ is the unit normal vector to $\partial\mathcal{B}$, and N_A are components of the corresponding 1-form $\mathbf{N}^\flat$. Similarly, for the second term one can write

$$\begin{aligned}\int_{\mathcal{B}} \frac{\partial W}{\partial F^a{}_{A|B}}\delta\varphi^a{}_{|A|B}dV &= -\int_{\mathcal{B}}\left(\frac{\partial W}{\partial F^a{}_{A|B}}\right)_{|B}\delta\varphi^a{}_{|A}dV + \int_{\partial\mathcal{B}}\frac{\partial W}{\partial F^a{}_{A|B}}N_B\delta\varphi^a{}_{|A}dA \\ &= \int_{\mathcal{B}}\left(\frac{\partial W}{\partial F^a{}_{A|B}}\right)_{|B|A}\delta\varphi^a dV - \int_{\partial\mathcal{B}}\left[\left(\frac{\partial W}{\partial F^a{}_{A|B}}N_B\right)_{|A} + \left(\frac{\partial W}{\partial F^a{}_{A|B}}\right)_{|B}N_A\right]\delta\varphi^a dA\,.\end{aligned} \tag{7.246}$$

In deriving the above relation we used the topological fact that boundary of a boundary is empty, i.e., $\partial\partial\mathcal{B} = \varnothing$. Therefore

$$\begin{aligned}\delta W = &-\int_{\mathcal{B}}\left[\frac{\partial W}{\partial F^a{}_A} - \left(\frac{\partial W}{\partial F^a{}_{A|B}}\right)_{|B}\right]_{|A}\delta\varphi^a dV + \int_{\partial\mathcal{B}}\left[\frac{\partial W}{\partial F^a{}_A} - \left(\frac{\partial W}{\partial F^a{}_{A|B}}\right)_{|B}\right]N_A\delta\varphi^a dA \\ &- \int_{\partial\mathcal{B}}\left(\frac{\partial W}{\partial F^a{}_{A|B}}N_B\right)_{|A}\delta\varphi^a dA\,.\end{aligned} \tag{7.247}$$

The first term in (7.247) implies that the first Piola-Kirchhoff stress in gradient elasticity has the following representation

$$P^{aA} = g^{ab}\left[\frac{\partial W}{\partial F^b{}_A} - \left(\frac{\partial W}{\partial F^b{}_{A|B}}\right)_{|B}\right]. \tag{7.248}$$

Following Toupin [229] we define a *hyper-stress* $H_a{}^{AB} = H_a{}^{BA} = \frac{\partial W}{\partial F^a{}_{A|B}}$. The integrand of the third term in (7.247) is simplified as

$$(H_a{}^{AB} N_B)_{|A} = H_a{}^{AB}{}_{|A} N_B - \mathfrak{B}_{AB} H_a{}^{AB}, \tag{7.249}$$

where $\mathfrak{B}_{AB} = \mathfrak{B}_{BA} = -N_{A|B}$ is the second fundamental form of the surface $\partial\mathcal{B}$ embedded in the Euclidean space (we assume that the undeformed body is embedded in a Euclidean ambient space). Therefore, traction in gradient elasticity is written as

$$T^a = P^{aA} N_A - H^{aAB}{}_{|B} N_A + H^{aAB} \mathfrak{B}_{AB}. \tag{7.250}$$

Similar to classical nonlinear elasticity, $\delta T = \rho_0 \langle\!\langle \mathbf{V}, D_t^{\mathbf{g}} \delta\varphi \rangle\!\rangle_{\mathbf{g}} = \frac{d}{dt} \rho_0 \langle\!\langle \mathbf{V}, \delta\varphi \rangle\!\rangle_{\mathbf{g}} - \rho_0 \langle\!\langle \mathbf{A}, \delta\varphi \rangle\!\rangle_{\mathbf{g}}$. Assuming that $\delta\varphi(X, t_1) = \delta\varphi(X, t_2) = 0$, one obtains

$$\delta \int_{t_1}^{t_2} \int_{\mathcal{B}} -T dV dt = \int_{t_1}^{t_2} \int_{\mathcal{B}} \rho_0 \langle\!\langle \mathbf{A}, \delta\varphi \rangle\!\rangle_{\mathbf{g}} dV dt. \tag{7.251}$$

Therefore, the balance of linear momentum (the Euler-Lagrange equations) reads $P^{aA}{}_{|A} = \rho_0 A^a$ (inclusion of body forces would be straightforward using the Lagrange-D'Alembert principle) and traction vector is given in (7.250).

The Cauchy stress $\sigma^{ab} = J^{-1} F^a{}_A P^{bA}$ in gradient elasticity has the following representation (note the typo in Toupin [229]'s Eq.(10.18))

$$\sigma^{ab} = J^{-1} g^{bc} \left[F^a{}_A \frac{\partial W}{\partial F^c{}_A} + F^a{}_{A|B} \frac{\partial W}{\partial F^c{}_{A|B}} \right] - h^{bac}{}_{|c}, \tag{7.252}$$

where $h^{bac} = J^{-1} F^a{}_A F^c{}_B H^{bAB}$.

Remark 7.5.1. When a gradient elastic solid is in a stress-free state the traction vector at every point on any surface vanishes. From (7.250) this implies that $P^{aA} - H^{aAB}{}_{|B} = 0$, and $H^{aAB} = 0$. Therefore, in a stress-free state both the (total) first Piola-Kirchhoff stress and hyper-stress vanish.

Balance of angular momentum. In the setting of Lagrangian mechanics of continua the balance of angular momentum is derived using Noether's theorem. Consider a flow $\psi_s : \mathcal{S} \to \mathcal{S}$ on the (Euclidean) ambient space. According to Noether's theorem any symmetry of the Lagrangian density corresponds to a conserved quantity. In particular, invariance of a Lagrangian density under rotations of the ambient space corresponds to the balance of angular momentum. For the balance of angular momentum in a Euclidean ambient space $\psi_s(x) = x + s\mathbf{\Omega}x$, where $\mathbf{\Omega}$ is an anti-symmetric matrix. As the kinetic energy density is invariant under rotations one only needs to require invariance of the energy function under flows of rotations of the ambient space, i.e.,

$$W(X, \mathbf{F}, \nabla\mathbf{F}, \mathbf{G}, \mathbf{g}) = W(X, \psi_{s*}\mathbf{F}, \psi_{s*}\nabla\mathbf{F}, \mathbf{G}, \psi_{s*}\mathbf{g}). \tag{7.253}$$

Taking derivative with respect to s of both sides and evaluating at $s = 0$, one obtains

$$g^{ac}\left[F^b{}_A \frac{\partial W}{\partial F^c{}_A} + F^b{}_{A|B} \frac{\partial W}{\partial F^c{}_{A|B}}\right]\Omega_{ab} = 0\,. \tag{7.254}$$

As $\Omega_{ba} = -\Omega_{ab}$, one concludes that $\Pi^{ab} = \Pi^{ba}$, or $\Pi^{[ab]} = 0$,[32] where

$$\Pi^{ab} = g^{ac}\left[F^b{}_A \frac{\partial W}{\partial F^c{}_A} + F^b{}_{A|B} \frac{\partial W}{\partial F^c{}_{A|B}}\right] = g^{ac}F^b{}_A \frac{\partial W}{\partial F^c{}_A} + H^{aAB}F^b{}_{A|B}\,. \tag{7.255}$$

In terms of the first Piola-Kirchhoff stress the balance of angular momentum reads

$$P^{[aA}F^{b]}{}_A + \left(H^{[aAB}F^{b]}{}_A\right)_{|B} = 0\,. \tag{7.256}$$

In terms of the Cauchy stress, $\sigma^{[ab]} + m^{bac}{}_{|c} = 0$, where $m^{bac} = h^{[ba]c}$ is Toupin's *couple-stress*.

[32]We use the standard notation $\Pi^{[ab]} = \frac{1}{2}\left(\Pi^{ab} - \Pi^{ba}\right)$.

Linearized balance of linear momentum. Linearizing the balance of linear momentum about a motion $\mathring{\varphi}$ one obtains $(\delta P^{aA})_{|A} + \rho_0 \delta B^a = \rho_0 \ddot{U}^a$. Note that $\delta(P^{aA}{}_{|A}) = \delta P^{aA}{}_{|A}$, where

$$\delta P^{aA} = \frac{\partial P^{aA}}{\partial F^b{}_B}\delta F^b{}_B + \frac{\partial P^{aA}}{\partial F^b{}_{B|C}}\delta F^b{}_{B|C} = \mathsf{A}^{aA}{}_b{}^B\, U^b{}_{|B} + \mathsf{B}^{aA}{}_b{}^{BC}\, U^b{}_{|B|C}\,, \tag{7.257}$$

and

$$\mathsf{A}^{aA}{}_b{}^B = \frac{\partial P^{aA}}{\partial F^b{}_B}, \quad \mathsf{B}^{aA}{}_b{}^{BC} = \frac{\partial P^{aA}}{\partial F^b{}_{B|C}}\,. \tag{7.258}$$

Following DiVincenzo [236] we call $\mathbf{A}$ and $\mathbf{B}$ *dynamic elastic constants*. Notice that $\mathsf{B}^{aAbBC} = \mathsf{B}^{aAbCB}$. Noting that $P^{aA} = g^{am}\frac{\partial W}{\partial F^m{}_A} - H^{aAM}{}_{|M}$, one can write

$$\begin{aligned}\delta P^{aA} &= g^{am}\frac{\partial^2 W}{\partial F^m{}_A \partial F^n{}_N}\delta F^n{}_N + g^{am}\frac{\partial^2 W}{\partial F^m{}_A \partial F^n{}_{N|M}}\delta F^n{}_{N|M} - \delta(H^{aAM}{}_{|M}) \\ &= g^{am}\frac{\partial^2 W}{\partial F^m{}_A \partial F^n{}_N}U^n{}_{|N} + g^{am}\frac{\partial^2 W}{\partial F^m{}_A \partial F^n{}_{N|M}}U^n{}_{|N|M} - (\delta H^{aAM})_{|M}\,.\end{aligned} \tag{7.259}$$

But

$$\begin{aligned}\delta H^{aAM} &= \frac{\partial H^{aAM}}{\partial F^c{}_C}\delta F^c{}_C + \frac{\partial H^{aAM}}{\partial F^c{}_{C|D}}\delta F^c{}_{C|D} \\ &= g^{am}\frac{\partial^2 W}{\partial F^c{}_C \partial F^m{}_{A|M}}U^c{}_{|C} + g^{am}\frac{\partial^2 W}{\partial F^c{}_{C|D} \partial F^m{}_{A|M}}U^c{}_{|C|D}\,.\end{aligned} \tag{7.260}$$

Let us define the following three *static elastic constants* [236]

$$\mathbb{A}_a{}^A{}_b{}^B = \frac{\partial^2 W}{\partial F^a{}_A \partial F^b{}_B}, \quad \mathbb{B}_a{}^A{}_b{}^{BC} = \frac{\partial^2 W}{\partial F^a{}_A \partial F^b{}_{B|C}}, \quad \mathbb{C}_a{}^{AB}{}_b{}^{CD} = \frac{\partial^2 W}{\partial F^a{}_{A|B} \partial F^b{}_{C|D}}\,. \tag{7.261}$$

Therefore, the static elastic constants satisfy the following symmetries: $\mathbb{A}_a{}^A{}_b{}^B = \mathbb{A}_b{}^B{}_a{}^A$, $\mathbb{B}_a{}^A{}_b{}^{BC} = \mathbb{B}_a{}^A{}_b{}^{CB}$, and $\mathbb{C}_a{}^{AB}{}_b{}^{CD} = \mathbb{C}_a{}^{BA}{}_b{}^{CD} = \mathbb{C}_a{}^{BA}{}_b{}^{DC} = \mathbb{C}_b{}^{DC}{}_a{}^{BA}$ (note that $\mathbb{C}$ has 21 and 171 independent components in 2D and 3D, respectively). Thus, $\delta H^{aAM} = \mathbb{B}_b{}^{BaAM}U^b{}_{|B} + \mathbb{C}_b{}^{BCaAM}U^b{}_{|B|C}$, and hence

$$(\delta H^{aAM})_{|M} = (\mathbb{B}_b{}^{BaAM}U^b{}_{|B} + \mathbb{C}_b{}^{BCaAM}U^b{}_{|B|C})_{|M}\,. \tag{7.262}$$

Therefore

$$\begin{aligned} \mathsf{A}^{aA}{}_{b}{}^{B} &= \mathbb{A}^{aA}{}_{b}{}^{B} - \mathbb{B}_{b}{}^{BaAM}{}_{|M}\,, \\ \mathsf{B}^{aA}{}_{b}{}^{BC} &= \mathbb{B}^{aA}{}_{b}{}^{BC} - \mathbb{B}_{b}{}^{BaAC} - \mathbb{C}^{aAM}{}_{b}{}^{BC}{}_{|M}\,. \end{aligned} \tag{7.263}$$

Or equivalently

$$\begin{aligned} \mathsf{A}^{aAbB} &= \mathbb{A}^{aAbB} - \mathbb{B}^{bBaAM}{}_{|M}\,, \\ \mathsf{B}^{aAbBC} &= \mathbb{B}^{aAbBC} - \mathbb{B}^{bBaAC} - \mathbb{C}^{aAMbBC}{}_{|M}\,. \end{aligned} \tag{7.264}$$

In deriving the second relation we ignored the term $U^{b}{}_{|B|C|M}$ in $\delta H^{aAM}{}_{|M}$ as we are assuming a second-gradient elasticity in which displacement derivatives of orders three or higher are neglected. Note that in [236] homogeneous solids were considered. Here, for cloaking purposes, one needs to consider inhomogeneous solids and that is why derivatives of the static elastic constants appear in (7.264).

Note that $(7.264)_2$ requires that $\mathbb{B}^{bBaAC} = \mathbb{B}^{bCaAB} = \mathbb{B}^{bAaBC}$, reducing the number of independent components of $\mathbb{B}$ to 16 and 90 in 2D and 3D, respectively. After some manipulations and using the major symmetries of $\mathbb{C}$, one also finds that

$$\mathsf{B}^{aAbBC} + \mathsf{B}^{bBaAC} = -\left(\mathbb{C}^{bBMaAC} + \mathbb{C}^{aAMbBC}\right)_{|M} = -\left(\mathbb{C}^{aACbBM} + \mathbb{C}^{bBCaAM}\right)_{|M}\,. \tag{7.265}$$

Similarly, from $(7.264)_1$ and knowing that $\mathbb{A}$ has the major symmetries one obtains

$$\mathsf{A}^{aAbB} - \mathsf{A}^{bBaA} = \left(\mathbb{B}^{aAbBM} - \mathbb{B}^{bBaAM}\right)_{|M} = \left(\mathbb{B}^{aBbAM} - \mathbb{B}^{bBaAM}\right)_{|M}\,. \tag{7.266}$$

Linearized balance of angular momentum. The balance of angular momentum is equivalent to $\Pi^{ab} = \Pi^{ba}$, or $\Pi^{[ab]} = 0$. Suppose the reference motion is an isometric embedding of an initially stress-free body into the Euclidean space, i.e., $\mathring{F}^{a}{}_{A} = \delta^{a}_{A}$, which implies that $\mathring{F}^{a}{}_{A|B} = 0$. Assuming that $\mathring{P}^{aA} = 0$, one concludes that

$$\mathring{H}^{aAB}{}_{|B} = g^{ab}\frac{\partial W}{\partial F^{a}{}_{A}}\bigg|_{\mathbf{F}=\mathring{\mathbf{F}}} =: \mathring{P}^{aA}_{\text{cl.}}\,. \tag{7.267}$$

Therefore

$$\begin{aligned}\delta\Pi^{ab} &= \mathring{P}^{aA}_{\text{cl.}} U^b{}_{|A} + \mathring{H}^{aAB} U^b{}_{|A|B} + \mathbb{A}^{aA}{}_m{}^M \mathring{F}^b{}_A U^m{}_{|M} + \mathbb{B}^{aA}{}_m{}^{MN} \mathring{F}^b{}_A U^m{}_{|M|N} \\ &= \left(\mathbb{A}^{aM}{}_m{}^A \mathring{F}^b{}_A + \mathring{H}^{aAB}{}_{|B}\delta^b_m\right) U^m{}_{|A} + \left(\mathbb{B}^{aM}{}_m{}^{AB} \mathring{F}^b{}_M + \mathring{H}^{aAB}\delta^b_m\right) U^m{}_{|A|B}\,.\end{aligned} \tag{7.268}$$

Knowing that $\delta\Pi^{[ab]} = 0$, and that the first and the second covariant derivatives of the displacement field are independent, one concludes that

$$\begin{aligned}\mathbb{A}^{[aM}{}_m{}^A \mathring{F}^{b]}{}_M + \mathring{H}^{[aAB}{}_{|B}\delta^{b]}_m = 0, \\ \mathbb{B}^{[aM}{}_m{}^{AB} \mathring{F}^{b]}{}_M + \mathring{H}^{[aAB}\delta^{b]}_m = 0.\end{aligned} \tag{7.269}$$

Note that the issue with the acceleration term that was discussed in classical nonlinear elasticity in $\S$8.4.1 persists even in nonlinear gradient elasticity, and hence, we do not discuss transformation cloaking in nonlinear gradient elastodynamics.

Transformation cloaking in linearized gradient elastodynamics. We start from a virtual body that is made of a homogeneous, isotropic, and centro-symmetric gradient elastic solid. We then consider a cloaking transformation and try to find the elastic constants of the physical body induced from the cloaking transformation such that the balance of linear and angular momenta are respected in both the virtual and physical bodies. Let us start from the balance of linear momentum in the physical body, i.e., $\operatorname{Div}\mathbf{P} + \rho_0\mathbf{B} = \rho_0\mathbf{A}$. Its linearization reads $\delta\left(\operatorname{Div}\mathbf{P}\right) + \rho_0\delta\mathbf{B} = \rho_0\mathbf{A}$, where

$$\delta\left(\operatorname{Div}\mathbf{P}\right) = \operatorname{Div}\delta\mathbf{P} = \operatorname{Div}\left(\mathsf{\mathbf{A}} : \nabla\mathbf{U} + \mathsf{\mathbf{B}} : \nabla\nabla\mathbf{U}\right) = \left(\mathsf{A}^{aA}{}_b{}^B\, U^b{}_{|B} + \mathsf{B}^{aA}{}_b{}^{BC}\, U^b{}_{|B|C}\right)_{|A} \frac{\partial}{\partial x^a}. \tag{7.270}$$

Under a cloaking transformation $\Xi : \mathcal{B} \to \tilde{\mathcal{B}}$ and using the shifter map, $\operatorname{Div}\delta\mathbf{P}$ is transformed to

$$J_\Xi \left(\tilde{\mathsf{A}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}}\, \tilde{U}^{\tilde{b}}{}_{|\tilde{B}} + \tilde{\mathsf{B}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}\tilde{C}}\, \tilde{U}^{\tilde{b}}{}_{|\tilde{B}|\tilde{C}}\right)_{|\tilde{A}} \frac{\partial}{\partial \tilde{x}^{\tilde{a}}}, \tag{7.271}$$

where

$$
\begin{aligned}
\tilde{U}^{\tilde{a}} &= \mathsf{s}^{\tilde{a}}{}_a U^a\,,\\
\tilde{\mathsf{A}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}} &= J_{\Xi}^{-1}\mathsf{s}^{\tilde{a}}{}_a \bar{\bar{\bar{F}}}^{\tilde{A}}{}_A(\mathsf{s}^{-1})^b{}_{\tilde{b}}\bar{\bar{\bar{F}}}^{\tilde{B}}{}_B\ \mathsf{A}^{aA}{}_b{}^B + J_{\Xi}^{-1}\mathsf{s}^{\tilde{a}}{}_a \bar{\bar{\bar{F}}}^{\tilde{A}}{}_A(\mathsf{s}^{-1})^b{}_{\tilde{b}}\bar{\bar{\bar{F}}}^{\tilde{B}}{}_{B|C}\ \mathsf{B}^{aA}{}_b{}^{BC}\,,\\
\tilde{\mathsf{B}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}\tilde{C}} &= J_{\Xi}^{-1}\mathsf{s}^{\tilde{a}}{}_a \bar{\bar{\bar{F}}}^{\tilde{A}}{}_A(\mathsf{s}^{-1})^b{}_{\tilde{b}}\bar{\bar{\bar{F}}}^{\tilde{B}}{}_B\bar{\bar{\bar{F}}}^{\tilde{C}}{}_C\ \mathsf{B}^{aA}{}_b{}^{BC}\,.
\end{aligned}
\tag{7.272}
$$

Or, equivalently

$$
\begin{aligned}
\mathsf{A}^{aA}{}_b{}^B &= J_{\Xi}(\mathsf{s}^{-1})^a{}_{\tilde{a}}(\bar{\bar{\bar{F}}}^{-1})^A{}_{\tilde{A}}\mathsf{s}^{\tilde{b}}{}_b(\bar{\bar{\bar{F}}}^{-1})^B{}_{\tilde{B}}\ \tilde{\mathsf{A}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}} + J_{\Xi}(\mathsf{s}^{-1})^a{}_{\tilde{a}}(\bar{\bar{\bar{F}}}^{-1})^A{}_{\tilde{A}}\mathsf{s}^{\tilde{b}}{}_b(\bar{\bar{\bar{F}}}^{-1})^B{}_{\tilde{B}|\tilde{C}}\ \tilde{\mathsf{B}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}\tilde{C}}\,,\\
\mathsf{B}^{aA}{}_b{}^{BC} &= J_{\Xi}(\mathsf{s}^{-1})^a{}_{\tilde{a}}(\bar{\bar{\bar{F}}}^{-1})^A{}_{\tilde{A}}\mathsf{s}^{\tilde{b}}{}_b(\bar{\bar{\bar{F}}}^{-1})^B{}_{\tilde{B}}(\bar{\bar{\bar{F}}}^{-1})^C{}_{\tilde{C}}\ \tilde{\mathsf{B}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}\tilde{C}}\,.
\end{aligned}
\tag{7.273}
$$

To see this note that

$$
\begin{aligned}
&\left(\mathsf{A}^{aA}{}_b{}^B\ U^b{}_{|B} + \mathsf{B}^{aA}{}_b{}^{BC}\ U^b{}_{|B|C}\right)_{|A} = J_{\Xi}\Big[J_{\Xi}^{-1}\bar{\bar{\bar{F}}}^{\tilde{A}}{}_A\mathsf{A}^{aA}{}_b{}^B\ U^b{}_{|B} + J_{\Xi}^{-1}\bar{\bar{\bar{F}}}^{\tilde{A}}{}_A\mathsf{B}^{aA}{}_b{}^{BC}\ U^b{}_{|B|C}\Big]_{|\tilde{A}}\\
&= J_{\Xi}\Big[J_{\Xi}^{-1}\bar{\bar{\bar{F}}}^{\tilde{A}}{}_A\mathsf{A}^{aA}{}_b{}^B\bar{\bar{\bar{F}}}^{\tilde{B}}{}_B U^b{}_{|\tilde{B}} + J_{\Xi}^{-1}\bar{\bar{\bar{F}}}^{\tilde{A}}{}_A\mathsf{B}^{aA}{}_b{}^{BC}\bar{\bar{\bar{F}}}^{\tilde{B}}{}_B\bar{\bar{\bar{F}}}^{\tilde{C}}{}_C U^b{}_{|\tilde{B}|\tilde{C}} + J_{\Xi}^{-1}\bar{\bar{\bar{F}}}^{\tilde{A}}{}_A\mathsf{B}^{aA}{}_b{}^{BC}\bar{\bar{\bar{F}}}^{\tilde{B}}{}_{B|C}U^b{}_{|\tilde{B}}\Big]_{|\tilde{A}}\\
&= (\mathsf{s}^{-1})^a{}_{\tilde{a}}J_{\Xi}\left(\tilde{\mathsf{A}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}}\ \tilde{U}^{\tilde{b}}{}_{|\tilde{B}} + \tilde{\mathsf{B}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}\tilde{C}}\ \tilde{U}^{\tilde{b}}{}_{|\tilde{B}|\tilde{C}}\right)_{|\tilde{A}}\,,
\end{aligned}
\tag{7.274}
$$

where the following relations were used.

$$
\begin{aligned}
U^b{}_{|B} &= (\mathsf{s}^{-1})^b{}_{\tilde{b}}\bar{\bar{\bar{F}}}^{\tilde{B}}{}_B\ \tilde{U}^{\tilde{b}}{}_{|\tilde{B}},\\
U^b{}_{|B|C} &= (\mathsf{s}^{-1})^b{}_{\tilde{b}}\left[\bar{\bar{\bar{F}}}^{\tilde{B}}{}_{B|C}\ \tilde{U}^{\tilde{b}}{}_{|\tilde{B}} + \bar{\bar{\bar{F}}}^{\tilde{B}}{}_B\bar{\bar{\bar{F}}}^{\tilde{C}}{}_C\ \tilde{U}^{\tilde{b}}{}_{|\tilde{B}|\tilde{C}}\right].
\end{aligned}
\tag{7.275}
$$

We assume that the reference motion for both the physical and virtual bodies are isometric embeddings in the Euclidean ambient space, i.e., $\mathring{F}^a{}_A = \delta^a_A$ and $\mathring{\tilde{F}}^{\tilde{a}}{}_{\tilde{A}} = \delta^{\tilde{a}}_{\tilde{A}}$. This implies that $\mathring{F}^a{}_{A|B} = 0$, and $\mathring{\tilde{F}}^{\tilde{a}}{}_{\tilde{A}|\tilde{B}} = 0$. It is also assumed that there is no initial stress in either configuration, i.e., $\mathring{P}^a{}_A = 0$, $\mathring{H}^{aAB} = 0$, and $\mathring{\tilde{P}}^{\tilde{a}}{}_{\tilde{A}} = 0$, $\mathring{\tilde{H}}^{\tilde{a}\tilde{A}\tilde{B}} = 0$. Therefore, from (7.269) the balance of angular momentum in the physical and virtual bodies read (minor symmetries of $\mathbb{A}$, $\mathbb{B}$, $\tilde{\mathbb{A}}$, and $\tilde{\mathbb{B}}$)

$$\mathbb{A}^{[aM}{}_{m}{}^{A}\mathring{F}^{b]}{}_{M} = 0, \quad \mathbb{B}^{[aM}{}_{m}{}^{AB}\mathring{F}^{b]}{}_{M} = 0, \tag{7.276}$$

$$\tilde{\mathbb{A}}^{[\tilde{a}\tilde{M}}{}_{\tilde{m}}{}^{\tilde{A}}\mathring{\tilde{F}}^{\tilde{b}]}{}_{\tilde{M}} = 0, \quad \tilde{\mathbb{B}}^{[\tilde{a}\tilde{M}}{}_{\tilde{m}}{}^{\tilde{A}\tilde{B}}\mathring{\tilde{F}}^{\tilde{b}]}{}_{\tilde{M}} = 0. \tag{7.277}$$

The virtual body being uniform its elastic constants are (covariantly) constant, and hence, from (7.263) one concludes that

$$\tilde{\mathsf{A}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}} = \tilde{\mathbb{A}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}}, \qquad \tilde{\mathsf{B}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}\tilde{C}} = \tilde{\mathbb{B}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}\tilde{C}} - \tilde{\mathbb{B}}_{\tilde{b}}{}^{\tilde{B}\tilde{a}\tilde{A}\tilde{C}}. \tag{7.278}$$

Therefore, from $(7.277)_1$ one obtains

$$\tilde{\mathsf{A}}^{[\tilde{a}\tilde{M}}{}_{\tilde{m}}{}^{\tilde{A}}\mathring{\tilde{F}}^{\tilde{b}]}{}_{\tilde{M}} = 0. \tag{7.279}$$

The virtual body is assumed to be isotropic and non-chiral (centro-symmetric).[33] Knowing that an odd-order tensor cannot be isotropic and centro-symmetric [231, 237] one concludes that $\tilde{\mathbb{B}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}\tilde{C}} = 0$, and hence $\tilde{\mathsf{B}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}\tilde{C}} = 0$. In particular, $(7.277)_2$ is trivially satisfied. From $(7.273)_2$ one obtains $\mathsf{B}^{aA}{}_{b}{}^{BC} = 0$, and hence from $(7.264)_2$

$$\begin{aligned} &\mathbb{B}^{aA}{}_{b}{}^{BC} - \mathbb{B}_{b}{}^{BaAC} = \mathbb{C}^{aAM}{}_{b}{}^{BC}{}_{|M}, \\ &\mathsf{A}^{aA}{}_{b}{}^{B} = J_{\Xi}(\mathsf{s}^{-1})^{a}{}_{\tilde{a}}(\overset{\equiv}{F}{}^{-1})^{A}{}_{\tilde{A}}\mathsf{s}^{\tilde{b}}{}_{b}(\overset{\equiv}{F}{}^{-1})^{B}{}_{\tilde{B}}\,\tilde{\mathbb{A}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}}\,. \end{aligned} \tag{7.280}$$

Notice that the dynamic elastic constants $\mathbf{A}$ of the cloak possess the major symmetries, i.e., $\mathsf{A}^{aAbB} = \mathsf{A}^{bBaA} = J_{\Xi}(\mathsf{s}^{-1})^{a}{}_{\tilde{a}}(\overset{\equiv}{F}{}^{-1})^{A}{}_{\tilde{A}}(\mathsf{s}^{-1})^{b}{}_{\tilde{b}}(\overset{\equiv}{F}{}^{-1})^{B}{}_{\tilde{B}}\,\tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}$. Thus, using (7.266), one obtains $\mathbb{B}^{[aBb]AM}{}_{|M} = 0$, i.e.,

$$\mathbb{B}^{aBbAM}{}_{|M} = \mathbb{B}^{bBaAM}{}_{|M}\,. \tag{7.281}$$

[33]Note that only the virtual body is assumed to be centro-symmetric. There is no such constraint on the physical body; it can be both non-centro-symmetric and anisotropic.

From $(7.280)_1$ and $(7.276)_2$, one obtains

$$\mathbb{C}^{[aAMbBC}{}_{|M}\mathring{F}^{c]}{}_{A} = -\mathbb{B}^{bB[aAC}\mathring{F}^{c]}{}_{A}\,. \tag{7.282}$$

Also, from (7.265) and knowing that $\mathbf{B}$ vanishes identically, one obtains

$$\left(\mathbb{C}^{aACbBM} + \mathbb{C}^{bBCaAM}\right)_{|M} = 0\,. \tag{7.283}$$

Remark 7.5.2. From $(7.280)_1$ and $(7.276)_2$, one can write

$$\mathbb{B}^{[aAbBC}{}_{|C}\mathring{F}^{c]}{}_{A} - \mathbb{B}^{bB[aAC}{}_{|C}\mathring{F}^{c]}{}_{A} = -\mathbb{B}^{bB[aAC}{}_{|C}\mathring{F}^{c]}{}_{A} = \mathbb{C}^{[aAMbBC}{}_{|M|C}\mathring{F}^{c]}{}_{A}. \tag{7.284}$$

Therefore

$$\mathbb{B}^{bB[aAC}{}_{|C}\mathring{F}^{c]}{}_{A} = -\mathbb{C}^{[aAMbBC}{}_{|M|C}\mathring{F}^{c]}{}_{A}. \tag{7.285}$$

Note that from $(7.264)_1$ and $(7.280)_1$ we have

$$\begin{aligned}\mathbb{A}^{aAbB} &= \mathbb{B}^{bBaAM}{}_{|M} + J_{\Xi}(\mathsf{s}^{-1})^{a}{}_{\tilde{a}}(\bar{\bar{F}}^{-1})^{A}{}_{\tilde{A}}(\mathsf{s}^{-1})^{b}{}_{\tilde{b}}(\bar{\bar{F}}^{-1})^{B}{}_{\tilde{B}}\,\tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} \\ &= \mathbb{B}^{aAbBM}{}_{|M} - \mathbb{C}^{aAMbBN}{}_{|M|N} + J_{\Xi}(\mathsf{s}^{-1})^{a}{}_{\tilde{a}}(\bar{\bar{F}}^{-1})^{A}{}_{\tilde{A}}(\mathsf{s}^{-1})^{b}{}_{\tilde{b}}(\bar{\bar{F}}^{-1})^{B}{}_{\tilde{B}}\,\tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}.\end{aligned} \tag{7.286}$$

From the above relation and (7.276) one obtains

$$0 = \mathbb{A}^{[aAbB}\mathring{F}^{c]}{}_{A} = -\mathbb{C}^{[aAMbBN}{}_{|M|N}\mathring{F}^{c]}{}_{A} + J_{\Xi}(\mathsf{s}^{-1})^{[a}{}_{\tilde{a}}(\bar{\bar{F}}^{-1})^{A}{}_{\tilde{A}}(\mathsf{s}^{-1})^{b}{}_{\tilde{b}}(\bar{\bar{F}}^{-1})^{B}{}_{\tilde{B}}\,\tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\mathring{F}^{c]}{}_{A}. \tag{7.287}$$

Therefore

$$\mathbb{C}^{[aAMbBN}{}_{|M|N}\mathring{F}^{c]}{}_{A} = J_{\Xi}(\mathsf{s}^{-1})^{[a}{}_{\tilde{a}}(\bar{\bar{F}}^{-1})^{A}{}_{\tilde{A}}(\mathsf{s}^{-1})^{b}{}_{\tilde{b}}(\bar{\bar{F}}^{-1})^{B}{}_{\tilde{B}}\,\tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\mathring{F}^{c]}{}_{A}. \tag{7.288}$$

Note that in classical linearized elasticity the right-hand side had to vanish, e.g., Eq. (7.182), which was an obstruction to transformation cloaking.

From (7.288) and (7.285), the system of first-order PDEs for $\mathbb{B}$ read

$$\mathbb{B}^{bB[aAC}{}_{|C}\mathring{F}^{c]}{}_{A}=-J_{\Xi}(\mathsf{s}^{-1})^{[a}{}_{\tilde{a}}(\bar{\bar{\bar{F}}}^{-1})^{A}{}_{\tilde{A}}(\mathsf{s}^{-1})^{b}{}_{\tilde{b}}(\bar{\bar{\bar{F}}}^{-1})^{B}{}_{\tilde{B}}\,\tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\mathring{F}^{c]}{}_{A}\,. \tag{7.289}$$

In summary, one is given the homogeneous static elastic constants of the virtual body. Because of isotropy and centro-symmetry only $\tilde{\mathbb{A}}$ and $\tilde{\mathbb{C}}$ are nonzero. These are both (covariantly) constant tensors. The balance of angular momentum implies that $\tilde{\mathbb{A}}$ has the classical minor symmetries. So, it will have the same form as $\tilde{\mathbb{A}}$ in the classical linear elasticity. There is no relation between $\tilde{\mathbb{C}}$ and $\mathbb{C}$, and there are no constraints on $\tilde{\mathbb{C}}$ other than being positive-definite. The balance of linear momentum relates the dynamic elastic constants of the two problems. From $\tilde{\mathbf{B}}=\mathbf{0}$ in the virtual body one concludes that $\mathbf{B}=\mathbf{0}$ in the physical body. This gives a relation between $\mathbb{B}$ and $\mathbb{C}$ in the form of a system of PDEs. The balance of angular momentum in the physical body is written as a set of constraints in the form of a system of second-order PDEs for $\mathbb{C}$.

The impossibility of transformation cloaking in gradient elasticity. Next we show that transformation cloaking is not possible in gradient elasticity. We first show this for cylindrical and spherical cloaks. From $(7.276)_2$ and knowing that $\mathring{F}^{b}{}_{M}=\delta^{b}{}_{M}$, with an abuse of notation, one writes

$$\mathbb{B}^{abmAB}=\mathbb{B}^{bamAB}\,, \tag{7.290}$$

i.e., $\mathbb{B}$ must be symmetric with respect to the first two indices. This symmetry and (7.289), imply that the right-hand side of (7.289) is symmetric with respect to indices b and B as well. Thus

$$(\mathsf{s}^{-1})^{[a}{}_{\tilde{a}}(\bar{\bar{\bar{F}}}^{-1})^{A}{}_{\tilde{A}}(\mathsf{s}^{-1})^{b}{}_{\tilde{b}}(\bar{\bar{\bar{F}}}^{-1})^{B}{}_{\tilde{B}}\,\tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\mathring{F}^{c]}{}_{A}=(\mathsf{s}^{-1})^{[a}{}_{\tilde{a}}(\bar{\bar{\bar{F}}}^{-1})^{A}{}_{\tilde{A}}(\mathsf{s}^{-1})^{B}{}_{\tilde{b}}(\bar{\bar{\bar{F}}}^{-1})^{b}{}_{\tilde{B}}\,\tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\mathring{F}^{c]}{}_{A}\,. \tag{7.291}$$

This is simplified and rearranged to read

$$\begin{aligned}&(\mathsf{s}^{-1})^b{}_{\tilde{b}}(\bar{\bar{\bar{F}}}^{-1})^B{}_{\tilde{B}}\left[(\mathsf{s}^{-1})^c{}_{\tilde{a}}(\bar{\bar{\bar{F}}}^{-1})^a{}_{\tilde{A}}-(\mathsf{s}^{-1})^a{}_{\tilde{a}}(\bar{\bar{\bar{F}}}^{-1})^c{}_{\tilde{A}}\right]\tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\\&=(\mathsf{s}^{-1})^B{}_{\tilde{b}}(\bar{\bar{\bar{F}}}^{-1})^b{}_{\tilde{B}}\left[(\mathsf{s}^{-1})^c{}_{\tilde{a}}(\bar{\bar{\bar{F}}}^{-1})^a{}_{\tilde{A}}-(\mathsf{s}^{-1})^a{}_{\tilde{a}}(\bar{\bar{\bar{F}}}^{-1})^c{}_{\tilde{A}}\right]\tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\,,\quad\forall\, b,B,c,a\in\{1,2,3\}\,.\end{aligned}\tag{7.292}$$

In particular, it is straightforward to verify that this condition cannot be satisfied for either a cylindrical or a spherical cloak. To see this, let us expand (7.292) for $b=1$, $B=3$, $a=1$, and $c=3$:

$$\begin{aligned}&(\mathsf{s}^{-1})^1{}_{\tilde{b}}(\bar{\bar{\bar{F}}}^{-1})^3{}_{\tilde{B}}\left[(\mathsf{s}^{-1})^3{}_{\tilde{a}}(\bar{\bar{\bar{F}}}^{-1})^1{}_{\tilde{A}}-(\mathsf{s}^{-1})^1{}_{\tilde{a}}(\bar{\bar{\bar{F}}}^{-1})^3{}_{\tilde{A}}\right]\tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\\&=(\mathsf{s}^{-1})^3{}_{\tilde{b}}(\bar{\bar{\bar{F}}}^{-1})^1{}_{\tilde{B}}\left[(\mathsf{s}^{-1})^3{}_{\tilde{a}}(\bar{\bar{\bar{F}}}^{-1})^1{}_{\tilde{A}}-(\mathsf{s}^{-1})^1{}_{\tilde{a}}(\bar{\bar{\bar{F}}}^{-1})^3{}_{\tilde{A}}\right]\tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\,.\end{aligned}\tag{7.293}$$

Knowing that in the spherical (or cylindrical) coordinates and for a radial cloak $\mathbf{s}^{-1}$ and $\bar{\bar{\bar{\mathbf{F}}}}^{-1}$ have diagonal representations, this is further simplified and reads

$$\begin{aligned}&(\mathsf{s}^{-1})^1{}_1(\bar{\bar{\bar{F}}}^{-1})^3{}_3\left[(\mathsf{s}^{-1})^3{}_3(\bar{\bar{\bar{F}}}^{-1})^1{}_1\tilde{\mathbb{A}}^{3113}-(\mathsf{s}^{-1})^1{}_1(\bar{\bar{\bar{F}}}^{-1})^3{}_3\tilde{\mathbb{A}}^{1313}\right]\\&=(\mathsf{s}^{-1})^3{}_3(\bar{\bar{\bar{F}}}^{-1})^1{}_1\left[(\mathsf{s}^{-1})^3{}_3(\bar{\bar{\bar{F}}}^{-1})^1{}_1\tilde{\mathbb{A}}^{3131}-(\mathsf{s}^{-1})^1{}_1(\bar{\bar{\bar{F}}}^{-1})^3{}_3\tilde{\mathbb{A}}^{1331}\right]\,.\end{aligned}\tag{7.294}$$

But note that $\tilde{\mathbb{A}}$ (the elastic constants of the virtual body) possesses both the minor and major symmetries. Moreover, $\tilde{\mathbb{A}}^{1331}=\mu>0$. Hence, one obtains

$$\begin{aligned}&(\mathsf{s}^{-1})^1{}_1(\bar{\bar{\bar{F}}}^{-1})^3{}_3\left[(\mathsf{s}^{-1})^3{}_3(\bar{\bar{\bar{F}}}^{-1})^1{}_1-(\mathsf{s}^{-1})^1{}_1(\bar{\bar{\bar{F}}}^{-1})^3{}_3\right]\\&=(\mathsf{s}^{-1})^3{}_3(\bar{\bar{\bar{F}}}^{-1})^1{}_1\left[(\mathsf{s}^{-1})^3{}_3(\bar{\bar{\bar{F}}}^{-1})^1{}_1-(\mathsf{s}^{-1})^1{}_1(\bar{\bar{\bar{F}}}^{-1})^3{}_3\right]\,.\end{aligned}\tag{7.295}$$

Therefore, $(\bar{\bar{\bar{F}}}^{-1})^3{}_3(\mathsf{s}^{-1})^1{}_1=(\mathsf{s}^{-1})^3{}_3(\bar{\bar{\bar{F}}}^{-1})^1{}_1$. Recalling that $\bar{\bar{\bar{\mathbf{F}}}}=\mathrm{diag}(f'(R),1,1)$, and $\mathbf{s}=\mathrm{diag}\,(1,R/f(R),1)$ and $\mathbf{s}=\mathrm{diag}\,(1,R/f(R),R/f(R))$, in the cylindrical and spherical coordinates, respectively, one must have $f(R)=R$, i.e., $\Xi=id$, which is not acceptable for a cloaking map.

One may now ask whether choosing a less symmetric cloaking map may make transformation cloaking possible. We next show that this is not the case.

Proposition 7.5.3. *Assuming that the virtual body is isotropic and centro-symmetric, elastodynamic* *transformation cloaking is not possible for gradient elastic solids in either* $2D$ *or* $3D$ *for a hole (cavity) of any shape.*

Proof. Without loss of generality, one may write (7.292) in Cartesian coordinates for which the shifter has a trivial representation, i.e., $\mathsf{s}^{\tilde{a}}{}_{a} = \delta^{\tilde{a}}_{a}$. In 2D, one has

$$\begin{aligned} &(\bar{\bar{\bar{F}}}^{-1})^{B}{}_{\tilde{B}} \left[(\bar{\bar{\bar{F}}}^{-1})^{a}{}_{\tilde{A}} \tilde{\mathbb{A}}^{c\tilde{A}b\tilde{B}} - (\bar{\bar{\bar{F}}}^{-1})^{c}{}_{\tilde{A}} \tilde{\mathbb{A}}^{a\tilde{A}b\tilde{B}} \right] \\ &= (\bar{\bar{\bar{F}}}^{-1})^{b}{}_{\tilde{B}} \left[(\bar{\bar{\bar{F}}}^{-1})^{a}{}_{\tilde{A}} \tilde{\mathbb{A}}^{c\tilde{A}B\tilde{B}} - (\bar{\bar{\bar{F}}}^{-1})^{c}{}_{\tilde{A}} \tilde{\mathbb{A}}^{a\tilde{A}B\tilde{B}} \right], \quad \forall\, b, B, c, a \in \{1, 2\}. \end{aligned} \tag{7.296}$$

Let us consider an arbitrary cloaking transformation with the following components

$$\bar{\bar{\bar{\mathbf{F}}}}^{-1} = \begin{bmatrix} \mathsf{F}_{11} & \mathsf{F}_{12} \\ \mathsf{F}_{21} & \mathsf{F}_{22} \end{bmatrix}. \tag{7.297}$$

Equation (7.296) is expanded to read

$$\lambda \left(\mathsf{F}_{12} - \mathsf{F}_{21}\right)^2 + \mu \left[\left(\mathsf{F}_{11} - \mathsf{F}_{22}\right)^2 + 2\left(\mathsf{F}_{12}^2 + \mathsf{F}_{21}^2\right) \right] = 0\,. \tag{7.298}$$

We know that $\mu > 0$, and $2\mu + 3\lambda > 0$ for the energy function to be positive-definite.[34] Thus, multiplying the above identity by 3, one obtains $3\lambda \left(\mathsf{F}_{12} - \mathsf{F}_{21}\right)^2 + 3\mu[\left(\mathsf{F}_{11} - \mathsf{F}_{22}\right)^2 + 2(\mathsf{F}_{12}^2 + \mathsf{F}_{21}^2)] = 0$, which can be rewritten as

$$(3\lambda + 2\mu)\left(\mathsf{F}_{12} - \mathsf{F}_{21}\right)^2 + 3\mu\left(\mathsf{F}_{11} - \mathsf{F}_{22}\right)^2 + \mu\left[3\left(\mathsf{F}_{12} + \mathsf{F}_{21}\right)^2 + \left(\mathsf{F}_{12} - \mathsf{F}_{21}\right)^2\right] = 0\,. \tag{7.299}$$

Note that the coefficient of each term is positive. Therefore, $\mathsf{F}_{12} = \mathsf{F}_{21} = 0$, and $\mathsf{F}_{11} = \mathsf{F}_{22}$, i.e.,

[34] Note that

$$\begin{aligned} \delta W &= \frac{1}{2} \frac{\partial W}{\partial F^a{}_A \partial F^b{}_B} U^a{}_{|A} U^b{}_{|B} + \frac{\partial W}{\partial F^a{}_A \partial F^b{}_{B|C}} U^a{}_{|A} U^b{}_{|B|C} + \frac{1}{2} \frac{\partial W}{\partial F^a{}_{A|B} \partial F^b{}_{C|D}} U^a{}_{|A|B} U^b{}_{|C|D} \\ &= \frac{1}{2} \mathbb{A}^{aAbB} U_{a|A} U_{b|B} + \mathbb{B}^{aAbBC} U_{a|A} U_{b|B|C} + \frac{1}{2} \mathbb{C}^{aABbCD} U_{a|A|B} U_{b|C|D}. \end{aligned}$$

Positive-definiteness of energy requires that $\delta W > 0$ for any pair $(U_{a|A}, U_{a|A|B}) \neq (0, 0)$. In particular, when $U_{a|A} \neq 0$, and $U_{a|A|B} = 0$, $\mathbb{A}^{aAbB} U_{a|A} U_{b|B} > 0$, which implies that $\mathbb{A}$ must be positive-definite. In the case of isotropic solids this is equivalent to $\mu > 0$, and $3\lambda + 2\mu > 0$.

$\bar{\bar{\bar{\mathbf{F}}}} = \alpha\mathbf{I}$, where $\mathbf{I}$ is the identity matrix and $\alpha > 0$ is a scalar.

In 3D, the balance of angular momentum in the physical body reads

$$\begin{aligned}&(\bar{\bar{\bar{F}}}^{-1})^{B}{}_{\tilde{B}}\left[(\bar{\bar{\bar{F}}}^{-1})^{a}{}_{\tilde{A}}\tilde{\mathbb{A}}^{c\tilde{A}b\tilde{B}}-(\bar{\bar{\bar{F}}}^{-1})^{c}{}_{\tilde{A}}\tilde{\mathbb{A}}^{a\tilde{A}b\tilde{B}}\right]\\&=(\bar{\bar{\bar{F}}}^{-1})^{b}{}_{\tilde{B}}\left[(\bar{\bar{\bar{F}}}^{-1})^{a}{}_{\tilde{A}}\tilde{\mathbb{A}}^{c\tilde{A}B\tilde{B}}-(\bar{\bar{\bar{F}}}^{-1})^{c}{}_{\tilde{A}}\tilde{\mathbb{A}}^{a\tilde{A}B\tilde{B}}\right],\quad\forall\, b,B,c,a\in\{1,2,3\}.\end{aligned}\tag{7.300}$$

This relation for $\{b \neq B\}$ and $\{c \neq a\}$ is nontrivial and gives six linearly independent algebraic equations. Let us consider an arbitrary cloaking transformation with the following components

$$\bar{\bar{\bar{\mathbf{F}}}}^{-1}=\begin{bmatrix}\mathsf{F}_{11} & \mathsf{F}_{12} & \mathsf{F}_{13}\\ \mathsf{F}_{21} & \mathsf{F}_{22} & \mathsf{F}_{23}\\ \mathsf{F}_{31} & \mathsf{F}_{32} & \mathsf{F}_{33}\end{bmatrix}.\tag{7.301}$$

Equation (7.300) is expanded for $i \neq j \neq k \in \{1,2,3\}$ and reads

$$\lambda\left(\mathsf{F}_{ij}-\mathsf{F}_{ji}\right)^2+\mu\left[2\left(\mathsf{F}_{ij}^2+\mathsf{F}_{ji}^2\right)+\mathsf{F}_{ik}^2+\mathsf{F}_{jk}^2+\left(\mathsf{F}_{ii}-\mathsf{F}_{jj}\right)^2\right]=0\,,\tag{7.302}$$

$$\lambda\left(\mathsf{F}_{ij}-\mathsf{F}_{ji}\right)\left(\mathsf{F}_{ik}-\mathsf{F}_{ki}\right)+\mu\left[\mathsf{F}_{ij}\mathsf{F}_{ik}+\mathsf{F}_{jj}\mathsf{F}_{kj}+\mathsf{F}_{kk}\mathsf{F}_{jk}+2\mathsf{F}_{ji}\mathsf{F}_{ki}-\mathsf{F}_{ii}\left(\mathsf{F}_{jk}+\mathsf{F}_{kj}\right)\right]=0\,.\tag{7.303}$$

After some algebraic manipulations (7.302) is rewritten as

$$(3\lambda+2\mu)\left(\mathsf{F}_{ij}-\mathsf{F}_{ji}\right)^2+\mu\left[3\left(\mathsf{F}_{ij}+\mathsf{F}_{ji}\right)^2+\left(\mathsf{F}_{ij}-\mathsf{F}_{ji}\right)^2\right]+3\mu\left[\mathsf{F}_{ik}^2+\mathsf{F}_{jk}^2+\left(\mathsf{F}_{ii}-\mathsf{F}_{jj}\right)^2\right]=0\,.\tag{7.304}$$

Note that the coefficient of each term is positive. Thus, $\mathsf{F}_{ij} = 0$ for $i \neq j$, and $\mathsf{F}_{11} = \mathsf{F}_{22} = \mathsf{F}_{33}$. These trivially satisfy (7.303). One concludes that $\bar{\bar{\bar{\mathbf{F}}}} = \alpha\mathbf{I}$, where $\mathbf{I}$ is the identity matrix and $\alpha > 0$ is a scalar. Knowing that the cloaking map restricted to the outer boundary of the cloak is the identity map, one concludes that $\alpha = 1$, and hence, $\Xi = id$, which is clearly not an acceptable cloaking transformation. □

7.5.2 Elastodynamic Transformation Cloaking in Generalized Cosserat Solids

We assume that the microstructure in the deformed configuration is described by three (or two in 2D) linearly independent vectors $\{\underset{\mathfrak{a}}{\mathbf{d}}(x,t),\ \mathfrak{a}=1,2,3\}$, which are called director fields or directors [227]. The directors in the reference configuration are denoted by $\big\{\underset{\mathfrak{a}}{\mathbf{D}}(X),\ \mathfrak{a}=1,2,3\big\}$. These are not material vectors in the sense that $\underset{\mathfrak{a}}{\mathbf{d}}(x,t)\neq(\varphi_*\underset{\mathfrak{a}}{\mathbf{D}})(x,t)$. The kinematics of an oriented body is described by the pair $\big(\varphi(X,t),\underset{\mathfrak{a}}{\mathbf{d}}(X,t)\big)$, where $\underset{\mathfrak{a}}{\mathbf{d}}(X,t)$ is short for $\underset{\mathfrak{a}}{\mathbf{d}}(\varphi(X,t),\underset{\mathfrak{a}}{\mathbf{D}}(X))$ [229, 238]. An oriented body with (deformable) directors is called a generalized Cosserat solid. If the directors are rigid, i.e.,

$$\underset{\mathfrak{a}}{\mathsf{d}}^a(X,t)\,\underset{\mathfrak{b}}{\mathsf{d}}^b(X,t)g_{ab}(\varphi(X,t))=\underset{\mathfrak{ab}}{\mathfrak{g}}(X)\,, \tag{7.305}$$

for some symmetric, positive-definite, and time-independent matrix $\underset{\mathfrak{ab}}{\mathfrak{g}}$, the oriented body is referred to as a Cosserat solid. The reciprocal of $\underset{\mathfrak{a}}{\mathbf{D}}$ is denoted by $\overset{\mathfrak{a}}{\mathbf{\Theta}}$ such that $\underset{\mathfrak{a}}{\mathsf{D}}^A\overset{\mathfrak{a}}{\Theta}_B=\delta^A_B$ and $\underset{\mathfrak{a}}{\mathsf{D}}^A\overset{\mathfrak{b}}{\Theta}_A=\delta^{\mathfrak{b}}_{\mathfrak{a}}$. Similarly, the reciprocal of $\underset{\mathfrak{a}}{\mathbf{d}}$ is denoted by $\overset{\mathfrak{a}}{\boldsymbol{\vartheta}}$ such that $\underset{\mathfrak{a}}{\mathsf{d}}^a\overset{\mathfrak{a}}{\vartheta}_b=\delta^a_b$ and $\underset{\mathfrak{a}}{\mathsf{d}}^a\overset{\mathfrak{b}}{\vartheta}_a=\delta^{\mathfrak{b}}_{\mathfrak{a}}$. The referential directors vary from point to point and $\underset{\mathfrak{a}}{\mathsf{D}}^A{}_{|B}=W^A{}_{CB}\underset{\mathfrak{a}}{\mathsf{D}}^C$, where $W^A{}_{CB}$ is called the wryness of the director field and is defined as

$$W^A{}_{BC}=\underset{\mathfrak{a}}{\mathsf{D}}^A{}_{|C}\overset{\mathfrak{a}}{\Theta}_B=-\underset{\mathfrak{a}}{\mathsf{D}}^A\overset{\mathfrak{a}}{\Theta}_{B|C}\,. \tag{7.306}$$

The director gradient $\underset{\mathfrak{a}}{\mathbf{F}}=\nabla^{\mathbf{G}}\underset{\mathfrak{a}}{\mathbf{d}}$ with components $\underset{\mathfrak{a}}{\mathsf{F}}^a{}_A=\underset{\mathfrak{a}}{\mathsf{d}}^a{}_{|A}$ is related to the relative wryness as

$$\underset{\mathfrak{a}}{\mathsf{F}}^a{}_A=w^a{}_{bA}\,\underset{\mathfrak{a}}{\mathsf{d}}^b\,, \tag{7.307}$$

where

$$w^a{}_{bA}=\underset{\mathfrak{a}}{\mathsf{d}}^a{}_{|A}\overset{\mathfrak{a}}{\vartheta}_b\,. \tag{7.308}$$

One can define the following director metrics: $\underset{\mathfrak{ab}}{\mathfrak{G}}=\underset{\mathfrak{a}}{\mathsf{D}}^A\underset{\mathfrak{b}}{\mathsf{D}}^BG_{AB}$, and $\underset{\mathfrak{ab}}{\mathfrak{g}}=\underset{\mathfrak{a}}{\mathsf{d}}^a\underset{\mathfrak{b}}{\mathsf{d}}^bg_{ab}$. Using these metrics, $\overset{\mathfrak{a}}{\Theta}_A=\overset{\mathfrak{ab}}{\mathfrak{G}}\,G_{AB}\underset{\mathfrak{b}}{\mathsf{D}}^B$, and $\overset{\mathfrak{a}}{\vartheta}_a=\overset{\mathfrak{ab}}{\mathfrak{g}}\,g_{ab}\underset{\mathfrak{b}}{\mathsf{d}}^b$.

The kinetic energy density of a Cosserat solid is written as [229]

$$T = \frac{1}{2}\rho V^a V^b g_{ab} + \frac{1}{2}\overset{\mathfrak{ab}}{\nu}\, \underset{\mathfrak{a}}{\dot{\mathsf{d}}}^a \underset{\mathfrak{b}}{\dot{\mathsf{d}}}^b g_{ab}, \tag{7.309}$$

where $\overset{\mathfrak{ab}}{\nu} = \overset{\mathfrak{ba}}{\nu}$ is the micro-mass moment of inertia, and $\underset{\mathfrak{a}}{\dot{\mathbf{d}}}(X,t) = \frac{\partial}{\partial t}\underset{\mathfrak{a}}{\mathbf{d}}(X,t)$ is the director velocity. The energy density is written as $W = W(X, \mathbf{F}, \underset{\mathfrak{a}}{\mathbf{F}}, \mathbf{G}, \mathbf{g})$.[35]

Balance of linear momentum. We next derive the governing equations of generalized Cosserat elasticity using Hamilton's principle of least action. The variation of the kinetic energy is written as

$$\begin{aligned}\delta T &= \rho_0 \langle\!\langle \mathbf{V}, D_t^{\mathbf{g}}\delta\varphi \rangle\!\rangle_{\mathbf{g}} + \overset{\mathfrak{ab}}{\nu}\, \langle\!\langle \underset{\mathfrak{a}}{\dot{\mathbf{d}}}, D_t^{\mathbf{g}}\delta\underset{\mathfrak{b}}{\mathbf{d}} \rangle\!\rangle_{\mathbf{g}} \\ &= \frac{d}{dt}\left(\rho_0 \langle\!\langle \mathbf{V}, \delta\varphi \rangle\!\rangle_{\mathbf{g}} + \overset{\mathfrak{ab}}{\nu}\, \langle\!\langle \underset{\mathfrak{a}}{\dot{\mathbf{d}}}, \delta\underset{\mathfrak{b}}{\mathbf{d}} \rangle\!\rangle_{\mathbf{g}}\right) - \rho_0 \langle\!\langle \mathbf{A}, \delta\varphi \rangle\!\rangle_{\mathbf{g}} - \overset{\mathfrak{ab}}{\nu}\, \langle\!\langle \underset{\mathfrak{a}}{\ddot{\mathbf{d}}}, \delta\underset{\mathfrak{b}}{\mathbf{d}} \rangle\!\rangle_{\mathbf{g}},\end{aligned} \tag{7.310}$$

where $\underset{\mathfrak{a}}{\ddot{\mathbf{d}}} = D_t^{\mathbf{g}}\underset{\mathfrak{a}}{\dot{\mathbf{d}}}$ is the director acceleration. Assuming that $\delta\varphi(X,t_1) = \delta\varphi(X,t_2) = 0$, and $\delta\underset{\mathfrak{b}}{\mathbf{d}}(X,t_1) = \delta\underset{\mathfrak{b}}{\mathbf{d}}(X,t_2) = \mathbf{0}$ one obtains

$$\delta\int_{t_1}^{t_2}\int_{\mathcal{B}} -T\,dV\,dt = \int_{t_1}^{t_2}\int_{\mathcal{B}}\left(\rho_0 \langle\!\langle \mathbf{A}, \delta\varphi \rangle\!\rangle_{\mathbf{g}} + \overset{\mathfrak{ab}}{\nu}\, \langle\!\langle \underset{\mathfrak{a}}{\ddot{\mathbf{d}}}, \delta\underset{\mathfrak{b}}{\mathbf{d}} \rangle\!\rangle_{\mathbf{g}}\right) dV\,dt. \tag{7.311}$$

The variation of the energy density is calculated as

$$\delta W = \nabla^{\varphi_\epsilon}_{\frac{\partial}{\partial\epsilon}} W\Big|_{\epsilon=0} = \frac{\partial W}{\partial \mathbf{F}} : \delta\mathbf{F} + \frac{\partial W}{\partial \underset{\mathfrak{a}}{\mathbf{F}}} : \delta\underset{\mathfrak{a}}{\mathbf{F}} = \frac{\partial W}{\partial F^a{}_A}\delta F^a{}_A + \frac{\partial W}{\partial \underset{\mathfrak{a}}{\mathsf{F}}{}^a{}_A}\delta\underset{\mathfrak{a}}{\mathsf{F}}{}^a{}_A\,. \tag{7.312}$$

[35] Note that the energy function can have an explicit dependence on the director field, i.e., $W = W(X, \mathbf{F}, \underset{\mathfrak{a}}{\mathbf{d}}, \underset{\mathfrak{a}}{\mathbf{F}}, \mathbf{G}, \mathbf{g})$. The partial derivative $\frac{\partial W}{\partial \underset{\mathfrak{a}}{\mathbf{d}}}$ is a micro body force that we do not consider in this chapter.

Note that $\delta F^a{}_A = U^a{}_{|A}$ and $\delta \underset{\mathfrak{a}}{\mathsf{F}}{}^a{}_A = \delta \underset{\mathfrak{a}}{\mathsf{d}}{}^a{}_{|A}$. Thus

$$
\begin{aligned}
\int_{\mathcal{B}} \delta W dV &= \int_{\mathcal{B}} \left(\frac{\partial W}{\partial F^a{}_A} \delta \varphi^a{}_{|A} + \frac{\partial W}{\partial \underset{\mathfrak{a}}{\mathsf{F}}{}^a{}_A} \delta \underset{\mathfrak{a}}{\mathsf{d}}{}^a{}_{|A} \right) dV \\
&= - \int_{\mathcal{B}} \left[\left(\frac{\partial W}{\partial F^a{}_A} \right)_{|A} \delta \varphi^a + \left(\frac{\partial W}{\partial \underset{\mathfrak{a}}{\mathsf{F}}{}^a{}_A} \right)_{|A} \delta \underset{\mathfrak{a}}{\mathsf{d}}{}^a \right] dV \\
&\quad + \int_{\partial\mathcal{B}} \left(\frac{\partial W}{\partial F^a{}_A} N_A \delta \varphi^a + \frac{\partial W}{\partial \underset{\mathfrak{a}}{\mathsf{F}}{}^a{}_A} N_A \delta \underset{\mathfrak{a}}{\mathsf{d}}{}^a \right) dA\,.
\end{aligned}
\tag{7.313}
$$

Therefore, the balance of linear momentum and the balance of micro linear momentum read

$$
P^{aA}{}_{|A} = \rho_0 A^a, \quad \overset{\mathfrak{a}}{\mathsf{H}}{}^{aA}{}_{|A} = \overset{\mathfrak{ab}}{\nu}\, \underset{\mathfrak{b}}{\ddot{\mathsf{d}}}{}^a\,,
\tag{7.314}
$$

where

$$
P^{aA} = g^{ab} \frac{\partial W}{\partial F^b{}_A}, \quad \overset{\mathfrak{a}}{\mathsf{H}}{}^{aA} = g^{ab} \frac{\partial W}{\partial \underset{\mathfrak{a}}{\mathsf{F}}{}^b{}_A}\,.
\tag{7.315}
$$

$\overset{\mathfrak{a}}{\mathsf{H}}{}^{aA}$ is called the hyperstress tensor. The traction and micro-traction are defined as $T^a = P^{aA} N_A$ and $\overset{\mathfrak{a}}{\mathsf{T}}{}^a = \overset{\mathfrak{a}}{\mathsf{H}}{}^{aA} N_A$, respectively.

Balance of angular momentum. In order to derive the balance of angular momentum in a Euclidean ambient space consider the flow $\psi_s(x) = x + s\mathbf{\Omega} x$, where $\mathbf{\Omega}$ is an anti-symmetric matrix. As the kinetic energy density (8.42) is invariant under rotations one only needs to require invariance of the energy function under flows of rotations of the ambient space, i.e.,

$$
W(X, \mathbf{F}, \underset{\mathfrak{a}}{\mathsf{F}}, \mathbf{G}, \mathbf{g}) = W(X, \psi_{s*}\mathbf{F}, \psi_{s*}\underset{\mathfrak{a}}{\mathsf{F}}, \mathbf{G}, \psi_{s*}\mathbf{g}).
\tag{7.316}
$$

Note that under a rotation $R^a{}_b$ of the Euclidean ambient space the directors are transformed as $\underset{\mathfrak{a}}{\mathsf{d}}{}'^a = R^a{}_b \underset{\mathfrak{a}}{\mathsf{d}}{}^b$. Taking derivative with respect to s of both sides of (7.316) and evaluating at $s = 0$, one obtains

$$
g^{ac} \left[F^b{}_A \frac{\partial W}{\partial F^c{}_A} + \underset{\mathfrak{a}}{\mathsf{F}}{}^b{}_A \frac{\partial W}{\partial \underset{\mathfrak{a}}{\mathsf{F}}{}^c{}_A} \right] \Omega_{ab} = 0\,.
\tag{7.317}
$$

As $\Omega_{ba} = -\Omega_{ab}$, one concludes that $\Pi^{ab} = \Pi^{ba}$, or $\Pi^{[ab]} = 0$, where

$$\Pi^{ab} = g^{ac}\left[F^b{}_A \frac{\partial W}{\partial F^c{}_A} + \underset{\mathfrak{a}}{\mathsf{F}}^b{}_A \frac{\partial W}{\partial \underset{\mathfrak{a}}{\mathsf{F}}^c{}_A}\right] = F^b{}_A P^{aA} + \underset{\mathfrak{a}}{\mathsf{F}}^b{}_A \overset{\mathfrak{a}}{\mathsf{H}}^{aA}\,. \tag{7.318}$$

Thus, the balance of angular momentum reads

$$P^{[aA} F^{b]}{}_A + \overset{\mathfrak{a}}{\mathsf{H}}^{[aA} \underset{\mathfrak{a}}{\mathsf{F}}^{b]}{}_A = 0\,. \tag{7.319}$$

Linearized balance of linear momentum. Linearizing the balance of linear momentum and the balance of micro linear momentum about a pair $(\mathring{\varphi}, \underset{\mathfrak{a}}{\mathring{\mathbf{d}}})$ one obtains

$$(\delta P^{aA})_{|A} = \rho_0 \ddot{U}^a, \quad (\delta \overset{\mathfrak{a}}{\mathsf{H}}^{aA})_{|A} = \overset{\mathfrak{ab}}{\nu}\, \underset{\mathfrak{b}}{\ddot{\mathfrak{U}}}{}^a, \tag{7.320}$$

where $\underset{\mathfrak{a}}{\mathfrak{U}} = \delta \underset{\mathfrak{a}}{\mathbf{d}}$ is the director displacement field. Note that

$$\begin{aligned} \delta P^{aA} &= \mathbb{A}^{aA}{}_b{}^B\, U^b{}_{|B} + \overset{\mathfrak{a}}{\mathbb{B}}{}^{aA}{}_b{}^B\, \underset{\mathfrak{a}}{\mathfrak{U}}{}^b{}_{|B}, \\ \delta \overset{\mathfrak{a}}{\mathsf{H}}^{aA} &= \overset{\mathfrak{a}}{\mathbb{B}}{}^{aA}{}_b{}^B\, U^b{}_{|B} + \overset{\mathfrak{ab}}{\mathbb{C}}{}^{aA}{}_b{}^B\, \underset{\mathfrak{b}}{\mathfrak{U}}{}^b{}_{|B}\,, \end{aligned} \tag{7.321}$$

where

$$\mathbb{A}_a{}^A{}_b{}^B = \frac{\partial^2 W}{\partial F^a{}_A \partial F^b{}_B}, \quad \overset{\mathfrak{a}}{\mathbb{B}}{}_a{}^A{}_b{}^B = \frac{\partial^2 W}{\partial F^a{}_A \partial \underset{\mathfrak{a}}{\mathsf{F}}^b{}_B}, \quad \overset{\mathfrak{ab}}{\mathbb{C}}{}_a{}^A{}_b{}^B = \frac{\partial^2 W}{\partial \underset{\mathfrak{a}}{\mathsf{F}}^a{}_A \partial \underset{\mathfrak{b}}{\mathsf{F}}^b{}_B}\,. \tag{7.322}$$

The elastic constants satisfy the following symmetries: $\mathbb{A}^{aAbB} = \mathbb{A}^{bBaA}$ and $\overset{\mathfrak{ab}}{\mathbb{C}}{}^{aAbB} = \overset{\mathfrak{ba}}{\mathbb{C}}{}^{bBaA}$.

Linearized balance of angular momentum. Suppose the reference motion is an isometric embedding of an initially stress-free body into the Euclidean space, i.e., $\mathring{F}^a{}_A = \delta^a_A$, $\mathring{P}^{aA} = 0$, and $\overset{\mathfrak{a}}{\mathring{\mathsf{H}}}{}^{aA} = 0$. We will linearize the balance of angular momentum about this motion and about a director field $\underset{\mathfrak{a}}{\mathring{\mathsf{d}}}{}^a = \delta^a_A \underset{\mathfrak{a}}{\mathsf{D}}^A$. Thus, $\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^a{}_A = \delta^a_B \underset{\mathfrak{a}}{\mathsf{D}}^B{}_{|A} = \mathring{w}^a{}_{mA} \underset{\mathfrak{a}}{\mathring{\mathsf{d}}}{}^m = \delta^a_C W^C{}_{MA} \underset{\mathfrak{a}}{\mathsf{D}}^M$. Linearization of Π^{ab} in (7.318) one obtains

$$\delta\Pi^{ab} = \left(\mathbb{A}^{aA}{}_c{}^B \mathring{F}^b{}_A + \overset{\mathfrak{a}}{\mathbb{B}}{}^{aA}{}_c{}^B \underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^b{}_A\right) U^c{}_{|B} + \left(\overset{\mathfrak{b}}{\mathbb{B}}{}^{aA}{}_c{}^B \mathring{F}^b{}_A + \overset{\mathfrak{ab}}{\mathbb{C}}{}^{aA}{}_c{}^B \underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^b{}_A\right) \underset{\mathfrak{b}}{\mathfrak{U}}{}^c{}_{|B}\,. \tag{7.323}$$

Knowing that $\delta\Pi^{[ab]} = 0$ and that the displacement and director displacement fields are independent, one obtains

$$\begin{aligned} &\mathbb{A}^{[aA}{}_{c}{}^{B}\,\mathring{F}^{b]}{}_{A} + \overset{\mathfrak{a}}{\mathbb{B}}{}^{[aA}{}_{c}{}^{B}\,\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^{b]}{}_{A} = 0, \\ &\overset{\mathfrak{b}}{\mathbb{B}}{}^{[aA}{}_{c}{}^{B}\,\mathring{F}^{b]}{}_{A} + \overset{\mathfrak{ab}}{\mathbb{C}}{}^{[aA}{}_{c}{}^{B}\,\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^{b]}{}_{A} = 0. \end{aligned} \tag{7.324}$$

Or equivalently

$$\begin{aligned} &\mathbb{A}^{[aAbB}\,\mathring{F}^{c]}{}_{A} + \overset{\mathfrak{a}}{\mathbb{B}}{}^{[aAbB}\,\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^{c]}{}_{A} = 0, \\ &\overset{\mathfrak{a}}{\mathbb{B}}{}^{[aAbB}\,\mathring{F}^{c]}{}_{A} + \overset{\mathfrak{ba}}{\mathbb{C}}{}^{[aAbB}\,\underset{\mathfrak{b}}{\mathring{\mathsf{F}}}{}^{c]}{}_{A} = 0. \end{aligned} \tag{7.325}$$

It is seen that the wryness of the reference configuration enters the linearized balance of angular momentum. In other words, in addition to the elastic constants, one needs some information on the non-uniformity of the director field in the stress-free reference configuration.

Transformation cloaking in generalized Cosserat elastodynamics. It is straightforward to show that the issue with the acceleration term observed in classical nonlinear elasticity persists even in nonlinear Cosserat elasticity, and hence, we do not discuss transformation cloaking in nonlinear Cosserat elastodynamics. We start from a virtual body that is made of a homogeneous, isotropic Cosserat elastic solid. We consider a cloaking transformation and try to calculate the elastic constants of the physical body induced from the cloaking transformation such that the balance of linear and angular momenta are respected in both the virtual and physical bodies.

The divergence term $(\delta P^{aA})_{|A}$ in the balance of linear momentum in the physical body is written as

$$\left(\mathbb{A}^{aA}{}_{b}{}^{B}\,U^{b}{}_{|B} + \overset{\mathfrak{a}}{\mathbb{B}}{}^{aA}{}_{b}{}^{B}\,\underset{\mathfrak{a}}{\mathfrak{U}}{}^{b}{}_{|B}\right)_{|A}\frac{\partial}{\partial x^{a}}. \tag{7.326}$$

Under a cloaking transformation $\Xi : \mathcal{B} \to \tilde{\mathcal{B}}$ and using the shifter map this is transformed to

$$J_{\Xi}\left(\tilde{\mathbb{A}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}}\,\tilde{U}^{\tilde{b}}{}_{|\tilde{B}} + \overset{\mathfrak{a}}{\tilde{\mathbb{B}}}{}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}}\,\underset{\mathfrak{a}}{\tilde{\mathfrak{U}}}{}^{\tilde{b}}{}_{|\tilde{B}}\right)_{|\tilde{A}}\frac{\partial}{\partial \tilde{x}^{\tilde{a}}}, \tag{7.327}$$

where

$$\tilde{U}^{\tilde{a}} = \mathsf{s}^{\tilde{a}}{}_a U^a\,, \quad \tilde{\underset{\mathfrak{a}}{\mathfrak{U}}}^{\tilde{a}} = \mathsf{s}^{\tilde{a}}{}_a\, \underset{\mathfrak{a}}{\mathfrak{U}}^a,$$

$$\tilde{\mathbb{A}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}} = J_{\Xi}^{-1}\mathsf{s}^{\tilde{a}}{}_a \bar{\bar{\bar{F}}}^{\tilde{A}}{}_A (\mathsf{s}^{-1})^b{}_{\tilde{b}} \bar{\bar{\bar{F}}}^{\tilde{B}}{}_B\, \mathbb{A}^{aA}{}_b{}^B\,, \tag{7.328}$$

$$\tilde{\overset{\mathfrak{a}}{\mathbb{B}}}{}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}} = J_{\Xi}^{-1}\mathsf{s}^{\tilde{a}}{}_a \bar{\bar{\bar{F}}}^{\tilde{A}}{}_A (\mathsf{s}^{-1})^b{}_{\tilde{b}} \bar{\bar{\bar{F}}}^{\tilde{B}}{}_B\, \overset{\mathfrak{a}}{\mathbb{B}}{}^{aA}{}_b{}^B\,.$$

Or, equivalently

$$\mathbb{A}^{aA}{}_b{}^B = J_{\Xi}(\mathsf{s}^{-1})^a{}_{\tilde{a}}(\bar{\bar{\bar{F}}}^{-1})^A{}_{\tilde{A}}\mathsf{s}^{\tilde{b}}{}_b(\bar{\bar{\bar{F}}}^{-1})^B{}_{\tilde{B}}\, \tilde{\mathbb{A}}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}}\,,$$

$$\overset{\mathfrak{a}}{\mathbb{B}}{}^{aA}{}_b{}^B = J_{\Xi}(\mathsf{s}^{-1})^a{}_{\tilde{a}}(\bar{\bar{\bar{F}}}^{-1})^A{}_{\tilde{A}}\mathsf{s}^{\tilde{b}}{}_b(\bar{\bar{\bar{F}}}^{-1})^B{}_{\tilde{B}}\, \tilde{\overset{\mathfrak{a}}{\mathbb{B}}}{}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}}\,. \tag{7.329}$$

The divergence term $(\delta\overset{\mathfrak{a}}{\mathsf{H}}{}^{aA})_{|A}$ in the balance of micro linear momentum in the physical body is written as

$$\left(\overset{\mathfrak{a}}{\mathbb{B}}{}^{aA}{}_b{}^B\, U^b{}_{|B} + \overset{\mathfrak{ab}}{\mathbb{C}}{}^{aA}{}_b{}^B\, \underset{\mathfrak{b}}{\mathfrak{U}}^b{}_{|B}\right)_{|A} \frac{\partial}{\partial x^a}. \tag{7.330}$$

Under the cloaking transformation and using the shifter map this is transformed to read

$$J_{\Xi}\left(\tilde{\overset{\mathfrak{a}}{\mathbb{B}}}{}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}}\, \tilde{U}^{\tilde{b}}{}_{|\tilde{B}} + \tilde{\overset{\mathfrak{ab}}{\mathbb{C}}}{}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}}\, \tilde{\underset{\mathfrak{a}}{\mathfrak{U}}}^{\tilde{b}}{}_{|\tilde{B}}\right)_{|\tilde{A}} \frac{\partial}{\partial \tilde{x}^{\tilde{a}}}, \tag{7.331}$$

where

$$\tilde{\overset{\mathfrak{ab}}{\mathbb{C}}}{}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}} = J_{\Xi}^{-1}\mathsf{s}^{\tilde{a}}{}_a \bar{\bar{\bar{F}}}^{\tilde{A}}{}_A (\mathsf{s}^{-1})^b{}_{\tilde{b}} \bar{\bar{\bar{F}}}^{\tilde{B}}{}_B\, \overset{\mathfrak{ab}}{\mathbb{C}}{}^{aA}{}_b{}^B\,, \tag{7.332}$$

and the other transformed quantities are given in (7.334). Equivalently

$$\overset{\mathfrak{ab}}{\mathbb{C}}{}^{aA}{}_b{}^B = J_{\Xi}(\mathsf{s}^{-1})^a{}_{\tilde{a}}(\bar{\bar{\bar{F}}}^{-1})^A{}_{\tilde{A}}\mathsf{s}^{\tilde{b}}{}_b(\bar{\bar{\bar{F}}}^{-1})^B{}_{\tilde{B}}\, \tilde{\overset{\mathfrak{ab}}{\mathbb{C}}}{}^{\tilde{a}\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}}\,. \tag{7.333}$$

The micro-mass moment of inertia is transformed as $\tilde{\overset{\mathfrak{ab}}{\nu}} = J_{\Xi}^{-1}\overset{\mathfrak{ab}}{\nu}$. Similar to classical linear elasticity the mass density is transformed as $\tilde{\rho}_0 = J_{\Xi}^{-1}\rho_0$. In summary, the linearized balance of linear momentum and micro linear momentum are form-invariant under the following *cloaking transformations*:

$$\begin{aligned}
&\tilde{X} = \Xi(X), \quad \tilde{\mathbf{U}} = \mathbf{s} \circ \mathring{\varphi}\, \mathbf{U} \circ \Xi^{-1}, \;\; \tilde{\underset{\mathfrak{a}}{\mathfrak{U}}} = \mathbf{s} \circ \mathring{\varphi}\, \underset{\mathfrak{a}}{\mathfrak{U}} \circ \Xi^{-1}, \\
&\tilde{\rho}_0 = J_\Xi^{-1} \rho_0 \circ \Xi^{-1}, \;\; \tilde{\overset{\mathfrak{ab}}{\nu}} = J_\Xi^{-1} \overset{\mathfrak{ab}}{\nu} \circ \Xi^{-1}, \;\; \tilde{\mathbb{A}} = (J_\Xi^{-1} \mathbf{s}^{-1} \circ \mathring{\varphi}\, \bar{\bar{\bar{\mathbf{F}}}}\, \mathbf{s} \circ \mathring{\varphi}\, \mathbb{A}\, \bar{\bar{\bar{\mathbf{F}}}}^\star) \circ \Xi^{-1}, \qquad (7.334) \\
&\tilde{\overset{\mathfrak{a}}{\mathbb{B}}} = (J_\Xi^{-1} \mathbf{s}^{-1} \circ \mathring{\varphi}\, \bar{\bar{\bar{\mathbf{F}}}}\, \mathbf{s} \circ \mathring{\varphi}\, \overset{\mathfrak{a}}{\mathbb{B}}\, \bar{\bar{\bar{\mathbf{F}}}}^\star) \circ \Xi^{-1}, \;\; \tilde{\overset{\mathfrak{ab}}{\mathbb{C}}} = (J_\Xi^{-1} \mathbf{s}^{-1} \circ \mathring{\varphi}\, \bar{\bar{\bar{\mathbf{F}}}}\, \mathbf{s} \circ \mathring{\varphi}\, \overset{\mathfrak{ab}}{\mathbb{C}}\, \bar{\bar{\bar{\mathbf{F}}}}^\star) \circ \Xi^{-1}.
\end{aligned}$$

Note that under a cloaking transformation, the stress and hyperstress tensors are transformed as

$$\tilde{P}^{\tilde{a}\tilde{A}} = J_\Xi^{-1} \mathsf{s}^{\tilde{a}}{}_a \bar{\bar{\bar{F}}}^{\tilde{A}}{}_A P^{aA}, \quad \tilde{\overset{\mathfrak{a}}{\mathsf{H}}}^{\tilde{a}\tilde{A}} = J_\Xi^{-1} \mathsf{s}^{\tilde{a}}{}_a \bar{\bar{\bar{F}}}^{\tilde{A}}{}_A \overset{\mathfrak{a}}{\mathsf{H}}^{aA}. \qquad (7.335)$$

We assume that in the stress-free reference configuration of the virtual body the director field is uniform, i.e., $\tilde{\underset{\mathfrak{a}}{\mathsf{D}}}^{\tilde{A}}{}_{|\tilde{B}} = 0$, and hence, $\tilde{\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}}^{\tilde{a}}{}_{\tilde{A}} = 0$. In other words, in the virtual body the wryness vanishes. Therefore, the balance of angular momentum (7.325) in the virtual body reads

$$\tilde{\mathbb{A}}^{[\tilde{a}\tilde{A}\tilde{b}\tilde{B}} \tilde{\mathring{F}}^{\tilde{c}]}{}_{\tilde{A}} = 0, \quad \tilde{\overset{\mathfrak{a}}{\mathbb{B}}}^{[\tilde{a}\tilde{A}\tilde{b}\tilde{B}} \tilde{\mathring{F}}^{\tilde{c}]}{}_{\tilde{A}} = 0. \qquad (7.336)$$

Assuming that in the virtual body the balance of angular momentum (7.336) is satisfied, the balance of angular momentum in the physical body (7.325) requires that

$$\begin{aligned}
&(\mathsf{s}^{-1})^{[a}{}_{\tilde{a}} (\mathsf{s}^{-1})^{b}{}_{\tilde{b}} (\bar{\bar{\bar{F}}}^{-1})^{A}{}_{\tilde{A}} (\bar{\bar{\bar{F}}}^{-1})^{B}{}_{\tilde{B}} \left\{ \mathring{F}^{c]}{}_A \tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} + \underset{\mathfrak{a}}{\mathring{\mathsf{F}}}^{c]}{}_A\, \tilde{\overset{\mathfrak{a}}{\mathbb{B}}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} \right\} = 0, \\
&(\mathsf{s}^{-1})^{[a}{}_{\tilde{a}} (\mathsf{s}^{-1})^{b}{}_{\tilde{b}} (\bar{\bar{\bar{F}}}^{-1})^{A}{}_{\tilde{A}} (\bar{\bar{\bar{F}}}^{-1})^{B}{}_{\tilde{B}} \left\{ \mathring{F}^{c]}{}_A \tilde{\overset{\mathfrak{a}}{\mathbb{B}}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} + \underset{\mathfrak{b}}{\mathring{\mathsf{F}}}^{c]}{}_A \tilde{\overset{\mathfrak{ba}}{\mathbb{C}}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} \right\} = 0,
\end{aligned} \qquad (7.337)$$

where $\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}^{c}{}_A = \delta^c_C W^C{}_{MA} \underset{\mathfrak{a}}{\mathsf{D}}^M$. Note that $(\bar{\bar{\bar{F}}})^{\tilde{A}}{}_{A|B} = (\bar{\bar{\bar{F}}})^{\tilde{A}}{}_{B|A}$.

Remark 7.5.4. Note that $\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}^{a}{}_{A|B} = \underset{\mathfrak{a}}{\mathring{\mathsf{d}}}^{a}{}_{|A|B}$. Knowing that the reference configuration of the physical body is flat, covariant derivatives commute. Therefore, a necessary condition for a field $\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}^{a}{}_A$ to be the director gradient of a director field is

$$\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}^{a}{}_{A|B} = \underset{\mathfrak{a}}{\mathring{\mathsf{F}}}^{a}{}_{B|A}. \qquad (7.338)$$

When designing a cloak the field $\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}^{a}{}_A$ is not known a priori. It must satisfy both (7.337) and the compatibility equations (7.338).

Remark 7.5.5. In the literature it has been assumed that the virtual body is made of a classical linear elastic solid while the cloak is suggested to be made of a Cosserat elastic solid (without any discussion on how to calculate the Cosserat elastic constants). Note, however, that the virtual body cannot be made of a classical solid. If one assumes that the virtual body is classical, i.e., $\overset{\mathfrak{a}}{\tilde{\mathbb{B}}}$ and $\overset{\mathfrak{ab}}{\tilde{\mathbb{C}}}$ vanish, (7.337) reduces to the classical balance of angular momentum for the physical body (7.182), which is an obstruction to transformation cloaking. In other words, both the virtual body and the physical body (even outside the cloak) must be made of Cosserat linear elastic solids in order to achieve elastodynamic transformation cloaking.

Elastic constants of the virtual body. We assume that the virtual body is made of an elastic, homogeneous, and isotropic generalized Cosserat solid. The most general form of a fourth-order isotropic tensor is given by $a_1\delta_{ij}\delta_{kl} + a_2\delta_{ik}\delta_{jl} + a_3\delta_{il}\delta_{jk}$ for some scalars a_1, a_2, and a_3. Therefore, the elastic constants of the virtual body are written as

$$\begin{aligned}\tilde{\mathbb{A}}_{\tilde{a}}{}^{\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}} &= \lambda \tilde{G}^{\tilde{M}\tilde{A}}\tilde{G}^{\tilde{N}\tilde{B}}\mathring{F}^{\tilde{m}}{}_{\tilde{M}}\mathring{F}^{\tilde{n}}{}_{\tilde{N}}g_{\tilde{a}\tilde{m}}g_{\tilde{b}\tilde{n}} \\ &\quad + \mu\Big(\tilde{G}^{\tilde{M}\tilde{N}}\tilde{G}^{\tilde{A}\tilde{B}}\mathring{F}^{\tilde{m}}{}_{\tilde{M}}\mathring{F}^{\tilde{n}}{}_{\tilde{N}}g_{\tilde{a}\tilde{m}}g_{\tilde{b}\tilde{n}} + \tilde{G}^{\tilde{M}\tilde{B}}\tilde{G}^{\tilde{N}\tilde{A}}\mathring{F}^{\tilde{m}}{}_{\tilde{M}}\mathring{F}^{\tilde{n}}{}_{\tilde{N}}g_{\tilde{a}\tilde{m}}g_{\tilde{b}\tilde{n}}\Big)\,.\end{aligned} \tag{7.339}$$

This is a consequence of the minor symmetries. Also

$$\begin{aligned}\overset{\mathfrak{a}}{\tilde{\mathbb{B}}}{}_{\tilde{a}}{}^{\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}} = \overset{\mathfrak{a}}{\mathring{b}}_1 G^{\tilde{M}\tilde{A}}G^{\tilde{N}\tilde{B}}\mathring{F}^{\tilde{m}}{}_{\tilde{M}}\mathring{F}^{\tilde{n}}{}_{\tilde{N}}g_{\tilde{a}\tilde{m}}g_{\tilde{b}\tilde{n}} &+ \overset{\mathfrak{a}}{\mathring{b}}_2 G^{\tilde{M}\tilde{N}}G^{\tilde{A}\tilde{B}}\mathring{F}^{\tilde{m}}{}_{\tilde{M}}\mathring{F}^{\tilde{n}}{}_{\tilde{N}}g_{\tilde{a}\tilde{m}}g_{\tilde{b}\tilde{n}} \\ &+ \overset{\mathfrak{a}}{\mathring{b}}_3 G^{\tilde{M}\tilde{B}}G^{\tilde{N}\tilde{A}}\mathring{F}^{\tilde{m}}{}_{\tilde{M}}\mathring{F}^{\tilde{n}}{}_{\tilde{N}}g_{\tilde{a}\tilde{m}}g_{\tilde{b}\tilde{n}}\,.\end{aligned} \tag{7.340}$$

But note that $\overset{\mathfrak{a}}{\tilde{\mathbb{B}}}$ has the minor symmetries from $(7.336)_2$, and hence, $\overset{\mathfrak{a}}{\mathring{b}}_2 = \overset{\mathfrak{a}}{\mathring{b}}_3$. Similarly

$$\begin{aligned}\overset{\mathfrak{ab}}{\tilde{\mathbb{C}}}{}_{\tilde{a}}{}^{\tilde{A}}{}_{\tilde{b}}{}^{\tilde{B}} = \overset{\mathfrak{ab}}{c}_1 G^{\tilde{M}\tilde{A}}G^{\tilde{N}\tilde{B}}\mathring{F}^{\tilde{m}}{}_{\tilde{M}}\mathring{F}^{\tilde{n}}{}_{\tilde{N}}g_{\tilde{a}\tilde{m}}g_{\tilde{b}\tilde{n}} &+ \overset{\mathfrak{ab}}{c}_2 G^{\tilde{M}\tilde{N}}G^{\tilde{A}\tilde{B}}\mathring{F}^{\tilde{m}}{}_{\tilde{M}}\mathring{F}^{\tilde{n}}{}_{\tilde{N}}g_{\tilde{a}\tilde{m}}g_{\tilde{b}\tilde{n}} \\ &+ \overset{\mathfrak{ab}}{c}_3 G^{\tilde{M}\tilde{B}}G^{\tilde{N}\tilde{A}}\mathring{F}^{\tilde{m}}{}_{\tilde{M}}\mathring{F}^{\tilde{n}}{}_{\tilde{N}}g_{\tilde{a}\tilde{m}}g_{\tilde{b}\tilde{n}}\,,\end{aligned} \tag{7.341}$$

where $\overset{\mathfrak{ab}}{c}_i = \overset{\mathfrak{ba}}{c}_i$, $i = 1, 2, 3$, because $\overset{\mathfrak{ab}}{\tilde{\mathbb{C}}}{}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} = \overset{\mathfrak{ba}}{\tilde{\mathbb{C}}}{}^{\tilde{b}\tilde{B}\tilde{a}\tilde{A}}$. Knowing that $\mathring{F}^a{}_A = \delta^a_A$, one can identify the spatial and material manifolds with the same metric, and thus, with a slight abuse of notation, the elastic constants are simplified to read

$$
\begin{aligned}
\tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} &= \lambda \tilde{G}^{\tilde{a}\tilde{A}}\tilde{G}^{\tilde{b}\tilde{B}} + \mu(\tilde{G}^{\tilde{a}\tilde{b}}\tilde{G}^{\tilde{A}\tilde{B}} + \tilde{G}^{\tilde{a}\tilde{B}}\tilde{G}^{\tilde{b}\tilde{A}})\,, \\
\overset{\mathfrak{a}}{\tilde{\mathbb{B}}}{}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} &= \overset{\mathfrak{a}}{b}_1 G^{\tilde{a}\tilde{A}}G^{\tilde{b}\tilde{B}} + \overset{\mathfrak{a}}{b}_2(G^{\tilde{a}\tilde{b}}G^{\tilde{A}\tilde{B}} + G^{\tilde{a}\tilde{B}}G^{\tilde{b}\tilde{A}})\,, \\
\overset{\mathfrak{ab}}{\tilde{\mathbb{C}}}{}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} &= \overset{\mathfrak{ab}}{c}_1 G^{\tilde{a}\tilde{A}}G^{\tilde{b}\tilde{B}} + \overset{\mathfrak{ab}}{c}_2 G^{\tilde{a}\tilde{b}}G^{\tilde{A}\tilde{B}} + \overset{\mathfrak{ab}}{c}_3 G^{\tilde{a}\tilde{B}}G^{\tilde{b}\tilde{A}}\,.
\end{aligned}
\tag{7.342}
$$

Therefore, one has 15 and 26 independent elastic constants in dimensions two and three, respectively.

Positive-definiteness of the energy density. Knowing that in the initial configuration $\mathring{P}^{aA} = 0$ and $\overset{\mathfrak{a}}{\mathring{\mathsf{H}}}{}^{aA} = 0$, one can write

$$
\begin{aligned}
\delta W &= \frac{1}{2}\frac{\partial W}{\partial F^a{}_A \partial F^b{}_B} U^a{}_{|A} U^b{}_{|B} + \frac{\partial W}{\partial F^a{}_A \partial \underset{\mathfrak{a}}{\mathsf{F}}{}^b{}_B} U^a{}_{|A} \underset{\mathfrak{a}}{\mathfrak{U}}{}^b{}_{|B} + \frac{1}{2}\frac{\partial W}{\partial \underset{\mathfrak{a}}{\mathsf{F}}{}^a{}_A \partial \underset{\mathfrak{b}}{\mathsf{F}}{}^b{}_B} \underset{\mathfrak{a}}{\mathfrak{U}}{}^a{}_{|A} \underset{\mathfrak{b}}{\mathfrak{U}}{}^b{}_{|B} \\
&= \frac{1}{2}\mathbb{A}_a{}^A{}_b{}^B U^a{}_{|A} U^b{}_{|B} + \overset{\mathfrak{a}}{\mathbb{B}}_a{}^A{}_b{}^B U^a{}_{|A} \underset{\mathfrak{a}}{\mathfrak{U}}{}^b{}_{|B} + \frac{1}{2}\overset{\mathfrak{ab}}{\mathbb{C}}_a{}^A{}_b{}^B \underset{\mathfrak{a}}{\mathfrak{U}}{}^a{}_{|A} \underset{\mathfrak{b}}{\mathfrak{U}}{}^b{}_{|B} \\
&= \frac{1}{2}\mathbb{A}^{aAbB} U_{a|A} U_{b|B} + \overset{\mathfrak{a}}{\mathbb{B}}{}^{aAbB} U_{a|A} \underset{\mathfrak{a}}{\mathfrak{U}}_{b|B} + \frac{1}{2}\overset{\mathfrak{ab}}{\mathbb{C}}{}^{aAbB} \underset{\mathfrak{a}}{\mathfrak{U}}_{a|A} \underset{\mathfrak{b}}{\mathfrak{U}}_{b|B}.
\end{aligned}
\tag{7.343}
$$

Let us introduce the new indices $\gamma = \{aA\}$, $\Gamma = \{\mathfrak{a}aA\}$,[36] and define the new variables $X_\gamma = U_{a|A}$ and $Y_\Gamma = \underset{\mathfrak{a}}{\mathfrak{U}}_{a|A}$. Thus, the energy density is rewritten as

$$
\delta W = \frac{1}{2}\mathbb{A}^{\gamma\lambda} X_\gamma X_\lambda + \mathbb{B}^{\gamma\Gamma} X_\gamma Y_\Gamma + \frac{1}{2}\mathbb{C}^{\Gamma\Lambda} Y_\Gamma Y_\Lambda. \tag{7.344}
$$

Next let us define a new variable $\mathbf{Z} = \begin{Bmatrix} \mathbf{X} \\ \mathbf{Y} \end{Bmatrix}$. It is straightforward to show that

$$
\delta W = \frac{1}{2}\mathbf{Z}^{\mathsf{T}} \cdot \mathbb{D}\mathbf{Z}, \quad \mathbb{D} = \begin{bmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{B}^{\mathsf{T}} & \mathbb{C} \end{bmatrix}. \tag{7.345}
$$

[36]More specifically, $\{11, 12, \cdots, 1n, 21, 22, \cdots, nn\} \to \{1, 2, \cdots, n^2\}$ and $\{\mathfrak{1}11, \mathfrak{1}12, \cdots, \mathfrak{1}nn, \mathfrak{2}11, \cdots, \mathfrak{n}nn\} \to \{1, 2, \cdots, n^3\}$.

Note that $\mathbb{A}^{\gamma\lambda} = \mathbb{A}^{\lambda\gamma}$, and $\mathbb{C}^{\Gamma\Lambda} = \mathbb{C}^{\Lambda\Gamma}$, but $\mathbb{B}^{\gamma\Gamma} \neq \mathbb{B}^{\Gamma\gamma}$, in general. Note that $\mathbb{D}$ is symmetric, and hence, $\delta W > 0,\ \forall \mathbf{Z} \neq \mathbf{0}$ if and only if all the eigenvalues of $\mathbb{D}$ are positive ($\mathbb{D}$ is a square matrix of size 12 and 36 in dimensions two and three, respectively). For the energy density to be positive-definite it is necessary that $\mathbb{A}$ and $\mathbb{C}$ be positive-definite because $\mathbf{X}$ and $\mathbf{Y}$ are independent. $\mathbb{A}$ is positive-definite if and only if $\mu > 0$ and $3\lambda + 2\mu > 0$. Note that $\underset{\mathfrak{a}}{\mathfrak{U}}$, $\mathfrak{a} = 1, 2, 3$ are independent, and hence, as a consequence of positive-definiteness of $\mathbb{C}$, $3\overset{\mathfrak{aa}}{c}_1 + \overset{\mathfrak{aa}}{c}_2 + \overset{\mathfrak{aa}}{c}_3 > 0$, $\overset{\mathfrak{aa}}{c}_2 + \overset{\mathfrak{aa}}{c}_3 > 0$, and $\overset{\mathfrak{aa}}{c}_2 - \overset{\mathfrak{aa}}{c}_3 > 0$, (no summation on $\mathfrak{a}$). In particular, $\overset{\mathfrak{aa}}{c}_2 > 0$. $\mathbb{A}$ and $\mathbb{C}$ are positive-definite, and hence, invertible. From Schur's complement condition [239], positive-definiteness of $\mathbb{D}$ is now equivalent to positive-definiteness of either $\mathbb{C} - \mathbb{B}^{\mathsf{T}}\mathbb{A}^{-1}\mathbb{B}$ or $\mathbb{A} - \mathbb{B}\mathbb{C}^{-1}\mathbb{B}^{\mathsf{T}}$.

Suppose $\delta W > 0$ for any $(U^a{}_{|A}, \underset{\mathfrak{a}}{\mathfrak{U}}^a{}_{|A}) \neq (0, 0)$. In the virtual body

$$\delta\tilde{W} = \frac{1}{2}\tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\tilde{U}_{\tilde{a}|\tilde{A}}\tilde{U}_{\tilde{b}|\tilde{B}} + \overset{\mathfrak{a}}{\tilde{\mathbb{B}}}{}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\tilde{U}_{\tilde{a}|\tilde{A}}\underset{\mathfrak{a}}{\tilde{\mathfrak{U}}}_{\tilde{b}|\tilde{B}} + \frac{1}{2}\overset{\mathfrak{ab}}{\tilde{\mathbb{C}}}{}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\underset{\mathfrak{a}}{\tilde{\mathfrak{U}}}_{\tilde{a}|\tilde{A}}\underset{\mathfrak{b}}{\tilde{\mathfrak{U}}}_{\tilde{b}|\tilde{B}}. \tag{7.346}$$

Using (7.334), the relations $\tilde{U}^{\tilde{b}}{}_{|\tilde{B}} = \mathsf{s}^{\tilde{b}}{}_b(\overset{\Xi}{\bar{F}}{}^{-1})^B{}_{\tilde{B}}U^b{}_{|B}$, $\underset{\mathfrak{a}}{\tilde{\mathfrak{U}}}^{\tilde{b}}{}_{|\tilde{B}} = \mathsf{s}^{\tilde{b}}{}_b(\overset{\Xi}{\bar{F}}{}^{-1})^B{}_{\tilde{B}}\underset{\mathfrak{a}}{\mathfrak{U}}^b{}_{|B}$, and the relation $\mathsf{s}^{\tilde{a}}{}_a\mathsf{s}^{\tilde{b}}{}_b\tilde{g}_{\tilde{a}\tilde{b}} = g_{ab}$, one can easily show that $\delta\tilde{W} = J_{\Xi}^{-1}\delta W > 0$. Therefore, the elastic constants of the physical body are positive-definite if and only if those of the virtual body are positive-definite. In other words, the cloaking transformations (7.334) preserve the positive-definiteness of the elastic constants.

Boundary conditions in the physical and virtual problems. Let $\partial\mathcal{B} = \mathcal{H} \cup \partial_o\mathcal{B}$, where $\mathcal{H}$ is the boundary of the hole and $\partial_o\mathcal{B}$ is the outer boundary of $\mathcal{B}$. Suppose $\partial_o\mathcal{B} = \partial_o\mathcal{B}_t \cup \partial_o\mathcal{B}_d$ such that the Neumann and Dirichlet boundary conditions are written as

$$\begin{cases} \mathbf{P}\mathbf{N} = \bar{\mathbf{t}}(X,t) \\ \overset{\mathfrak{a}}{\mathbf{H}}\mathbf{N} = \overset{\mathfrak{a}}{\bar{\mathbf{m}}}(X,t) \end{cases} \text{on } \partial_o\mathcal{B}_t, \qquad \begin{cases} \varphi(X,t) = \bar{\varphi}(X,t) \\ \underset{\mathfrak{a}}{\mathbf{d}}(X,t) = \underset{\mathfrak{a}}{\bar{\mathbf{d}}}(X,t) \end{cases} \text{on } \partial_o\mathcal{B}_d\,, \tag{7.347}$$

where $\mathbf{N}$ is the unit normal vector on $\partial_o\mathcal{B}_t$ and $\bar{\mathbf{t}}$, $\bar{\mathbf{m}}$, $\bar{\varphi}$, and $\underset{\mathfrak{a}}{\bar{\mathbf{d}}}$ are given. Under the mapping $\Xi : \mathcal{B} \to \tilde{\mathcal{B}}$, $\partial_o\tilde{\mathcal{B}} = \Xi(\partial_o\mathcal{B}) = \Xi(\partial_o\mathcal{B}_t) \cup \Xi(\partial_o\mathcal{B}_d)$. Suppose on $\partial\mathcal{H}$, $\mathbf{t} = \mathbf{P}\mathbf{N} = \bar{\mathbf{t}}$, and $\overset{\mathfrak{a}}{\mathbf{H}}\mathbf{N} = \overset{\mathfrak{a}}{\bar{\mathbf{m}}}$. Note that

$$\mathbf{s}\mathbf{t}dA = \mathbf{s}\bar{\mathbf{t}}dA = \tilde{\mathbf{t}}d\tilde{A}, \quad \mathbf{s}\overset{\mathfrak{a}}{\mathbf{m}}dA = \mathbf{s}\overset{\mathfrak{a}}{\bar{\mathbf{m}}}dA = \overset{\mathfrak{a}}{\tilde{\mathbf{m}}}d\tilde{A}\,. \tag{7.348}$$

Therefore, on $\partial_o\mathcal{B}_t$, $\tilde{\mathbf{t}} = \tilde{\mathbf{P}}\tilde{\mathbf{N}} = (dA/d\tilde{A})\bar{\mathbf{t}}$, and $\overset{\mathfrak{a}}{\tilde{\mathbf{m}}} = \overset{\mathfrak{a}}{\tilde{\mathbf{H}}}\tilde{\mathbf{N}} = (dA/d\tilde{A})\overset{\mathfrak{a}}{\bar{\mathbf{m}}}$. We know that $\tilde{N}_{\tilde{A}}d\tilde{A} = J_\Xi(\overset{\Xi}{F}{}^{-1})^A{}_{\tilde{A}}N_A dA$. Therefore (note that $\tilde{\mathbf{N}}$ is a $\tilde{\mathbf{G}}$-unit one-form)

$$\tilde{G}^{\tilde{A}\tilde{B}}\tilde{N}_{\tilde{A}}\tilde{N}_{\tilde{B}}d\tilde{A}^2 = d\tilde{A}^2 = J_\Xi^2\left[(\overset{\Xi}{F}{}^{-1})^A{}_{\tilde{A}}(\overset{\Xi}{F}{}^{-1})^B{}_{\tilde{B}}\tilde{G}^{\tilde{A}\tilde{B}}N_AN_B\right]dA^2\,. \tag{7.349}$$

Thus

$$\frac{dA}{d\tilde{A}} = J_\Xi^{-1}\left[(\overset{\Xi}{C}{}^{-1})^{AB}N_AN_B\right]^{-\frac{1}{2}}, \tag{7.350}$$

where $\overset{\Xi}{C}_{AB} = \overset{\Xi}{F}{}^{\tilde{A}}{}_A\overset{\Xi}{F}{}^{\tilde{B}}{}_B\tilde{G}_{\tilde{A}\tilde{B}}$, and $(\overset{\Xi}{C}{}^{-1})^{AB} = (\overset{\Xi}{F}{}^{-1})^A{}_{\tilde{A}}(\overset{\Xi}{F}{}^{-1})^B{}_{\tilde{B}}\tilde{G}^{\tilde{A}\tilde{B}}$. Therefore, the traction boundary condition on both $\partial\tilde{\mathcal{H}}$ and $\partial_o\tilde{\mathcal{B}}_t$ reads

$$\tilde{\mathbf{t}} = J_\Xi^{-1}\left[(\overset{\Xi}{C}{}^{-1})^{AB}N_AN_B\right]^{-\frac{1}{2}}\mathbf{s}\bar{\mathbf{t}}\,, \quad \overset{\mathfrak{a}}{\tilde{\mathbf{m}}} = J_\Xi^{-1}\left[(\overset{\Xi}{C}{}^{-1})^{AB}N_AN_B\right]^{-\frac{1}{2}}\mathbf{s}\overset{\mathfrak{a}}{\bar{\mathbf{m}}}\,. \tag{7.351}$$

Knowing that in $\mathcal{B}\setminus\mathcal{C}$, $\Xi = id$, we have

$$\tilde{\mathbf{t}} = \mathbf{s}\bar{\mathbf{t}}, \quad \overset{\mathfrak{a}}{\tilde{\mathbf{m}}} = \mathbf{s}\overset{\mathfrak{a}}{\bar{\mathbf{m}}} \quad \text{on} \quad \partial_o\tilde{\mathcal{B}}_t\,. \tag{7.352}$$

The hole in the physical body is assumed to be traction free, i.e.,

$$\mathbf{P}\mathbf{N} = \mathbf{t} = \mathbf{0}, \quad \overset{\mathfrak{a}}{\mathbf{H}}\mathbf{N} = \overset{\mathfrak{a}}{\mathbf{m}} = \mathbf{0}, \quad \text{on} \quad \partial\mathcal{H}\,. \tag{7.353}$$

Using (7.351) in the virtual body one has

$$\tilde{\mathbf{P}}\tilde{\mathbf{N}} = \tilde{\mathbf{t}} = \mathbf{0}, \quad \overset{\mathfrak{a}}{\tilde{\mathbf{H}}}\tilde{\mathbf{N}} = \overset{\mathfrak{a}}{\tilde{\mathbf{m}}} = \mathbf{0} \quad \text{on} \quad \partial\tilde{\mathcal{H}}\,. \tag{7.354}$$

In other words, the traction-free boundary condition on the surface of the hole $\mathcal{H}$ implies that the surface of the transformed (infinitesimal) hole $\tilde{\mathcal{H}}$ in the virtual body is traction-free as well.

On $\partial_o\tilde{\mathcal{B}}_d = \Xi(\partial_o\mathcal{B}_d)$, one assumes that the virtual and physical problems have the same Dirichlet boundary condition, i.e.,

$$\tilde{\varphi}(\tilde{X},t) = \bar{\varphi}\circ\Xi^{-1}(\tilde{X},t), \quad \tilde{\mathbf{d}}_{\mathfrak{a}}(\tilde{X},t) = \bar{\mathbf{d}}_{\mathfrak{a}}\circ\Xi^{-1}(\tilde{X},t) \;\text{ on }\; \partial_o\tilde{\mathcal{B}}_d\,. \tag{7.355}$$

Moreover, we note that $\partial(\mathcal{B}\setminus\mathcal{C}) = \partial_o\mathcal{B}\cup\partial_o\mathcal{C}$, and similarly, $\partial(\tilde{\mathcal{B}}\setminus\tilde{\mathcal{C}}) = \partial_o\tilde{\mathcal{B}}\cup\partial_o\tilde{\mathcal{C}}$, where $\partial_o\mathcal{C}$ is the outer boundary of the cloak. As Ξ is defined to be the identity map in $\mathcal{B}\setminus\mathcal{C}$, from (7.355), $\partial_o\mathcal{B}_d$ and $\partial_o\tilde{\mathcal{B}}_d$ have the same displacement boundary conditions. Notice that $T\Xi|_{\partial_o\mathcal{B}_t} = \overset{\Xi}{\mathbf{F}}|_{\partial_o\mathcal{B}_t} = id$, and hence $J_\Xi|_{\partial_o\mathcal{B}_t} = 1$, which implies that the traction boundary conditions on $\partial_o\mathcal{B}_t$ and $\partial_o\tilde{\mathcal{B}}_t$ are identical. Thus, $\partial_o\mathcal{B}$ and $\partial_o\tilde{\mathcal{B}}$ have the same traction and displacement boundary conditions. Note that $\partial\mathcal{C} = \partial\mathcal{H}\cup\partial_o\mathcal{C}$, where $\partial\mathcal{H}$ is the boundary of the hole. On $\partial_o\mathcal{C}$ one can write

$$\tilde{\mathbf{t}} = J_{\Xi}^{-1}\left[(\overset{\Xi}{C}{}^{-1})^{AB}N_AN_B\right]^{-\frac{1}{2}}\mathbf{st}\,. \tag{7.356}$$

Let us assume that in addition to $\Xi|_{\partial_o\mathcal{C}} = id$, the tangent map is the identity map as well, i.e., $T\Xi|_{\partial_o\mathcal{C}} = \overset{\Xi}{\mathbf{F}}|_{\partial_o\mathcal{C}} = id$.[37] For such maps $J_\Xi|_{\partial_o\mathcal{C}} = 1$, and hence

$$\tilde{\mathbf{t}} = \left[\delta^A_{\tilde{A}}\delta^B_{\tilde{B}}\tilde{G}^{\tilde{A}\tilde{B}}N_AN_B\right]^{-\frac{1}{2}}\mathbf{st} \;\;\text{ on }\; \partial_o\mathcal{C}\,. \tag{7.357}$$

Note that $\mathbf{G}$ and $\tilde{\mathbf{G}}$ are induced Euclidean metrics and because Ξ and $T\Xi$ are both identity on $\partial_o\mathcal{C}$, $\delta^A_{\tilde{A}}\delta^B_{\tilde{B}}\tilde{G}^{\tilde{A}\tilde{B}}N_AN_B = G^{AB}N_AN_B = 1$, and hence

$$\tilde{\mathbf{t}} = \mathbf{st} \;\;\text{ on }\; \partial_o\tilde{\mathcal{C}}\,. \tag{7.358}$$

[37] This condition has been ignored in the existing works on elastodynamic transformation cloaking. In particular, borrowing the cloaking transformation of Pendry *et al.* [142] from electromagnetism is not acceptable as it does not satisfy this condition.

Similarly

$$\overset{a}{\tilde{\mathbf{m}}} = \mathbf{s}\overset{a}{\mathbf{m}} \quad \text{on} \;\; \partial_o\tilde{\mathcal{C}}\,, \tag{7.359}$$

that is, $\partial_o\mathcal{C}$ and $\partial_o\tilde{\mathcal{C}}$ have identical traction boundary conditions. Outside the cloak the elastic constants of the two problems are obviously identical from (7.334). One needs to assume that outside the cloak the two bodies have identical mechanical properties. In particular, outside the cloak $\overset{\circ}{\underset{a}{\mathsf{F}}} = \mathbf{0}$, i.e., the director field in the physical body outside the cloak is uniform. This means that on $\mathcal{B}\setminus\mathcal{C}$, $\underset{a}{\mathsf{D}} = \tilde{\underset{a}{\mathsf{D}}}$. This holds on the outer boundary of the cloak as well, i.e., $\overset{\circ}{\tilde{\underset{a}{\mathsf{F}}}} = \mathbf{0}$, and $\underset{a}{\mathsf{D}} = \tilde{\underset{a}{\mathsf{D}}}$ on $\partial_o\mathcal{C}$. Therefore, $\mathcal{B}\setminus\mathcal{C}$ and $\tilde{\mathcal{B}}\setminus\tilde{\mathcal{C}}$ are made of the same generalized Cosserat solid (have identical elastic constants and have the same director fields in their undeformed configurations), and are subject to the same body forces and boundary conditions. It then immediately follows that $\varphi(X,t) = \tilde{\varphi}(\tilde{X},t)$ and $\underset{a}{\mathsf{d}}(X,t) = \tilde{\underset{a}{\mathsf{d}}}(\tilde{X},t)$ on $\mathcal{B}\setminus\mathcal{C}$, and hence, the current configurations of the two bodies are identical outside the cloak. This, in turn, renders cloaking possible as the virtual body is isotropic and homogeneous, and contains an infinitesimal hole, which has a low scattering effect on the incident waves.

The following summarizes the construction of an elastic cloak in a linear elastic generalized Cosserat solid. Consider a diffeomorphism $\Xi : \mathcal{B} \to \tilde{\mathcal{B}}$ that shrinks a hole in the physical body $\mathcal{B}$ to an infinitesimal hole in the virtual body $\tilde{\mathcal{B}}$. The hole is surrounded by a cloak $\mathcal{C}$ in the physical body $\mathcal{B}$. Assume that $\Xi|_{\mathcal{B}\setminus\mathcal{C}} = id$ and that on the outer boundary of the cloak $T\Xi = id$. Assume that the displacement vectors in the physical and virtual bodies, mass densities, body forces, and the elastic constants are related as given in (7.334). Outside the cloak the physical body is homogeneous and isotropic and has a constitutive equation identical to that of the virtual body. One assumes that the two problems have identical boundary conditions on the outer boundaries of $\mathcal{B}$ and $\tilde{\mathcal{B}}$, and the physical hole and the virtual hole are both traction free. Under the above assumptions the two boundary-value problems are equivalent. In other words, the governing equations of the physical problem are satisfied if and only if those of the virtual problem are satisfied. In addition, the two bodies have identical current configurations outside the cloak $\mathcal{C}$.

The impossibility of transformation cloaking in generalized Cosserat elasticity. In this section we prove that in dimension two transformation cloaking is not possible in linear generalized Cosserat elasticity. A corollary of this result is that transformation cloaking cannot be possible in any subclass of generalized Cosserat solids, and in particular, transformation cloaking is not possible in Cosserat elasticity. We start our discussion by first looking at the example of a cylindrical hole covered by a cylindrical cloak. It has been claimed in the literature that a cylindrical cloak would have to be made of a Cosserat solid. We show that this is not possible. In other words, (generalized) Cosserat elasticity allowing for a non-symmetric Cauchy stress does not imply that transformation cloaking can be achieved in (generalized) Cosserat elasticity.

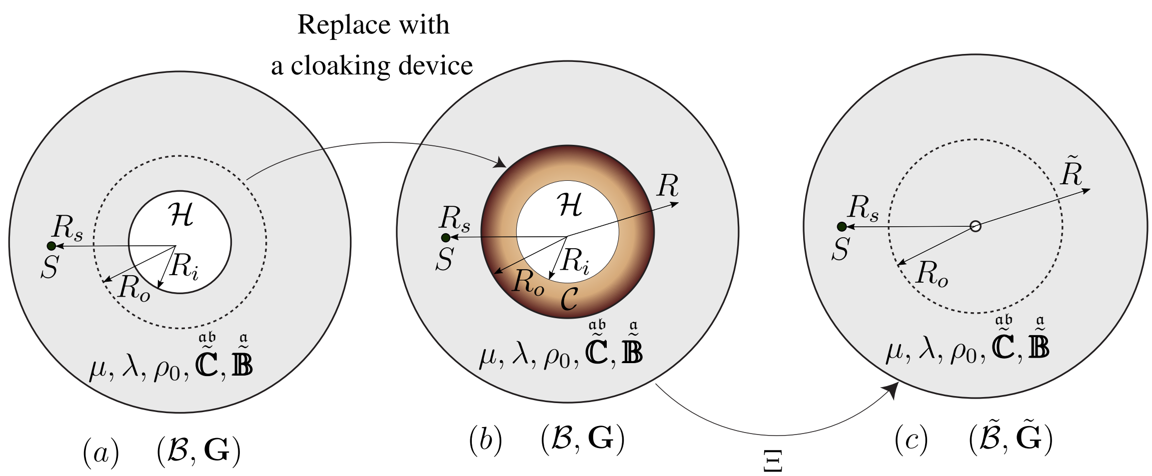


Figure 7.6: *Cloaking an object inside a hole $\mathcal{H}$ from elastic waves generated by a source S located at a distance R_s from the center of the hole. The system (a) is an isotropic, homogeneous, generalized Cosserat medium with elastic properties μ, λ, ρ_0, $\overset{\mathfrak{a}}{\tilde{\mathbb{B}}}$, $\overset{\mathfrak{ab}}{\tilde{\mathbb{C}}}$, and $\overset{\mathfrak{ab}}{\tilde{\nu}}$ containing a finite hole $\mathcal{H}$. The system (c) is isotropic and homogeneous generalized Cosserat solid with the same elastic properties as the medium in the system (a). The cloak $\mathcal{C}$ in (b) has the elastic properties $\mathbb{A}$, $\overset{\mathfrak{a}}{\mathbb{B}}$, $\overset{\mathfrak{ab}}{\mathbb{C}}$, and $\overset{\mathfrak{ab}}{\nu}$. The configuration (b) is mapped to the new reference configuration (c) such that the hole $\mathcal{H}$ is mapped to an infinitesimal hole. The cloaking transformation is the identity mapping in $\mathcal{B} \setminus \mathcal{C}$.*

Example (A generalized Cosserat cylindrical cloak). Consider an infinitely-long hollow solid cylinder that in its stress-free reference configuration has inner and outer radii R_i and R_o, respectively. Let us transform the reference configuration to the reference configuration of another body (virtual body) that is a hollow cylinder with inner and outer radii ϵ and R_o, respectively, using a cloaking map

$\Xi(R, \Theta, Z) = (f(R), \Theta, Z)$ such that $f(R_o) = R_o$. For such a map we have

$$\overset{\Xi}{\mathbf{F}} = \begin{bmatrix} f'(R) & 0 \\ 0 & 1 \end{bmatrix}. \tag{7.360}$$

Note that the shifter map is given as

$$\mathsf{s} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{R}{f(R)} \end{bmatrix}. \tag{7.361}$$

Following our discussion in §7.5.2, we note that Ξ must satisfy $T\Xi|_{\partial_o\mathcal{C}} = \overset{\Xi}{\mathbf{F}}|_{\partial_o\mathcal{C}} = id$, i.e., $f'(R_o) = 1$. Therefore, the simplest form of $f(R)$ is a quadratic polynomial given by

$$f(R) = -\frac{R_o^2(R_i - \epsilon)}{(R_o - R_i)^2} + \frac{R_o^2 + R_i^2 - 2R_o\epsilon}{(R_o - R_i)^2} R - \frac{R_i - \epsilon}{(R_o - R_i)^2} R^2 . \tag{7.362}$$

Note that $\rho = J_\Xi\, \tilde{\rho} \circ \Xi$ and $\overset{\mathfrak{ab}}{\nu} = J_\Xi\, \overset{\mathfrak{ab}}{\tilde{\nu}} \circ \Xi$, and therefore, the mass density and the micro-mass moment of inertia in $\mathcal{B} \setminus \mathcal{C}$ are homogeneous and are equal to ρ_0 and $\overset{\mathfrak{ab}}{\tilde{\nu}}$, respectively. The mass density in the cloaking device is inhomogeneous and is given by

$$\rho_{\mathcal{C}}(R) = \frac{f(R)f'(R)}{R}\rho_0 , \qquad R_i \leq R \leq R_o . \tag{7.363}$$

Hence, (7.363) implies that

$$\rho_{\mathcal{C}}(R) = \frac{[(R_o^2 + R_i^2 - 2R_o\epsilon)R - (R_o^2 + R^2)(R_i - \epsilon)]}{[R_o^2 + R_i^2 - 2R_o\epsilon - 2R(R_i - \epsilon)]^{-1} R(R_o - R_i)^4}\rho_0 , \quad R_i \leq R \leq R_o . \tag{7.364}$$

Similarly ($\mathfrak{a}, \mathfrak{b} \in \{1, 2\}$)

$$\overset{\mathfrak{ab}}{\nu}_{\mathcal{C}}(R) = \frac{[(R_o^2 + R_i^2 - 2R_o\epsilon)R - (R_o^2 + R^2)(R_i - \epsilon)]}{[R_o^2 + R_i^2 - 2R_o\epsilon - 2R(R_i - \epsilon)]^{-1} R(R_o - R_i)^4}\overset{\mathfrak{ab}}{\tilde{\nu}} , \quad R_i \leq R \leq R_o . \tag{7.365}$$

From (7.329), we find the Cosserat elastic constants of the cloak as follows ($\mathfrak{a}, \mathfrak{b} \in \{1, 2\}$)[38]

$$\hat{\mathbb{A}} = \left[\hat{\mathbb{A}}^{aAbB}\right] = \begin{bmatrix} \begin{bmatrix} \frac{(\lambda+2\mu)f(R)}{Rf'(R)} & 0 \\ 0 & \lambda \end{bmatrix} & \begin{bmatrix} 0 & \frac{\mu R f'(R)}{f(R)} \\ \mu & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & \mu \\ \frac{\mu f(R)}{Rf'(R)} & 0 \end{bmatrix} & \begin{bmatrix} \lambda & 0 \\ 0 & \frac{(\lambda+2\mu)Rf'(R)}{f(R)} \end{bmatrix} \end{bmatrix}, \tag{7.366}$$

$$\overset{\mathfrak{a}}{\hat{\mathbb{B}}} = \left[\overset{\mathfrak{a}}{\hat{\mathbb{B}}}{}^{aAbB}\right] = \begin{bmatrix} \begin{bmatrix} \frac{(\overset{\mathfrak{a}}{b}_1+2\overset{\mathfrak{a}}{b}_2)f(R)}{Rf'(R)} & 0 \\ 0 & \overset{\mathfrak{a}}{b}_1 \end{bmatrix} & \begin{bmatrix} 0 & \frac{\overset{\mathfrak{a}}{b}_2 R f'(R)}{f(R)} \\ \overset{\mathfrak{a}}{b}_2 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & \overset{\mathfrak{a}}{b}_2 \\ \frac{\overset{\mathfrak{a}}{b}_2 f(R)}{Rf'(R)} & 0 \end{bmatrix} & \begin{bmatrix} \overset{\mathfrak{a}}{b}_1 & 0 \\ 0 & \frac{(\overset{\mathfrak{a}}{b}_1+2\overset{\mathfrak{a}}{b}_2)Rf'(R)}{f(R)} \end{bmatrix} \end{bmatrix}, \tag{7.367}$$

$$\overset{\mathfrak{ab}}{\hat{\mathbb{C}}} = \left[\overset{\mathfrak{ab}}{\hat{\mathbb{C}}}{}^{aAbB}\right] = \begin{bmatrix} \begin{bmatrix} \frac{(\overset{\mathfrak{ab}}{c}_1+\overset{\mathfrak{ab}}{c}_2+\overset{\mathfrak{ab}}{c}_3)f(R)}{Rf'(R)} & 0 \\ 0 & \overset{\mathfrak{ab}}{c}_1 \end{bmatrix} & \begin{bmatrix} 0 & \frac{\overset{\mathfrak{ab}}{c}_2 R f'(R)}{f(R)} \\ \overset{\mathfrak{ab}}{c}_3 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & \overset{\mathfrak{ab}}{c}_3 \\ \frac{\overset{\mathfrak{ab}}{c}_2 f(R)}{Rf'(R)} & 0 \end{bmatrix} & \begin{bmatrix} \overset{\mathfrak{ab}}{c}_1 & 0 \\ 0 & \frac{(\overset{\mathfrak{ab}}{c}_1+\overset{\mathfrak{ab}}{c}_2+\overset{\mathfrak{ab}}{c}_3)Rf'(R)}{f(R)} \end{bmatrix} \end{bmatrix}. \tag{7.368}$$

Note that the first two indices specify the submatrix and the last two identify the components of that submatrix. We next consider a cylindrical cloak and find the distribution of those (initial) director gradient fields that are compatible with the balance of angular momentum. $(7.337)_1$ gives the following equations for $\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^c{}_A = \underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^c{}_A(R,\Theta)$:

$$Rf(R)\left[(\overset{1}{b}_1+2\overset{1}{b}_2)\underset{1}{\mathring{\mathsf{F}}}{}^{\theta}{}_R + (\overset{2}{b}_1+2\overset{2}{b}_2)\underset{2}{\mathring{\mathsf{F}}}{}^{\theta}{}_R\right] - \left[\overset{1}{b}_1\underset{1}{\mathring{\mathsf{F}}}{}^{r}{}_\Theta + \overset{2}{b}_1\underset{2}{\mathring{\mathsf{F}}}{}^{r}{}_\Theta\right] f'(R) = 0\,. \tag{7.369}$$

$$Rf(R)\left[\overset{1}{b}_1\underset{1}{\mathring{\mathsf{F}}}{}^{\theta}{}_R + \overset{2}{b}_1\underset{2}{\mathring{\mathsf{F}}}{}^{\theta}{}_R\right] - \left[(\overset{1}{b}_1+2\overset{1}{b}_2)\underset{1}{\mathring{\mathsf{F}}}{}^{r}{}_\Theta + (\overset{2}{b}_1+2\overset{2}{b}_2)\underset{2}{\mathring{\mathsf{F}}}{}^{r}{}_\Theta\right] f'(R) = 0\,. \tag{7.370}$$

[38] Note that the physical components of the elasticity tensor $\hat{\mathbb{A}}^{aAbB}$ are related to the components of the elasticity tensor as $\hat{\mathbb{A}}^{aAbB} = \sqrt{g_{aa}}\sqrt{G_{AA}}\sqrt{g_{bb}}\sqrt{G_{BB}}\mathbb{A}^{aAbB}$ (no summation).

$$Rf'(R)\left[\overset{1}{b}_2\underset{1}{\mathring{\mathsf{F}}}^{\theta}{}_{\Theta}+\overset{2}{b}_2\underset{2}{\mathring{\mathsf{F}}}^{\theta}{}_{\Theta}\right]-f(R)\left[\overset{1}{b}_2\underset{1}{\mathring{\mathsf{F}}}^{r}{}_{R}+\overset{2}{b}_2\underset{2}{\mathring{\mathsf{F}}}^{r}{}_{R}\right]=\mu[f(R)-Rf'(R)]\,. \tag{7.371}$$

$(7.337)_2$ gives the following equations

$$Rf(R)\left[(\overset{11}{c}_1+\overset{11}{c}_2+\overset{11}{c}_3)\underset{1}{\mathring{\mathsf{F}}}^{\theta}{}_{R}+(\overset{12}{c}_1+\overset{12}{c}_2+\overset{12}{c}_3)\underset{2}{\mathring{\mathsf{F}}}^{\theta}{}_{R}\right]-\left[\overset{11}{c}_1\underset{1}{\mathring{\mathsf{F}}}^{r}{}_{\Theta}+\overset{12}{c}_1\underset{2}{\mathring{\mathsf{F}}}^{r}{}_{\Theta}\right]f'(R)=0\,. \tag{7.372}$$

$$Rf(R)\left[(\overset{12}{c}_1+\overset{12}{c}_2+\overset{12}{c}_3)\underset{1}{\mathring{\mathsf{F}}}^{\theta}{}_{R}+(\overset{22}{c}_1+\overset{22}{c}_2+\overset{22}{c}_3)\underset{2}{\mathring{\mathsf{F}}}^{\theta}{}_{R}\right]-\left[\overset{12}{c}_1\underset{1}{\mathring{\mathsf{F}}}^{r}{}_{\Theta}+\overset{22}{c}_1\underset{2}{\mathring{\mathsf{F}}}^{r}{}_{\Theta}\right]f'(R)=0\,. \tag{7.373}$$

$$Rf(R)\left[\overset{11}{c}_1\underset{1}{\mathring{\mathsf{F}}}^{\theta}{}_{R}+\overset{12}{c}_1\underset{2}{\mathring{\mathsf{F}}}^{\theta}{}_{R}\right]-f'(R)\left[(\overset{11}{c}_1+\overset{11}{c}_2+\overset{11}{c}_3)\underset{1}{\mathring{\mathsf{F}}}^{r}{}_{\Theta}+(\overset{12}{c}_1+\overset{12}{c}_2+\overset{12}{c}_3)\underset{2}{\mathring{\mathsf{F}}}^{r}{}_{\Theta}\right]=0\,. \tag{7.374}$$

$$Rf(R)\left[\overset{12}{c}_1\underset{1}{\mathring{\mathsf{F}}}^{\theta}{}_{R}+\overset{22}{c}_1\underset{2}{\mathring{\mathsf{F}}}^{\theta}{}_{R}\right]-f'(R)\left[(\overset{12}{c}_1+\overset{12}{c}_2+\overset{12}{c}_3)\underset{1}{\mathring{\mathsf{F}}}^{r}{}_{\Theta}+(\overset{22}{c}_1+\overset{22}{c}_2+\overset{22}{c}_3)\underset{2}{\mathring{\mathsf{F}}}^{r}{}_{\Theta}\right]=0\,. \tag{7.375}$$

$$Rf'(R)\left[\overset{11}{c}_3\underset{1}{\mathring{\mathsf{F}}}^{\theta}{}_{\Theta}+\overset{12}{c}_3\underset{2}{\mathring{\mathsf{F}}}^{\theta}{}_{\Theta}\right]-f(R)\left[\overset{11}{c}_2\underset{1}{\mathring{\mathsf{F}}}^{r}{}_{R}+\overset{12}{c}_2\underset{2}{\mathring{\mathsf{F}}}^{r}{}_{R}\right]=\overset{1}{b}_2[f(R)-Rf'(R)]\,. \tag{7.376}$$

$$Rf'(R)\left[\overset{11}{c}_2\underset{1}{\mathring{\mathsf{F}}}^{\theta}{}_{\Theta}+\overset{12}{c}_2\underset{2}{\mathring{\mathsf{F}}}^{\theta}{}_{\Theta}\right]-f(R)\left[\overset{11}{c}_3\underset{1}{\mathring{\mathsf{F}}}^{r}{}_{R}+\overset{12}{c}_3\underset{2}{\mathring{\mathsf{F}}}^{r}{}_{R}\right]=\overset{1}{b}_2[f(R)-Rf'(R)]\,. \tag{7.377}$$

$$Rf'(R)\left[\overset{12}{c}_3\underset{1}{\mathring{\mathsf{F}}}^{\theta}{}_{\Theta}+\overset{22}{c}_3\underset{2}{\mathring{\mathsf{F}}}^{\theta}{}_{\Theta}\right]-f(R)\left[\overset{12}{c}_2\underset{1}{\mathring{\mathsf{F}}}^{r}{}_{R}+\overset{22}{c}_2\underset{2}{\mathring{\mathsf{F}}}^{r}{}_{R}\right]=\overset{2}{b}_2[f(R)-Rf'(R)]\,. \tag{7.378}$$

$$Rf'(R)\left[\overset{12}{c}_2\underset{1}{\mathring{\mathsf{F}}}^{\theta}{}_{\Theta}+\overset{22}{c}_2\underset{2}{\mathring{\mathsf{F}}}^{\theta}{}_{\Theta}\right]-f(R)\left[\overset{12}{c}_3\underset{1}{\mathring{\mathsf{F}}}^{r}{}_{R}+\overset{22}{c}_3\underset{2}{\mathring{\mathsf{F}}}^{r}{}_{R}\right]=\overset{2}{b}_2[f(R)-Rf'(R)]\,. \tag{7.379}$$

Note that the algebraic equations governing the diagonal and off-diagonal director gradients are uncoupled. We have six equations (7.369)-(7.370), and (7.372)-(7.375) for the four off-diagonal terms and five equations (7.371), and (7.376)-(7.379) for the diagonal terms. We first show that the determinant of the coefficient matrix of the linear system (7.372)-(7.375) is non-zero, and hence,

$\underset{1}{\mathring{\mathsf{F}}}{}^r{}_\Theta = \underset{2}{\mathring{\mathsf{F}}}{}^r{}_\Theta = 0$, and $\underset{1}{\mathring{\mathsf{F}}}{}^\theta{}_R = \underset{2}{\mathring{\mathsf{F}}}{}^\theta{}_R = 0$. Note that the determinant of the coefficient matrix reads

$$R^2 f^2(R) f'^2(R) \left[-(\overset{12}{c}_2 + \overset{12}{c}_3)^2 + (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3)\right]\left[-(2\overset{12}{c}_1 + \overset{12}{c}_2 + \overset{12}{c}_3)^2 + (2\overset{11}{c}_1 + \overset{11}{c}_2 + \overset{11}{c}_3)(2\overset{22}{c}_1 + \overset{22}{c}_2 + \overset{22}{c}_3)\right]. \tag{7.380}$$

It turns out that this determinant cannot vanish. This is a consequence of the positive-definiteness of $\mathbb{C}$. To see this let us assume that $U^a{}_{|A} = 0$, and $\underset{a}{\mathfrak{U}}{}^1{}_{|2} = \underset{a}{\mathfrak{U}}{}^2{}_{|1} = 0$. For this class of deformations $\delta W > 0$ implies positive-definiteness of the following matrix:

$$\begin{bmatrix} \overset{11}{c}_1 + \overset{11}{c}_2 + \overset{11}{c}_3 & \overset{11}{c}_1 & \overset{12}{c}_1 + \overset{12}{c}_2 + \overset{12}{c}_3 & \overset{12}{c}_1 \\ \overset{11}{c}_1 & \overset{11}{c}_1 + \overset{11}{c}_2 + \overset{11}{c}_3 & \overset{12}{c}_1 & \overset{12}{c}_1 + \overset{12}{c}_2 + \overset{12}{c}_3 \\ \overset{12}{c}_1 + \overset{12}{c}_2 + \overset{12}{c}_3 & \overset{12}{c}_1 & \overset{22}{c}_1 + \overset{22}{c}_2 + \overset{22}{c}_3 & \overset{22}{c}_1 \\ \overset{12}{c}_1 & \overset{12}{c}_1 + \overset{12}{c}_2 + \overset{12}{c}_3 & \overset{22}{c}_1 & \overset{22}{c}_1 + \overset{22}{c}_2 + \overset{22}{c}_3 \end{bmatrix}. \tag{7.381}$$

In particular, its determinant must be positive, i.e.,

$$\left[-(\overset{12}{c}_2 + \overset{12}{c}_3)^2 + (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3)\right]\left[-(2\overset{12}{c}_1 + \overset{12}{c}_2 + \overset{12}{c}_3)^2 + (2\overset{11}{c}_1 + \overset{11}{c}_2 + \overset{11}{c}_3)(2\overset{22}{c}_1 + \overset{22}{c}_2 + \overset{22}{c}_3)\right] > 0\,. \tag{7.382}$$

Therefore

$$\underset{1}{\mathring{\mathsf{F}}}{}^r{}_\Theta = \underset{2}{\mathring{\mathsf{F}}}{}^r{}_\Theta = 0\,, \qquad \underset{1}{\mathring{\mathsf{F}}}{}^\theta{}_R = \underset{2}{\mathring{\mathsf{F}}}{}^\theta{}_R = 0\,. \tag{7.383}$$

Note that (7.369) and (7.370) are now trivially satisfied.

The determinant of the coefficient matrix of the linear system (7.376)-(7.379) reads

$$-R^2 f^2(R) f'^2(R) \left[(\overset{12}{c}_2 - \overset{12}{c}_3)^2 - (\overset{11}{c}_2 - \overset{11}{c}_3)(\overset{22}{c}_2 - \overset{22}{c}_3)\right]\left[(\overset{12}{c}_2 + \overset{12}{c}_3)^2 - (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3)\right]\,. \tag{7.384}$$

Positive-definiteness of energy requires that this determinant be non-vanishing. To see this let us assume that $U^a{}_{|A} = 0$, and $\underset{a}{\mathfrak{U}}{}^1{}_{|1} = \underset{a}{\mathfrak{U}}{}^2{}_{|2} = 0$. For this class of deformations $\delta W > 0$ implies positive-definiteness of the following matrix.

$$\begin{bmatrix} \overset{11}{c}_2 & \overset{11}{c}_3 & \overset{12}{c}_2 & \overset{12}{c}_3 \\ \overset{11}{c}_3 & \overset{11}{c}_2 & \overset{12}{c}_3 & \overset{12}{c}_2 \\ \overset{12}{c}_2 & \overset{12}{c}_3 & \overset{22}{c}_2 & \overset{22}{c}_3 \\ \overset{12}{c}_3 & \overset{12}{c}_2 & \overset{22}{c}_3 & \overset{22}{c}_2 \end{bmatrix} . \tag{7.385}$$

In particular, its determinant must be positive, and hence

$$\left[(\overset{12}{c}_2 - \overset{12}{c}_3)^2 - (\overset{11}{c}_2 - \overset{11}{c}_3)(\overset{22}{c}_2 - \overset{22}{c}_3)\right] \left[(\overset{12}{c}_2 + \overset{12}{c}_3)^2 - (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3)\right] > 0 \,. \tag{7.386}$$

Therefore

$$\underset{1}{\mathring{\mathsf{F}}}{}^r{}_R = -\frac{\left[\overset{2}{b}_2(\overset{12}{c}_2 + \overset{12}{c}_3) - \overset{1}{b}_2(\overset{22}{c}_2 + \overset{22}{c}_3)\right](f(R) - Rf'(R))}{f(R)\left[(\overset{12}{c}_2 + \overset{12}{c}_3)^2 - (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3)\right]} \,, \tag{7.387}$$

$$\underset{1}{\mathring{\mathsf{F}}}{}^\theta{}_\Theta = \frac{\left[\overset{2}{b}_2(\overset{12}{c}_2 + \overset{12}{c}_3) - \overset{1}{b}_2(\overset{22}{c}_2 + \overset{22}{c}_3)\right](f(R) - Rf'(R))}{Rf'(R)\left[(\overset{12}{c}_2 + \overset{12}{c}_3)^2 - (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3)\right]} \,, \tag{7.388}$$

$$\underset{2}{\mathring{\mathsf{F}}}{}^r{}_R = \frac{\left[\overset{2}{b}_2(\overset{11}{c}_2 + \overset{11}{c}_3) - \overset{1}{b}_2(\overset{12}{c}_2 + \overset{12}{c}_3)\right](f(R) - Rf'(R))}{f(R)\left[(\overset{12}{c}_2 + \overset{12}{c}_3)^2 - (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3)\right]} \,, \tag{7.389}$$

$$\underset{2}{\mathring{\mathsf{F}}}{}^\theta{}_\Theta = \frac{\left[\overset{2}{b}_2(\overset{11}{c}_2 + \overset{11}{c}_3) - \overset{1}{b}_2(\overset{12}{c}_2 + \overset{12}{c}_3)\right](Rf'(R) - f(R))}{Rf'(R)\left[(\overset{12}{c}_2 + \overset{12}{c}_3)^2 - (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3)\right]} \,. \tag{7.390}$$

Note that

$$\underset{1}{\mathring{\mathsf{F}}}{}^r{}_R = -\frac{Rf'(R)}{f(R)}\underset{1}{\mathring{\mathsf{F}}}{}^\theta{}_\Theta, \quad \underset{2}{\mathring{\mathsf{F}}}{}^r{}_R = -\frac{Rf'(R)}{f(R)}\underset{2}{\mathring{\mathsf{F}}}{}^\theta{}_\Theta \,. \tag{7.391}$$

From (7.383), it is immediate to see that (7.369) and (7.370) are automatically satisfied, while (7.371) imposes the following constraint on the elastic constants of the virtual body:

$$2(\overset{1}{b}_2)^2(\overset{22}{c}_2 + \overset{22}{c}_3) - 4\overset{2}{b}_2\overset{1}{b}_2(\overset{12}{c}_2 + \overset{12}{c}_3) + 2(\overset{2}{b}_2)^2(\overset{11}{c}_2 + \overset{11}{c}_3) + \mu\left[(\overset{12}{c}_2 + \overset{12}{c}_3)^2 - (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3)\right] = 0 \,. \tag{7.392}$$

Let us assume that $U^a{}_{|A} = \delta^a_1\delta^2_A$, and $\underset{a}{\mathfrak{U}}{}^1{}_{|1} = \underset{a}{\mathfrak{U}}{}^2{}_{|2} = 0$. For this class of deformations $\delta W > 0$ implies positive-definiteness of the following matrix.

$$
\begin{bmatrix}
\mu & \overset{1}{b}_2 & \overset{1}{b}_2 & \overset{2}{b}_2 & \overset{2}{b}_2 \\
\overset{1}{b}_2 & \overset{11}{c}_2 & \overset{11}{c}_3 & \overset{12}{c}_2 & \overset{12}{c}_3 \\
\overset{1}{b}_2 & \overset{11}{c}_3 & \overset{11}{c}_2 & \overset{12}{c}_3 & \overset{12}{c}_2 \\
\overset{2}{b}_2 & \overset{12}{c}_2 & \overset{12}{c}_3 & \overset{22}{c}_2 & \overset{22}{c}_3 \\
\overset{2}{b}_2 & \overset{12}{c}_3 & \overset{12}{c}_2 & \overset{22}{c}_3 & \overset{22}{c}_2
\end{bmatrix} . \tag{7.393}
$$

Therefore, its determinant must be positive, and hence

$$
\begin{aligned}
&\left[(\overset{12}{c}_2 - \overset{12}{c}_3)^2 - (\overset{11}{c}_2 - \overset{11}{c}_3)(\overset{22}{c}_2 - \overset{22}{c}_3)\right] \Big\{ \mu \left[(\overset{12}{c}_2 + \overset{12}{c}_3)^2 - (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3)\right] \\
&\quad - 2\left[2\overset{1}{b}_2\overset{2}{b}_2(\overset{12}{c}_2 + \overset{12}{c}_3) - (\overset{2}{b}_2)^2\,(\overset{11}{c}_2 + \overset{11}{c}_3) - (\overset{1}{b}_2)^2\,(\overset{22}{c}_2 + \overset{22}{c}_3)\right] \Big\} > 0\,.
\end{aligned} \tag{7.394}
$$

In particular

$$
\mu\left[(\overset{12}{c}_2 + \overset{12}{c}_3)^2 - (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3)\right] \; - 2\left[2\overset{1}{b}_2\overset{2}{b}_2(\overset{12}{c}_2 + \overset{12}{c}_3) - (\overset{2}{b}_2)^2\,(\overset{11}{c}_2 + \overset{11}{c}_3) - (\overset{1}{b}_2)^2\,(\overset{22}{c}_2 + \overset{22}{c}_3)\right] \neq 0. \tag{7.395}
$$

Therefore, (7.392) violates positive-definiteness of the energy, and hence cloaking is not possible.

Remark 7.5.6. Even if one accepts a positive-semidefinite energy, the director gradient field given by (7.387) - (7.390) is not compatible. In other words, the director gradient given by (7.387) - (7.390) does not correspond to a single-valued director field in the physical body, in general. Necessary compatibility equations for the director gradients are written as

$$
\begin{aligned}
&\underset{1}{\overset{\circ}{\mathsf{F}}}{}^r{}_{\Theta|R} = \underset{1}{\overset{\circ}{\mathsf{F}}}{}^r{}_{R|\Theta} = 0, \quad \underset{1}{\overset{\circ}{\mathsf{F}}}{}^\theta{}_{\Theta|R} = \underset{1}{\overset{\circ}{\mathsf{F}}}{}^\theta{}_{R|\Theta} = 0 \\
&\underset{2}{\overset{\circ}{\mathsf{F}}}{}^r{}_{\Theta|R} = \underset{2}{\overset{\circ}{\mathsf{F}}}{}^r{}_{R|\Theta} = 0, \quad \underset{2}{\overset{\circ}{\mathsf{F}}}{}^\theta{}_{\Theta|R} = \underset{2}{\overset{\circ}{\mathsf{F}}}{}^\theta{}_{R|\Theta} = 0\,.
\end{aligned} \tag{7.396}
$$

Note that $\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^a{}_{A|B} = \partial \underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^a{}_A/\partial X^B - \Gamma^C{}_{AB}\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^a{}_C + \gamma^a{}_{bc}\mathring{F}^b{}_B\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^c{}_A$. Thus

$$\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^r{}_{R|\Theta} = \underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^r{}_{R,\Theta} - R\,\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^\theta{}_R - \frac{1}{R}\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^r{}_\Theta\,, \tag{7.397}$$

$$\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^r{}_{\Theta|R} = \underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^r{}_{\Theta,R} - \frac{1}{R}\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^r{}_\Theta\,, \tag{7.398}$$

$$\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^\theta{}_{\Theta|R} = \underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^\theta{}_{\Theta,R} + \frac{1}{R}\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^\theta{}_\Theta - \frac{1}{R}\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^\theta{}_\Theta = \underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^\theta{}_{\Theta,R}\,, \tag{7.399}$$

$$\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^\theta{}_{R|\Theta} = \underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^\theta{}_{R,\Theta} + \frac{1}{R}\left[\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^r{}_R - \underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^\theta{}_\Theta\right]. \tag{7.400}$$

Hence, $\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^r{}_{R|\Theta} = \underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^r{}_{\Theta|R}$, and $\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^\theta{}_{\Theta|R} = \underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^\theta{}_{R|\Theta}$ imply that

$$\begin{aligned} &\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^r{}_{R,\Theta} - \underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^r{}_{\Theta,R} = R\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^\theta{}_R\,,\\ &\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^\theta{}_{\Theta,R} - \underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^\theta{}_{R,\Theta} = \frac{1}{R}\left[\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^r{}_R - \underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^\theta{}_\Theta\right]. \end{aligned} \tag{7.401}$$

Now using (7.383), one obtains

$$\begin{aligned} &\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^r{}_{R,\Theta} = 0\,,\\ &(R\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^\theta{}_\Theta)_{,R} = \underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^r{}_R\,. \end{aligned} \tag{7.402}$$

Note that in this example $(7.402)_1$ is trivially satisfied. From (7.391) and $(7.402)_2$ one obtains

$$\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^\theta{}_\Theta = \frac{C_{\mathfrak{a}}}{Rf(R)}\,, \tag{7.403}$$

for constants $C_{\mathfrak{a}}$. If $C_{\mathfrak{a}} \neq 0$, (7.403) gives the following ODEs for $f(R)$.

$$Rf(R)f'(R) + k_{\mathfrak{a}}f'(R) - f^2(R) = 0\,, \tag{7.404}$$

where $k_{\mathfrak{a}}$ are constant. Knowing that $f(R_o) = R_o$, and $f'(R_o) = 1$, one concludes that $k_{\mathfrak{a}} = 0$, and thus, $f(R) = R$. If $C_{\mathfrak{a}} = 0$, then either $f(R) = R$, or

$$\overset{2}{b}_2(\overset{12}{c}_2 + \overset{12}{c}_3) - \overset{1}{b}_2(\overset{22}{c}_2 + \overset{22}{c}_3) = 0, \quad \overset{2}{b}_2(\overset{11}{c}_2 + \overset{11}{c}_3) - \overset{1}{b}_2(\overset{12}{c}_2 + \overset{12}{c}_3) = 0. \tag{7.405}$$

Knowing that $\overset{11}{c}_2 + \overset{11}{c}_3 > 0$, and $\overset{22}{c}_2 + \overset{22}{c}_3 > 0$, the above relations are written as

$$\overset{1}{b}_2 = \frac{\overset{12}{c}_2 + \overset{12}{c}_3}{\overset{22}{c}_2 + \overset{22}{c}_3}\overset{2}{b}_2, \quad \overset{2}{b}_2 = \frac{\overset{12}{c}_2 + \overset{12}{c}_3}{\overset{11}{c}_2 + \overset{11}{c}_3}\overset{1}{b}_2. \tag{7.406}$$

Thus

$$\overset{1}{b}_2 \frac{(\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3) - (\overset{12}{c}_2 + \overset{12}{c}_3)^2}{(\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3)} = 0, \quad \overset{2}{b}_2 \frac{(\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3) - (\overset{12}{c}_2 + \overset{12}{c}_3)^2}{(\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3)} = 0. \tag{7.407}$$

From (7.382), one concludes that $\overset{1}{b}_2 = \overset{2}{b}_2 = 0$. Substituting this into (7.371), one concludes that $f(R) = R$. Therefore, transformation cloaking is not possible in this example for any linear generalized Cosserat solid.

We next prove that the impossibility of transformation cloaking in dimension two for linear generalized Cosserat elasticity is independent of the shape of the hole (cavity). We write (7.337) in Cartesian coordinates for which the shifter is written as $\mathsf{s}^{\tilde a}{}_a = \delta^{\tilde a}{}_a$. Knowing that $\mathring{F}^c{}_A = \delta^c{}_A$, (7.337) is simplified to read $\forall\, b, B, c, a \in \{1,2,3\}$

$$\begin{aligned}
&(\overset{\equiv}{F}{}^{-1})^{[c}{}_{\tilde A}(\overset{\equiv}{F}{}^{-1})^{B}{}_{\tilde B}\tilde{\mathbb{A}}^{a]\tilde A b \tilde B} + (\overset{\equiv}{F}{}^{-1})^{A}{}_{\tilde A}(\overset{\equiv}{F}{}^{-1})^{B}{}_{\tilde B}\underset{\mathfrak{a}}{\mathring{\mathsf{F}}}{}^{[c}{}_{A}\,\overset{\mathfrak{a}}{\tilde{\mathbb{B}}}{}^{a]\tilde A b \tilde B} = 0\,,\\
&(\overset{\equiv}{F}{}^{-1})^{[c}{}_{\tilde A}(\overset{\equiv}{F}{}^{-1})^{B}{}_{\tilde B}\overset{\mathfrak{a}}{\tilde{\mathbb{B}}}{}^{a]\tilde A b \tilde B} + (\overset{\equiv}{F}{}^{-1})^{A}{}_{\tilde A}(\overset{\equiv}{F}{}^{-1})^{B}{}_{\tilde B}\underset{\mathfrak{b}}{\mathring{\mathsf{F}}}{}^{[c}{}_{A}\overset{\mathfrak{ba}}{\tilde{\mathbb{C}}}{}^{a]\tilde A b \tilde B} = 0\,.
\end{aligned} \tag{7.408}$$

First let us consider an arbitrary cloaking map and director gradient with the following components in 2D

$$\overset{\equiv}{\mathbf{F}}{}^{-1} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}, \qquad \underset{\mathfrak{a}}{\mathring{\mathsf{F}}} = \begin{bmatrix} \underset{\mathfrak{a}}{\mathsf{F}}{}^{11} & \underset{\mathfrak{a}}{\mathsf{F}}{}^{12} \\ \underset{\mathfrak{a}}{\mathsf{F}}{}^{21} & \underset{\mathfrak{a}}{\mathsf{F}}{}^{22} \end{bmatrix}. \tag{7.409}$$

Note that $\overset{\equiv}{\mathbf{F}}{}^{-1}$ is compatible, and hence, $F_{11,2} = F_{12,1}$, and $F_{22,1} = F_{21,2}$. Similarly, the compatibility equations for the director gradient are $\underset{\mathfrak{a}}{\mathsf{F}}{}^{11}{}_{,2} = \underset{\mathfrak{a}}{\mathsf{F}}{}^{12}{}_{,1}$, and $\underset{\mathfrak{a}}{\mathsf{F}}{}^{22}{}_{,1} = \underset{\mathfrak{a}}{\mathsf{F}}{}^{21}{}_{,2}$. Expanding $(7.408)_1$, for

$i \neq j \in \{1, 2\}$ one obtains

$$\begin{aligned}
&-F_{ii}F_{ij}\underset{1}{\mathsf{F}}^{ii}(\overset{1}{b}_1+\overset{1}{b}_2)+\underset{1}{\mathsf{F}}^{jj}\left[F_{ii}F_{ji}(\overset{1}{b}_1+2\overset{1}{b}_2)+F_{ij}F_{jj}\overset{1}{b}_2\right]-F_{ii}F_{ij}\underset{2}{\mathsf{F}}^{ii}(\overset{2}{b}_1+\overset{2}{b}_2)\\
&\quad+\underset{2}{\mathsf{F}}^{jj}\left[F_{ii}F_{ji}(\overset{2}{b}_1+2\overset{2}{b}_2)+F_{ij}F_{jj}\overset{2}{b}_2\right]-\underset{1}{\mathsf{F}}^{ij}(F_{ii}F_{jj}\overset{1}{b}_1+F_{ij}F_{ji}\overset{1}{b}_2)+\underset{1}{\mathsf{F}}^{ji}\left[F_{ii}^2(\overset{1}{b}_1+2\overset{1}{b}_2)+F_{ij}^2\overset{1}{b}_2\right]\\
&\quad-\underset{2}{\mathsf{F}}^{ij}(F_{ii}F_{jj}\overset{2}{b}_1+F_{ij}F_{ji}\overset{2}{b}_2)+\underset{2}{\mathsf{F}}^{ji}\left[F_{ii}^2(\overset{2}{b}_1+2\overset{2}{b}_2)+F_{ij}^2\overset{2}{b}_2\right]\\
&\quad=F_{ii}(F_{ij}-F_{ji})\lambda-(F_{ij}F_{jj}-F_{ii}F_{ij}+2F_{ii}F_{ji})\mu\,,
\end{aligned}\tag{7.410}$$

$$\begin{aligned}
&-\underset{1}{\mathsf{F}}^{ii}(F_{ii}F_{jj}\overset{1}{b}_2+F_{ij}F_{ji}\overset{1}{b}_1)+\underset{1}{\mathsf{F}}^{jj}\left[F_{ji}^2(\overset{1}{b}_1+2\overset{1}{b}_2)+F_{jj}^2\overset{1}{b}_2\right]-\underset{2}{\mathsf{F}}^{ii}(F_{ii}F_{jj}\overset{2}{b}_2+F_{ij}F_{ji}\overset{2}{b}_1)\\
&\quad+\underset{2}{\mathsf{F}}^{jj}\left[F_{ji}^2(\overset{2}{b}_1+2\overset{2}{b}_2)+F_{jj}^2\overset{2}{b}_2\right]-F_{ji}F_{jj}\underset{1}{\mathsf{F}}^{ij}(\overset{1}{b}_1+\overset{1}{b}_2)+\underset{1}{\mathsf{F}}^{ji}\left[F_{ii}F_{ji}(\overset{1}{b}_1+2\overset{1}{b}_2)+F_{ij}F_{jj}\overset{1}{b}_2\right]\\
&\quad-F_{ji}F_{jj}\underset{2}{\mathsf{F}}^{ij}(\overset{2}{b}_1+\overset{2}{b}_2)+\underset{2}{\mathsf{F}}^{ji}\left[F_{ii}F_{ji}(\overset{2}{b}_1+2\overset{2}{b}_2)+F_{ij}F_{jj}\overset{2}{b}_2\right]\\
&\quad=F_{ji}(F_{ij}-F_{ji})\lambda-(2F_{ji}^2+F_{jj}^2-F_{ii}F_{jj})\mu\,,
\end{aligned}\tag{7.411}$$

On the other hand, expanding $(7.408)_2$ one obtains

$$\begin{aligned}
&-F_{11}F_{12}\underset{1}{\mathsf{F}}^{11}(\overset{11}{c}_1+\overset{11}{c}_3)+\underset{1}{\mathsf{F}}^{22}\left[F_{11}F_{21}\,\overset{11}{c}_\Sigma+F_{12}F_{22}\overset{11}{c}_2\right]-F_{11}F_{12}\underset{2}{\mathsf{F}}^{11}(\overset{12}{c}_1+\overset{12}{c}_3)+\underset{2}{\mathsf{F}}^{22}\left[F_{11}F_{21}\,\overset{12}{c}_\Sigma+F_{12}F_{22}\overset{12}{c}_2\right]\\
&\quad-\underset{1}{\mathsf{F}}^{12}(F_{11}F_{22}\overset{11}{c}_1+F_{12}F_{21}\overset{11}{c}_3)+\underset{1}{\mathsf{F}}^{21}\left[F_{11}^2\,\overset{11}{c}_\Sigma+F_{12}^2\overset{11}{c}_2\right]-\underset{2}{\mathsf{F}}^{12}(F_{11}F_{22}\overset{12}{c}_1+F_{12}F_{21}\overset{12}{c}_3)\\
&\quad+\underset{2}{\mathsf{F}}^{21}\left[F_{11}^2\,\overset{12}{c}_\Sigma+F_{12}^2\overset{12}{c}_2\right]=F_{11}(F_{12}-F_{21})\overset{1}{b}_1+(F_{11}F_{12}-2F_{11}F_{21}-F_{12}F_{22})\overset{1}{b}_2\,,
\end{aligned}\tag{7.412}$$

$$\begin{aligned}
&-\underset{1}{\mathsf{F}}^{11}\left[F_{11}^2\,\overset{11}{c}_2+F_{12}^2\,\overset{11}{c}_\Sigma\right]+\underset{1}{\mathsf{F}}^{22}(F_{11}F_{22}\overset{11}{c}_3+F_{12}F_{21}\overset{11}{c}_1)-\underset{2}{\mathsf{F}}^{11}\left[F_{11}^2\,\overset{12}{c}_2+F_{12}^2\,\overset{12}{c}_\Sigma\right]+\underset{2}{\mathsf{F}}^{22}(F_{11}F_{22}\overset{12}{c}_3+F_{12}F_{21}\overset{12}{c}_1)\\
&\quad-\underset{1}{\mathsf{F}}^{12}[F_{11}F_{21}\overset{11}{c}_2+F_{12}F_{22}\,\overset{11}{c}_\Sigma]+F_{11}F_{12}\underset{1}{\mathsf{F}}^{21}(\overset{11}{c}_1+\overset{11}{c}_3)-\underset{2}{\mathsf{F}}^{12}[F_{11}F_{21}\overset{12}{c}_2+F_{12}F_{22}\,\overset{12}{c}_\Sigma]\\
&\quad+F_{11}F_{12}\underset{2}{\mathsf{F}}^{21}(\overset{12}{c}_1+\overset{12}{c}_3)=F_{12}(F_{12}-F_{21})\overset{1}{b}_1+(F_{11}^2+2F_{12}^2-F_{11}F_{22})\overset{1}{b}_2\,,
\end{aligned}\tag{7.413}$$

$$
\begin{aligned}
&-\underset{1}{\mathsf{F}}^{11}[F_{11}F_{22}\overset{11}{c}_3 + F_{12}F_{21}\overset{11}{c}_1] + \underset{1}{\mathsf{F}}^{22}\left[F_{21}^2\,\overset{11}{c}_\Sigma + F_{22}^2\overset{11}{c}_2\right] - \underset{2}{\mathsf{F}}^{11}[F_{11}F_{22}\overset{12}{c}_3 + F_{12}F_{21}\overset{12}{c}_1] + \underset{2}{\mathsf{F}}^{22}\left(F_{21}^2\overset{12}{c}_\Sigma + F_{22}^2\overset{12}{c}_2\right)\\
&\qquad - F_{21}F_{22}\underset{1}{\mathsf{F}}^{12}(\overset{11}{c}_1 + \overset{11}{c}_3) + \underset{1}{\mathsf{F}}^{21}[F_{11}F_{21}\overset{11}{c}_\Sigma + F_{12}F_{22}\overset{11}{c}_2] - F_{21}F_{22}\underset{2}{\mathsf{F}}^{12}(\overset{12}{c}_1 + \overset{12}{c}_3)\\
&\qquad + \underset{2}{\mathsf{F}}^{21}[F_{11}F_{21}\overset{12}{c}_\Sigma + F_{12}F_{22}\overset{12}{c}_2] = F_{21}(F_{12} - F_{21})\overset{1}{b}_1 + (F_{11}F_{22} - 2F_{21}^2 - F_{22}^2)\overset{1}{b}_2\,,
\end{aligned}
\tag{7.414}
$$

$$
\begin{aligned}
&-\underset{1}{\mathsf{F}}^{11}[F_{11}F_{21}\overset{11}{c}_2 + F_{12}F_{22}\,\overset{11}{c}_\Sigma] + F_{21}F_{22}\underset{1}{\mathsf{F}}^{22}(\overset{11}{c}_1 + \overset{11}{c}_3) - \underset{2}{\mathsf{F}}^{11}[F_{11}F_{21}\overset{12}{c}_2 + F_{12}F_{22}\,\overset{12}{c}_\Sigma] + F_{21}F_{22}\underset{2}{\mathsf{F}}^{22}(\overset{12}{c}_1 + \overset{12}{c}_3)\\
&\qquad - \underset{1}{\mathsf{F}}^{12}\left(F_{21}^2\overset{11}{c}_2 + F_{22}^2\,\overset{11}{c}_\Sigma\right) + \underset{1}{\mathsf{F}}^{21}(F_{11}F_{22}\overset{11}{c}_1 + F_{12}F_{21}\overset{11}{c}_3) - \underset{2}{\mathsf{F}}^{12}\left(F_{21}^2\overset{12}{c}_2 + F_{22}^2\,\overset{12}{c}_\Sigma\right)\\
&\qquad + \underset{2}{\mathsf{F}}^{21}(F_{11}F_{22}\overset{12}{c}_1 + F_{12}F_{21}\overset{12}{c}_3) = F_{22}(F_{12} - F_{21})\overset{1}{b}_1 + (F_{11}F_{21} - F_{21}F_{22} + 2F_{12}F_{22})\overset{1}{b}_2\,,
\end{aligned}
\tag{7.415}
$$

where $\overset{\mathfrak{ab}}{c}_\Sigma = \overset{\mathfrak{ab}}{c}_1 + \overset{\mathfrak{ab}}{c}_2 + \overset{\mathfrak{ab}}{c}_3$. Another independent set of four equations are generated from (7.412)-(7.415) using the transformation $\{\,\overset{12}{c} \to \overset{22}{c},\ \overset{11}{c} \to \overset{12}{c},\ \overset{1}{b} \to \overset{2}{b}\,\}$. The determinant of the coefficient matrix reads

$$
\begin{aligned}
(F_{12}F_{21} - F_{11}F_{22})^8 \left[(\overset{12}{c}_2 + \overset{12}{c}_3)^2 - (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3)\right]^2 &\left[(\overset{12}{c}_2 - \overset{12}{c}_3)^2 - (\overset{11}{c}_2 - \overset{11}{c}_3)(\overset{22}{c}_2 - \overset{22}{c}_3)\right]\\
&\left[(\overset{12}{c}_1 + \overset{12}{c}_\Sigma)^2 - (\overset{11}{c}_1 + \overset{11}{c}_\Sigma)(\overset{22}{c}_1 + \overset{22}{c}_\Sigma)\right]\ .
\end{aligned}
\tag{7.416}
$$

Note that from the positive-definiteness of energy, this determinant is non-zero, and hence the system

(7.412)-(7.415) (along with four more equations) has a unique solution, which reads

$$\underset{1}{\mathsf{F}}^{11} = \frac{F_{22}(F_{11}-F_{22})\overset{1}{\mathsf{t}}_3 - F_{12}F_{21}\overset{1}{\mathsf{t}}_2 + F_{21}^2\overset{1}{\mathsf{t}}_1}{F_{11}F_{22}-F_{12}F_{21}}\,, \quad \underset{1}{\mathsf{F}}^{22} = \frac{F_{11}(F_{22}-F_{11})\overset{1}{\mathsf{t}}_3 - F_{12}F_{21}\overset{1}{\mathsf{t}}_2 + F_{12}^2\overset{1}{\mathsf{t}}_1}{F_{11}F_{22}-F_{12}F_{21}}\,, \tag{7.417}$$

$$\underset{1}{\mathsf{F}}^{12} = \frac{F_{12}F_{22}\overset{1}{\mathsf{t}}_3 - F_{11}F_{21}\overset{1}{\mathsf{t}}_1 + F_{11}F_{12}\overset{1}{\mathsf{t}}_4}{F_{11}F_{22}-F_{12}F_{21}}\,, \qquad \underset{1}{\mathsf{F}}^{21} = \frac{F_{21}F_{11}\overset{1}{\mathsf{t}}_3 - F_{12}F_{22}\overset{1}{\mathsf{t}}_1 + F_{21}F_{22}\overset{1}{\mathsf{t}}_4}{F_{11}F_{22}-F_{12}F_{21}}\,, \tag{7.418}$$

$$\underset{2}{\mathsf{F}}^{11} = \frac{F_{22}(F_{11}-F_{22})\overset{2}{\mathsf{t}}_3 - F_{12}F_{21}\overset{2}{\mathsf{t}}_2 + F_{21}^2\overset{2}{\mathsf{t}}_1}{F_{11}F_{22}-F_{12}F_{21}}\,, \quad \underset{2}{\mathsf{F}}^{22} = \frac{F_{11}(F_{22}-F_{11})\overset{2}{\mathsf{t}}_3 - F_{12}F_{21}\overset{2}{\mathsf{t}}_2 + F_{12}^2\overset{2}{\mathsf{t}}_1}{F_{11}F_{22}-F_{12}F_{21}}\,, \tag{7.419}$$

$$\underset{2}{\mathsf{F}}^{12} = \frac{F_{12}F_{22}\overset{2}{\mathsf{t}}_3 - F_{11}F_{21}\overset{2}{\mathsf{t}}_1 + F_{11}F_{12}\overset{2}{\mathsf{t}}_4}{F_{11}F_{22}-F_{12}F_{21}}\,, \qquad \underset{2}{\mathsf{F}}^{21} = \frac{F_{21}F_{11}\overset{2}{\mathsf{t}}_3 - F_{12}F_{22}\overset{2}{\mathsf{t}}_1 + F_{21}F_{22}\overset{2}{\mathsf{t}}_4}{F_{11}F_{22}-F_{12}F_{21}}\,, \tag{7.420}$$

where $\overset{a}{\mathsf{t}}_i$, $i = 1, \cdots, 4$, $\mathfrak{a} = 1, 2$ are constants given as

$$\overset{1}{\mathsf{t}}_1 = \frac{(\overset{1}{b}_1 + \overset{1}{b}_2)(\overset{22}{c}_1 + \overset{22}{c}_\Sigma) - (\overset{2}{b}_1 + \overset{2}{b}_2)(\overset{12}{c}_1 + \overset{12}{c}_\Sigma)}{(\overset{12}{c}_1 + \overset{12}{c}_\Sigma)^2 - (\overset{11}{c}_1 + \overset{11}{c}_\Sigma)(\overset{22}{c}_1 + \overset{22}{c}_\Sigma)} + \frac{\overset{2}{b}_2(\overset{12}{c}_2 + \overset{12}{c}_3) - \overset{1}{b}_2(\overset{22}{c}_2 + \overset{22}{c}_3)}{(\overset{12}{c}_2 + \overset{12}{c}_3)^2 - (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3)}\,, \tag{7.421}$$

$$\overset{2}{\mathsf{t}}_1 = \frac{(\overset{2}{b}_1 + \overset{2}{b}_2)(\overset{11}{c}_1 + \overset{11}{c}_\Sigma) - (\overset{1}{b}_1 + \overset{1}{b}_2)(\overset{12}{c}_1 + \overset{12}{c}_\Sigma)}{(\overset{12}{c}_1 + \overset{12}{c}_\Sigma)^2 - (\overset{11}{c}_1 + \overset{11}{c}_\Sigma)(\overset{22}{c}_1 + \overset{22}{c}_\Sigma)} + \frac{\overset{1}{b}_2(\overset{12}{c}_2 + \overset{12}{c}_3) - \overset{2}{b}_2(\overset{11}{c}_2 + \overset{11}{c}_3)}{(\overset{12}{c}_2 + \overset{12}{c}_3)^2 - (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3)}\,, \tag{7.422}$$

$$\begin{aligned}
\overset{1}{\mathsf{t}}_2 =& \frac{\overset{1}{b}_1(\overset{22}{c}_1 + \overset{22}{c}_\Sigma) - \overset{2}{b}_1(\overset{12}{c}_1 + \overset{12}{c}_\Sigma)}{(\overset{12}{c}_1 + \overset{12}{c}_\Sigma)^2 - (\overset{11}{c}_1 + \overset{11}{c}_\Sigma)(\overset{22}{c}_1 + \overset{22}{c}_\Sigma)} \\
&- 2\overset{2}{b}_2 \frac{\overset{12}{c}_\Sigma(\overset{12}{c}_1 + \overset{12}{c}_\Sigma)(\overset{12}{c}_2 + \overset{12}{c}_3) - (\overset{22}{c}_1\overset{11}{c}_\Sigma + \overset{11}{c}_1\overset{22}{c}_\Sigma)(\overset{12}{c}_2 + \overset{12}{c}_3) - \overset{12}{c}_\Sigma(\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3)}{[(\overset{12}{c}_2 + \overset{12}{c}_3)^2 - (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3)]\,[(\overset{12}{c}_1 + \overset{12}{c}_\Sigma)^2 - (\overset{11}{c}_1 + \overset{11}{c}_\Sigma)(\overset{22}{c}_1 + \overset{22}{c}_\Sigma)]} \\
&+ 2\overset{1}{b}_2 \frac{[2\,\overset{12}{c}_1\overset{12}{c}_\Sigma - \overset{11}{c}_\Sigma(\overset{22}{c}_1 + \overset{22}{c}_\Sigma)](\overset{22}{c}_2 + \overset{22}{c}_3) + \overset{22}{c}_\Sigma(\overset{12}{c}_2 + \overset{12}{c}_3)^2}{[(\overset{12}{c}_2 + \overset{12}{c}_3)^2 - (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3)]\,[(\overset{12}{c}_1 + \overset{12}{c}_\Sigma)^2 - (\overset{11}{c}_1 + \overset{11}{c}_\Sigma)(\overset{22}{c}_1 + \overset{22}{c}_\Sigma)]}\,,
\end{aligned} \tag{7.423}$$

$$
\begin{aligned}
\overset{2}{\mathsf{t}}_2 =& \frac{\overset{2}{b}_1(\overset{11}{c}_1 + \overset{11}{c}_\Sigma) - \overset{1}{b}_1(\overset{12}{c}_1 + \overset{12}{c}_\Sigma)}{(\overset{12}{c}_1 + \overset{12}{c}_\Sigma)^2 - (\overset{11}{c}_1 + \overset{11}{c}_\Sigma)(\overset{22}{c}_1 + \overset{22}{c}_\Sigma)} \\
&+ 2\overset{2}{b}_2 \frac{[2\,\overset{12}{c}_1\overset{12}{c}_\Sigma - \overset{22}{c}_\Sigma(\overset{11}{c}_1 + \overset{11}{c}_\Sigma)](\overset{11}{c}_2 + \overset{11}{c}_3) + \overset{11}{c}_\Sigma(\overset{12}{c}_2 + \overset{12}{c}_3)^2}{[(\overset{12}{c}_2 + \overset{12}{c}_3)^2 - (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3)]\,[(\overset{12}{c}_1 + \overset{12}{c}_\Sigma)^2 - (\overset{11}{c}_1 + \overset{11}{c}_\Sigma)(\overset{22}{c}_1 + \overset{22}{c}_\Sigma)]} \\
&- 2\overset{1}{b}_2 \frac{\overset{12}{c}_\Sigma(\overset{12}{c}_1 + \overset{12}{c}_\Sigma)(\overset{12}{c}_2 + \overset{12}{c}_3) - (\overset{22}{c}_1\overset{11}{c}_\Sigma + \overset{11}{c}_1\overset{22}{c}_\Sigma)(\overset{12}{c}_2 + \overset{12}{c}_3) - \overset{12}{c}_\Sigma(\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3)}{[(\overset{12}{c}_2 + \overset{12}{c}_3)^2 - (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3)]\,[(\overset{12}{c}_1 + \overset{12}{c}_\Sigma)^2 - (\overset{11}{c}_1 + \overset{11}{c}_\Sigma)(\overset{22}{c}_1 + \overset{22}{c}_\Sigma)]}\,,
\end{aligned}
\tag{7.424}
$$

$$
\overset{1}{\mathsf{t}}_3 = \frac{\overset{1}{b}_2(\overset{22}{c}_2 + \overset{22}{c}_3) - \overset{2}{b}_2(\overset{12}{c}_2 + \overset{12}{c}_3)}{(\overset{12}{c}_2 + \overset{12}{c}_3)^2 - (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3)}\,, \quad \overset{2}{\mathsf{t}}_3 = \frac{\overset{2}{b}_2(\overset{11}{c}_2 + \overset{11}{c}_3) - \overset{1}{b}_2(\overset{12}{c}_2 + \overset{12}{c}_3)}{(\overset{12}{c}_2 + \overset{12}{c}_3)^2 - (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3)}\,, \tag{7.425}
$$

$$
\overset{1}{\mathsf{t}}_4 = \frac{(\overset{1}{b}_1 + \overset{1}{b}_2)(\overset{22}{c}_1 + \overset{22}{c}_\Sigma) - (\overset{2}{b}_1 + \overset{2}{b}_2)(\overset{12}{c}_1 + \overset{12}{c}_\Sigma)}{(\overset{12}{c}_1 + \overset{12}{c}_\Sigma)^2 - (\overset{11}{c}_1 + \overset{11}{c}_\Sigma)(\overset{22}{c}_1 + \overset{22}{c}_\Sigma)}\,, \quad \overset{2}{\mathsf{t}}_4 = \frac{(\overset{2}{b}_1 + \overset{2}{b}_2)(\overset{11}{c}_1 + \overset{11}{c}_\Sigma) - (\overset{1}{b}_1 + \overset{1}{b}_2)(\overset{12}{c}_1 + \overset{12}{c}_\Sigma)}{(\overset{12}{c}_1 + \overset{12}{c}_\Sigma)^2 - (\overset{11}{c}_1 + \overset{11}{c}_\Sigma)(\overset{22}{c}_1 + \overset{22}{c}_\Sigma)}\,. \tag{7.426}
$$

Substituting this solution into (7.410) and (7.411) one obtains the following two conditions to be satisfied by the elastic constants

$$
\mu = 2\left[\frac{2\overset{1}{b}_2\overset{2}{b}_2(\overset{12}{c}_2 + \overset{12}{c}_3) - (\overset{2}{b}_2)^2\,(\overset{11}{c}_2 + \overset{11}{c}_3) - (\overset{1}{b}_2)^2\,(\overset{22}{c}_2 + \overset{22}{c}_3)}{(\overset{12}{c}_2 + \overset{12}{c}_3)^2 - (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3)}\right], \tag{7.427}
$$

$$
\begin{aligned}
\lambda = 2&\left[\frac{2(\overset{1}{b}_1 + \overset{1}{b}_2)(\overset{2}{b}_1 + \overset{2}{b}_2)(\overset{12}{c}_1 + \overset{12}{c}_\Sigma) - (\overset{1}{b}_1 + \overset{1}{b}_2)^2\,(\overset{22}{c}_1 + \overset{22}{c}_\Sigma) - (\overset{2}{b}_1 + \overset{2}{b}_2)^2\,(\overset{11}{c}_1 + \overset{11}{c}_\Sigma)}{(\overset{12}{c}_1 + \overset{12}{c}_\Sigma)^2 - (\overset{11}{c}_1 + \overset{11}{c}_\Sigma)(\overset{22}{c}_1 + \overset{22}{c}_\Sigma)}\right] \\
&+ 2\left[\frac{(\overset{2}{b}_2)^2\,(\overset{11}{c}_2 + \overset{11}{c}_3) + (\overset{1}{b}_2)^2\,(\overset{22}{c}_2 + \overset{22}{c}_3) - 2\overset{1}{b}_2\overset{2}{b}_2(\overset{12}{c}_2 + \overset{12}{c}_3)}{(\overset{12}{c}_2 + \overset{12}{c}_3)^2 - (\overset{11}{c}_2 + \overset{11}{c}_3)(\overset{22}{c}_2 + \overset{22}{c}_3)}\right].
\end{aligned}
\tag{7.428}
$$

Note that (7.427) is identical to what we obtained for the cylindrical cloak example in (7.392), i.e., (7.427) violates positive-definiteness of the energy, and hence, cloaking is not possible.

The following proposition summarizes the discussions and calculations of this section.

Proposition 7.5.7. *Elastodynamic transformation cloaking is not possible for linear generalized Cosserat elastic solids in dimension two.*

Corollary 7.5.8. *Elastodynamic transformation cloaking is not possible for linear Cosserat elastic solids in dimension two.*

Example (A generalized Cosserat spherical cloak). Let us next consider a finite spherical cavity $\mathcal{H}$ with radius R_i embedded in an infinite isotropic homogeneous generalized Cosserat elastic medium with the elastic properties given in (7.342). Let (R, Θ, Φ) be the spherical coordinates, for which $R \geq 0$ with $R = 0$ corresponding to the center of the cavity, $0 \leq \Theta \leq \pi$, and $0 \leq \Phi \leq 2\pi$. Suppose that there is a wave source located at (R_p, Θ_p, Φ_p). The cloak $\mathcal{C}$ is a spherical shell with inner radius R_i and outer radius R_o surrounding the hole such that the source is located outside the cloaking region, i.e., $R_o < R_p$. Similar to the cylindrical cloak example, the reference configuration of the body $\mathcal{B}$ is mapped to that of $\tilde{\mathcal{B}}$ (the virtual body) using a mapping $\Xi : \mathcal{B} \to \tilde{\mathcal{B}}$. Note that Ξ is the identity outside the cloak, i.e., $R \geq R_o$ and is defined as $(\tilde{R}, \tilde{\Theta}, \tilde{\Phi}) = \Xi(R, \Theta, \Phi) = (f(R), \Theta, \Phi)$ for $R_i \leq R \leq R_o$ such that $f(R_i) = \epsilon$, $f(R_o) = R_o$, and $f'(R_o) = 1$.

The reference configurations of $\mathcal{B}$ and $\tilde{\mathcal{B}}$ are endowed with the induced metrics from $\mathbb{R}^3$, i.e., $\mathbf{G} = \mathrm{diag}(1, R^2, R^2 \sin^2 \Theta)$ and $\tilde{\mathbf{G}} = \mathrm{diag}(1, \tilde{R}^2, \tilde{R}^2 \sin^2 \tilde{\Theta})$, respectively. In the coordinates (r, θ, ϕ), the ambient space metric reads $\mathbf{g} = \mathrm{diag}(1, r^2, r^2 \sin^2 \theta)$. We assume that the virtual body $\tilde{\mathcal{B}}$ is isotropic and has the same elastic properties as the primary medium in the region $\mathcal{B} \setminus \mathcal{C}$. For the cloaking map $\overset{\Xi}{\mathbf{F}} = \mathrm{diag}(f'(R), 1, 1)$ in $R_i \leq R \leq R_o$. Note also that

$$\mathsf{s} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{R}{f(R)} & 0 \\ 0 & 0 & \frac{R}{f(R)} \end{bmatrix} . \tag{7.429}$$

Noting that $\rho = J_\Xi \tilde{\rho} \circ \Xi = J_\Xi \rho_0$, the mass density of the cloak is obtained as

$$\rho_{\mathcal{C}}(R) = \frac{f(R)^2 f'(R)}{R^2} \rho_0 , \qquad R_i \leq R \leq R_o . \tag{7.430}$$

Also

$$\overset{\mathfrak{ab}}{\nu}_{\mathcal{C}}(R) = \frac{f(R)^2 f'(R)}{R^2}\overset{\mathfrak{ab}}{\tilde{\nu}}\,, \qquad R_i \leq R \leq R_o\,. \tag{7.431}$$

Using (7.329) and (7.333), the elastic constants of the cloak are calculated and read ($\mathfrak{a}, \mathfrak{b} \in \{1, 2, 3\}$)

$$\hat{\mathbb{A}} = \begin{bmatrix} \begin{bmatrix} \frac{(\lambda+2\mu)f(R)^2}{R^2 f'(R)} & 0 & 0 \\ 0 & \frac{\lambda f(R)}{R} & 0 \\ 0 & 0 & \frac{\lambda f(R)}{R} \end{bmatrix} & \begin{bmatrix} 0 & \mu f'(R) & 0 \\ \frac{\mu f(R)}{R} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & \mu f'(R) \\ 0 & 0 & 0 \\ \frac{\mu f(R)}{R} & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & \frac{\mu f(R)}{R} & 0 \\ \frac{\mu f(R)^2}{R^2 f'(R)} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} \frac{\lambda f(R)}{R} & 0 & 0 \\ 0 & (\lambda+2\mu) f'(R) & 0 \\ 0 & 0 & \lambda f'(R) \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu f'(R) \\ 0 & \mu f'(R) & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & \frac{\mu f(R)}{R} \\ 0 & 0 & 0 \\ \frac{\mu f(R)^2}{R^2 f'(R)} & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \mu f'(R) \\ 0 & \mu f'(R) & 0 \end{bmatrix} & \begin{bmatrix} \frac{\lambda f(R)}{R} & 0 & 0 \\ 0 & \lambda f'(R) & 0 \\ 0 & 0 & (\lambda+2\mu) f'(R) \end{bmatrix} \end{bmatrix}, \tag{7.432}$$

$$\hat{\overset{\mathfrak{a}}{\mathbb{B}}} = \begin{bmatrix} \begin{bmatrix} \frac{\overset{\mathfrak{a}}{b}_{\Sigma} f(R)^2}{R^2 f'(R)} & 0 & 0 \\ 0 & \frac{\overset{\mathfrak{a}}{b}_1 f(R)}{R} & 0 \\ 0 & 0 & \frac{\overset{\mathfrak{a}}{b}_1 f(R)}{R} \end{bmatrix} & \begin{bmatrix} 0 & \overset{\mathfrak{a}}{b}_2 f'(R) & 0 \\ \frac{\overset{\mathfrak{a}}{b}_2 f(R)}{R} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & \overset{\mathfrak{a}}{b}_2 f'(R) \\ 0 & 0 & 0 \\ \frac{\overset{\mathfrak{a}}{b}_2 f(R)}{R} & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & \frac{\overset{\mathfrak{a}}{b}_2 f(R)}{R} & 0 \\ \frac{\overset{\mathfrak{a}}{b}_2 f(R)^2}{R^2 f'(R)} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} \frac{\overset{\mathfrak{a}}{b}_1 f(R)}{R} & 0 & 0 \\ 0 & \overset{\mathfrak{a}}{b}_{\Sigma} f'(R) & 0 \\ 0 & 0 & \overset{\mathfrak{a}}{b}_1 f'(R) \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \overset{\mathfrak{a}}{b}_2 f'(R) \\ 0 & \overset{\mathfrak{a}}{b}_2 f'(R) & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & \frac{\overset{\mathfrak{a}}{b}_2 f(R)}{R} \\ 0 & 0 & 0 \\ \frac{\overset{\mathfrak{a}}{b}_2 f(R)^2}{R^2 f'(R)} & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \overset{\mathfrak{a}}{b}_2 f'(R) \\ 0 & \overset{\mathfrak{a}}{b}_2 f'(R) & 0 \end{bmatrix} & \begin{bmatrix} \frac{\overset{\mathfrak{a}}{b}_1 f(R)}{R} & 0 & 0 \\ 0 & \overset{\mathfrak{a}}{b}_1 f'(R) & 0 \\ 0 & 0 & \overset{\mathfrak{a}}{b}_{\Sigma} f'(R) \end{bmatrix} \end{bmatrix}, \tag{7.433}$$

$$
\overset{\mathfrak{ab}}{\hat{\mathbb{C}}} = \begin{bmatrix}
\begin{bmatrix} \frac{\overset{\mathfrak{ab}}{c}_{\Sigma} f(R)^2}{R^2 f'(R)} & 0 & 0 \\ 0 & \frac{\overset{\mathfrak{ab}}{c}_1 f(R)}{R} & 0 \\ 0 & 0 & \frac{\overset{\mathfrak{ab}}{c}_1 f(R)}{R} \end{bmatrix} &
\begin{bmatrix} 0 & \overset{\mathfrak{ab}}{c}_2 f'(R) & 0 \\ \frac{\overset{\mathfrak{ab}}{c}_3 f(R)}{R} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} &
\begin{bmatrix} 0 & 0 & \overset{\mathfrak{ab}}{c}_2 f'(R) \\ 0 & 0 & 0 \\ \frac{\overset{\mathfrak{ab}}{c}_3 f(R)}{R} & 0 & 0 \end{bmatrix} \\
\begin{bmatrix} 0 & \frac{\overset{\mathfrak{ab}}{c}_3 f(R)}{R} & 0 \\ \frac{\overset{\mathfrak{ab}}{c}_2 f(R)^2}{R^2 f'(R)} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} &
\begin{bmatrix} \frac{\overset{\mathfrak{ab}}{c}_1 f(R)}{R} & 0 & 0 \\ 0 & \overset{\mathfrak{ab}}{c}_{\Sigma} f'(R) & 0 \\ 0 & 0 & \overset{\mathfrak{ab}}{c}_1 f'(R) \end{bmatrix} &
\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \overset{\mathfrak{ab}}{c}_2 f'(R) \\ 0 & \overset{\mathfrak{ab}}{c}_3 f'(R) & 0 \end{bmatrix} \\
\begin{bmatrix} 0 & 0 & \frac{\overset{\mathfrak{ab}}{c}_3 f(R)}{R} \\ 0 & 0 & 0 \\ \frac{\overset{\mathfrak{ab}}{c}_2 f(R)^2}{R^2 f'(R)} & 0 & 0 \end{bmatrix} &
\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \overset{\mathfrak{ab}}{c}_3 f'(R) \\ 0 & \overset{\mathfrak{ab}}{c}_2 f'(R) & 0 \end{bmatrix} &
\begin{bmatrix} \frac{\overset{\mathfrak{ab}}{c}_1 f(R)}{R} & 0 & 0 \\ 0 & \overset{\mathfrak{ab}}{c}_1 f'(R) & 0 \\ 0 & 0 & \overset{\mathfrak{ab}}{c}_{\Sigma} f'(R) \end{bmatrix}
\end{bmatrix}, \tag{7.434}
$$

where $\overset{\mathfrak{ab}}{c}_{\Sigma} = \overset{\mathfrak{ab}}{c}_1 + \overset{\mathfrak{ab}}{c}_2 + \overset{\mathfrak{ab}}{c}_3$, and $\overset{\mathfrak{a}}{b}_{\Sigma} = \overset{\mathfrak{a}}{b}_1 + 2\overset{\mathfrak{a}}{b}_2$.

Similar to the cylindrical cloak example, the balance of angular momentum (7.337) for the diagonal and off-diagonal components of the director gradients are uncoupled. We consider the equations for the diagonal components of the director gradients in $(7.337)_2$ and show that they force the cloaking map to be the identity. For the choice $(a,c) = (2,3)$, $(7.337)_2$ is expanded and gives the following equations for the (circumferential and azimuthal) diagonal director gradient components

$$
\overset{11}{c}_2 \underset{1}{\mathsf{F}}{}^{\phi}{}_{\Phi} + \overset{12}{c}_2 \underset{2}{\mathsf{F}}{}^{\phi}{}_{\Phi} + \overset{13}{c}_2 \underset{3}{\mathsf{F}}{}^{\phi}{}_{\Phi} - \overset{11}{c}_3 \underset{1}{\mathsf{F}}{}^{\theta}{}_{\Theta} - \overset{12}{c}_3 \underset{2}{\mathsf{F}}{}^{\theta}{}_{\Theta} - \overset{13}{c}_3 \underset{3}{\mathsf{F}}{}^{\theta}{}_{\Theta} = 0\,, \tag{7.435}
$$

$$
\overset{11}{c}_3 \underset{1}{\mathsf{F}}{}^{\phi}{}_{\Phi} + \overset{12}{c}_3 \underset{2}{\mathsf{F}}{}^{\phi}{}_{\Phi} + \overset{13}{c}_3 \underset{3}{\mathsf{F}}{}^{\phi}{}_{\Phi} - \overset{11}{c}_2 \underset{1}{\mathsf{F}}{}^{\theta}{}_{\Theta} - \overset{12}{c}_2 \underset{2}{\mathsf{F}}{}^{\theta}{}_{\Theta} - \overset{13}{c}_2 \underset{3}{\mathsf{F}}{}^{\theta}{}_{\Theta} = 0\,, \tag{7.436}
$$

$$
\overset{12}{c}_2 \underset{1}{\mathsf{F}}{}^{\phi}{}_{\Phi} + \overset{22}{c}_2 \underset{2}{\mathsf{F}}{}^{\phi}{}_{\Phi} + \overset{23}{c}_2 \underset{3}{\mathsf{F}}{}^{\phi}{}_{\Phi} - \overset{12}{c}_3 \underset{1}{\mathsf{F}}{}^{\theta}{}_{\Theta} - \overset{22}{c}_3 \underset{2}{\mathsf{F}}{}^{\theta}{}_{\Theta} - \overset{23}{c}_3 \underset{3}{\mathsf{F}}{}^{\theta}{}_{\Theta} = 0\,, \tag{7.437}
$$

$$
\overset{12}{c}_3 \underset{1}{\mathsf{F}}{}^{\phi}{}_{\Phi} + \overset{22}{c}_3 \underset{2}{\mathsf{F}}{}^{\phi}{}_{\Phi} + \overset{23}{c}_3 \underset{3}{\mathsf{F}}{}^{\phi}{}_{\Phi} - \overset{12}{c}_2 \underset{1}{\mathsf{F}}{}^{\theta}{}_{\Theta} - \overset{22}{c}_2 \underset{2}{\mathsf{F}}{}^{\theta}{}_{\Theta} - \overset{23}{c}_2 \underset{3}{\mathsf{F}}{}^{\theta}{}_{\Theta} = 0\,, \tag{7.438}
$$

$$
\overset{13}{c}_2 \underset{1}{\mathsf{F}}{}^{\phi}{}_{\Phi} + \overset{23}{c}_2 \underset{2}{\mathsf{F}}{}^{\phi}{}_{\Phi} + \overset{33}{c}_2 \underset{3}{\mathsf{F}}{}^{\phi}{}_{\Phi} - \overset{13}{c}_3 \underset{1}{\mathsf{F}}{}^{\theta}{}_{\Theta} - \overset{23}{c}_3 \underset{2}{\mathsf{F}}{}^{\theta}{}_{\Theta} - \overset{33}{c}_3 \underset{3}{\mathsf{F}}{}^{\theta}{}_{\Theta} = 0\,, \tag{7.439}
$$

$$
\overset{13}{c}_3 \underset{1}{\mathsf{F}}{}^{\phi}{}_{\Phi} + \overset{23}{c}_3 \underset{2}{\mathsf{F}}{}^{\phi}{}_{\Phi} + \overset{33}{c}_3 \underset{3}{\mathsf{F}}{}^{\phi}{}_{\Phi} - \overset{13}{c}_2 \underset{1}{\mathsf{F}}{}^{\theta}{}_{\Theta} - \overset{23}{c}_2 \underset{2}{\mathsf{F}}{}^{\theta}{}_{\Theta} - \overset{33}{c}_2 \underset{3}{\mathsf{F}}{}^{\theta}{}_{\Theta} = 0\,. \tag{7.440}
$$

Choosing a class of deformations for which $U^a{}_{|A} = 0$, and $\underset{\mathfrak{a}}{\mathfrak{U}}{}^1{}_{|1} = \underset{\mathfrak{a}}{\mathfrak{U}}{}^2{}_{|2} = \underset{\mathfrak{a}}{\mathfrak{U}}{}^3{}_{|3} = 0$, the positive-definiteness of the energy function implies the positive definiteness of the following matrix

$$\begin{bmatrix} \overset{11}{c}_2 & \overset{11}{c}_3 & \overset{12}{c}_2 & \overset{12}{c}_3 & \overset{13}{c}_2 & \overset{13}{c}_3 \\ \overset{11}{c}_3 & \overset{11}{c}_2 & \overset{12}{c}_3 & \overset{12}{c}_2 & \overset{13}{c}_3 & \overset{13}{c}_2 \\ \overset{12}{c}_2 & \overset{12}{c}_3 & \overset{22}{c}_2 & \overset{22}{c}_3 & \overset{23}{c}_2 & \overset{23}{c}_3 \\ \overset{12}{c}_3 & \overset{12}{c}_2 & \overset{22}{c}_3 & \overset{22}{c}_2 & \overset{23}{c}_3 & \overset{23}{c}_2 \\ \overset{13}{c}_2 & \overset{13}{c}_3 & \overset{23}{c}_2 & \overset{23}{c}_3 & \overset{33}{c}_2 & \overset{33}{c}_3 \\ \overset{13}{c}_3 & \overset{13}{c}_2 & \overset{23}{c}_3 & \overset{23}{c}_2 & \overset{33}{c}_3 & \overset{33}{c}_2 \end{bmatrix} . \tag{7.441}$$

In particular, the determinant of (7.441) is non-vanishing, and thus, so is the determinant of the coefficient matrix of the system (7.435)-(7.440). Therefore, $\underset{1}{\mathsf{F}}{}^{\phi}{}_{\Phi} = \underset{2}{\mathsf{F}}{}^{\phi}{}_{\Phi} = \underset{3}{\mathsf{F}}{}^{\phi}{}_{\Phi} = 0$ and $\underset{1}{\mathsf{F}}{}^{\theta}{}_{\Theta} = \underset{2}{\mathsf{F}}{}^{\theta}{}_{\Theta} = \underset{3}{\mathsf{F}}{}^{\theta}{}_{\Theta} = 0$. Using this, (7.337)$_2$ for choices $(a,c) = (1,2)$ and $(a,c) = (1,3)$ gives the following equations for $\underset{1}{\mathsf{F}}{}^{r}{}_{R}$, $\underset{2}{\mathsf{F}}{}^{r}{}_{R}$, and $\underset{3}{\mathsf{F}}{}^{r}{}_{R}$:

$$f(R)\left(\overset{11}{c}_3\underset{1}{\mathsf{F}}{}^{r}{}_{R} + \overset{12}{c}_3\underset{2}{\mathsf{F}}{}^{r}{}_{R} + \overset{13}{c}_3\underset{3}{\mathsf{F}}{}^{r}{}_{R}\right) = \overset{1}{b}_2\left(Rf'(R) - f(R)\right) , \tag{7.442}$$

$$f(R)\left(\overset{11}{c}_2\underset{1}{\mathsf{F}}{}^{r}{}_{R} + \overset{12}{c}_2\underset{2}{\mathsf{F}}{}^{r}{}_{R} + \overset{13}{c}_2\underset{3}{\mathsf{F}}{}^{r}{}_{R}\right) = \overset{1}{b}_2\left(Rf'(R) - f(R)\right) , \tag{7.443}$$

$$f(R)\left(\overset{12}{c}_3\underset{1}{\mathsf{F}}{}^{r}{}_{R} + \overset{22}{c}_3\underset{2}{\mathsf{F}}{}^{r}{}_{R} + \overset{23}{c}_3\underset{3}{\mathsf{F}}{}^{r}{}_{R}\right) = \overset{2}{b}_2\left(Rf'(R) - f(R)\right) , \tag{7.444}$$

$$f(R)\left(\overset{12}{c}_2\underset{1}{\mathsf{F}}{}^{r}{}_{R} + \overset{22}{c}_2\underset{2}{\mathsf{F}}{}^{r}{}_{R} + \overset{23}{c}_2\underset{3}{\mathsf{F}}{}^{r}{}_{R}\right) = \overset{2}{b}_2\left(Rf'(R) - f(R)\right) , \tag{7.445}$$

$$f(R)\left(\overset{13}{c}_3\underset{1}{\mathsf{F}}{}^{r}{}_{R} + \overset{23}{c}_3\underset{2}{\mathsf{F}}{}^{r}{}_{R} + \overset{33}{c}_3\underset{3}{\mathsf{F}}{}^{r}{}_{R}\right) = \overset{3}{b}_2\left(Rf'(R) - f(R)\right) , \tag{7.446}$$

$$f(R)\left(\overset{13}{c}_2\underset{1}{\mathsf{F}}{}^{r}{}_{R} + \overset{23}{c}_2\underset{2}{\mathsf{F}}{}^{r}{}_{R} + \overset{33}{c}_2\underset{3}{\mathsf{F}}{}^{r}{}_{R}\right) = \overset{3}{b}_2\left(Rf'(R) - f(R)\right) . \tag{7.447}$$

The coefficient matrix of the system of homogeneous algebraic equations that can be obtained from (7.442)-(7.443), (7.444)-(7.445), and (7.446)-(7.447) reads

$$\begin{bmatrix} \overset{11}{c}_3 - \overset{11}{c}_2 & \overset{12}{c}_3 - \overset{12}{c}_2 & \overset{13}{c}_3 - \overset{13}{c}_2 \\ \overset{12}{c}_3 - \overset{12}{c}_2 & \overset{22}{c}_3 - \overset{22}{c}_2 & \overset{23}{c}_3 - \overset{23}{c}_2 \\ \overset{13}{c}_3 - \overset{13}{c}_2 & \overset{23}{c}_3 - \overset{23}{c}_2 & \overset{33}{c}_3 - \overset{33}{c}_2 \end{bmatrix} . \tag{7.448}$$

It is straightforward to see that the determinant of (7.441) being non-zero implies that the determinant of (7.448) is non-zero as well. Thus, $\underset{1}{\mathsf{F}}{}^{r}{}_{R} = \underset{2}{\mathsf{F}}{}^{r}{}_{R} = \underset{3}{\mathsf{F}}{}^{r}{}_{R} = 0$, and from (7.442)-(7.447), one concludes that $f(R) = R$, which, in turn, means that cloaking is not possible. Therefore, we have proved the following result.

Proposition 7.5.9. *Elastodynamic transformation cloaking is not possible for a spherical cavity using a spherical cloak in linear generalized Cosserat elastic solids.*

Corollary 7.5.10. *Elastodynamic transformation cloaking is not possible for a spherical cavity using a spherical cloak in linear Cosserat elastic solids.*

We suspect that transformation cloaking in dimension three is not possible for a cavity of any shape. The idea of proof is similar to that of 2D. However, in this case there are 108 equations for 27 unknown director gradients $\underset{\mathfrak{a}}{\mathsf{F}}{}^{a}{}_{A}$. We have not been able to solve this large system of equations but expect that they would violate positive-definiteness of the energy and also force the cloaking map to be the identity.

Conjecture 7.5.11. Elastodynamic transformation cloaking is not possible for linear generalized Cosserat elastic solids in dimension three.

CHAPTER 8

TRANSFORMATION CLOAKING IN ELASTIC PLATES

8.1 Introduction

Hiding objects from electromagnetic waves has been a subject of intense interest in recent years. Pendry *et al.* [142] and Leonhardt [143] studied the possibility of electromagnetic transformation cloaking, which was later experimentally verified for microwaves frequencies by Schurig *et al.* [144] and for optical wavelengths ($1.4-2.7\,\mu m$) by Ergin *et al.* [146]. The first ideas pertaining to cloaking in elasticity can be found in the works of Gurney [170] and Reissner and Morduchow [171] on reinforced holes in linear elastic sheets. Mansfield [172] systematically studied cloaking in the context of linear elasticity by introducing the concept of neutral holes. Mansfield considered a hole(s) in a sheet under a given far-field in-plane loading and showed that the hole(s) can be reinforced such that the stress field outside the hole(s) is identical to that of an uncut sheet under the same (far-field) loading. The shape of the boundary of the (neutral) hole and the characteristics of the reinforcement are determined based on the stress field of the uncut sheet, and thus, they explicitly depend on the applied far-filed loading. The main difference between electromagnetic and elastodynamic transformation cloaking is that unlike Maxwell's equations for electromagnetism (with only one configuration, i.e., the ambient space), the governing equations of elasticity are written with respect to two inherently different configurations (frames): a reference and a current configuration. This, in turn, leads to two-point tensors in the governing equations. Marsden and Hughes [43], Yavari and Ozakin [82], and Steigmann [178] showed that if formulated properly, the governing equations of nonlinear and linearized elasticity are covariant (invariant) under arbitrary time-dependent changes in the current configuration (frame). The governing equations of elasticity are invariant under any time-independent changes in the reference configuration (or referential coordinate transformations) as well [132, 46]. Nevertheless, as was shown in [41], referential or spatial covariance of the governing equations is not the direct underlying principle of transformation cloaking; a cloaking map is neither a referential nor a spatial change of coordinates. Rather, a cloaking transformation maps the boundary-value problem of an isotropic and homogeneous elastic body containing an infinitesimal hole (virtual problem) to that of a generally anisotropic and inhomogeneous elastic body with a finite hole surrounded by a cloak (physical problem). The cloak should be designed such that the physical and virtual problems have identical solutions (elastic measurements in the case of elastodynamics) outside the cloak.

The idea of cloaking has been thoroughly studied and is well understood in the context of conductivity [149, 150], electrical impedance tomography and electromagnetism [151, 152, 153, 154]. The possibility of cloaking has been examined in many other fields of science and engineering, e.g., acoustics [155, 156, 157, 158, 159, 160, 161, 162], optics [163], thermodynamics (i.e., design of thermal cloaks) [164, 240], diffusion [165], quantum mechanics [166], thermoelasticity [241, 242], seismology [179, 204], and elastodynamics [167, 199] (see the recent reviews [168, 169, 243] for a discussion of theses applications in some detail). Recently, we formulated both the nonlinear and linearized elastodynamic transformation cloaking problems (in 3D) in a mathematically precise form [41]. In this chapter, we provide a geometric formulation of transformation cloaking in elastic plates starting from nonlinear shell theory. In our opinion, none of the existing works in the literature has properly formulated the transformation cloaking problem in elastic plates. In particular, the boundary (and continuity) conditions on the hole surface and the outer surface of the cloak, and the restrictions they impose on cloaking transformations have not been discussed. Additionally, we derive several constraints that involve the cloaking transformation, along with the elastic parameters of the virtual plate, which impose further restrictions on the cloaking map. These constraints seem to have been ignored in the literature to this date.

Many solid mechanics workers traditionally have used the classical formulation of linear elasticity. This is appropriate for many practical engineering applications. In the case of elastodynamic transformation cloaking problem, however, as observed in [41], starting from linear elasticity is not appropriate. This is because linear elasticity does not distinguish between the reference and the current configurations and the corresponding changes of coordinates defined in these inherently different configurations. This has been a source of confusion in the recent literature of transformation cloaking in elastodynamics. Coordinate transformations in the reference and current configurations are physically very different: Local referential changes of frame are related to the local material symmetry group, whereas the global coordinate transformations in the ambient space are related to objectivity (or material frame indifference). This, in turn, implies that even in the case of small strains, any elastodynamic transformation cloaking study needs to be formulated in the nonlinear framework.

Examples of improper formulations of transformation cloaking in elastic plates can be seen in almost all the existing works in the literature. Colquitt *et al.* [244] studied transformation cloaking in Kirchhoff-Love plates subjected to time-harmonic out-of-plane displacements in the setting of linear elasticity and starting from a governing equation simplified for the case of an isotropic and homogeneous (Kirchhoff-Love) plate. In §8.4.1, we show in detail that their formulation for transformation cloaking in Kirchhoff-Love plates is, unfortunately, incorrect. In particular, their transformed rigidity tensor is incorrect and does not agree with that of an isotropic and homogeneous elastic plate when the cloaking map is the identity, and is not positive definite. Colquitt *et al.* [244] and many other researchers, e.g., Brun *et al.* [245], Jones *et al.* [246], Misseroni *et al.* [247], Zareei and Alam [248], Darabi *et al.* [249], and Liu and Zhu [250] start from the following equation

$$D^{(0)}\nabla^4_{\mathbf{X}}W(\mathbf{X}) - Ph\omega^2 W(\mathbf{X}) = 0\,, \quad \text{or} \quad \left(\nabla^4_{\mathbf{X}} - \frac{Ph}{D^{(0)}}\omega^2\right)W(\mathbf{X}) = 0\,, \quad \mathbf{X}\in\chi\subseteq\mathbb{R}^2\,, \tag{8.1}$$

where h is the plate thickness, P is the mass density, $W(\mathbf{X})$ is the amplitude and ω is the frequency of time-harmonic transverse waves, and $D^{(0)} = Eh^3/12(1-\nu^2)$ is the bending rigidity. They next transform the governing equation under an invertible transformation $\mathcal{F}:\chi\to\Omega$, where $\mathbf{x}=\mathcal{F}(\mathbf{X})$, $\mathbf{F}=\nabla_{\mathbf{X}}\mathbf{x}$, and $J=\det\mathbf{F}$, using [159, Lemma 2.1] twice and obtain

$$\left(\nabla\cdot J^{-1}\mathbf{F}\mathbf{F}^{\mathsf{T}}\nabla J\nabla\cdot J^{-1}\mathbf{F}\mathbf{F}^{\mathsf{T}}\nabla - \frac{Ph}{JD^{(0)}}\omega^2\right)W(\mathbf{x}) = 0\,, \quad \mathbf{x}\in\Omega\,. \tag{8.2}$$

We show that applying [159, Lemma 2.1] twice, one assumes that the gradients of (out-of-plane) displacements and the gradients of the Laplacian of displacements in the physical and virtual plates are related by the Piola transformation. Surprisingly, in none of the works that use this lemma to transform the biharmonic equation (8.1) is there any discussion of these strong assumptions and whether they are compatible with the fact that displacements in the physical and virtual plates are required to be equal. In particular, we see in §8.4.1 that taking these assumptions into account, the cloaking map is forced to isometrically transform the governing equations of the virtual plate to those of the physical plate. Therefore, this formulation of the cloaking problem does not result in any new information, and the physical and virtual plates are essentially the same elastic plate with the same mechanical response. Furthermore, we will discuss that ignoring these assumptions and what constraints they impose on the cloaking map has resulted in obtaining incorrect transformed fields for the physical problem.

In another related work, Colquitt *et al.* [251] presented a formulation for the cloaking of the out-of-plane shear waves for the Helmholtz equation. In Remark. 8.4.6, we illustrate that their cloaking scheme is, unfortunately, incorrect. Similar to [244], they use [159, Lemma 2.1] to transform the Helmholtz equation for the isotropic and homogeneous medium without considering the restrictions that are imposed on the cloaking map by this lemma. We show that their cloaking map does not satisfy these restrictions, and therefore, will not result in cloaking.

More recently, Pomot *et al.* [252] pointed out some shortcomings of Colquitt *et al.* [244]'s formulation of flexural cloaking[1] and attempted to formulate the cloaking problem. However, their formulation is not consistent. In particular, it is not clear what type of coordinate transformation is being used as they do not distinguish between spatial and referential coordinate transformations. Furthermore, their formulation is missing the important constraint (8.6) that puts some severe restrictions on the cloaking map. In particular, their claim that "the transformed equation does not verify in general the equilibrium equation $N_{IJ,J} + S_I = 0$." is incorrect. We show in §8.4.1 and §8.4.2 that the equilibrium equations for a finitely-deformed plate for a generic radial cloaking transformation do not put any extra constraints on the cloaking map as long as the cloaking map satisfies the *cloaking compatibility equations* (8.151). Finally, there is no discussion on the boundary conditions and what

[1] They, however, do not provide a clear mathematical reasoning as to why Colquitt *et al.* [244]'s formulation is incorrect.

restrictions they impose on the cloaking map. In particular, their cloaking map is inadmissible, because its derivative is not the identity on the outer boundary of the cloak. Pomot *et al.* [252] consider two spaces E and e related by the mapping Φ (which they call the geometrical transformation) such that $F = \nabla_x X$, denotes the Jacobian of Φ, $J = \det \Phi$, and $\Gamma_{Iij} = \frac{\partial^2 X_I}{\partial x_i \partial x_j}$, which they call the Hessian matrix. The flexural rigidities, mass densities, and infinitesimal displacements in E and e, are, denoted by (D_{IJKL}, P, U) and (D_{ijkl}, ρ, u), respectively. They write the total energy density in the transformed space as $\mathcal{E} = \mathcal{W} + \mathcal{T}$, where $\mathcal{W} = \int_e \frac{1}{2} u_{,ij} D_{ijkl}(\mathbf{x}) u_{,kl} \mathrm{dv}$, and $\mathcal{T} = \int_e \frac{1}{2} \rho(\mathbf{x}) h \dot{u}^2 \mathrm{dv}$, with h being the plate thickness. In the initial space, they write the Hamiltonian as $\mathcal{E}_0 = \mathcal{W}_0 + \mathcal{T}_0$, where

$$\begin{aligned} \mathcal{W}_0 &= \int_E \frac{1}{2} \left(U_{,IJ} F_{Ii} F_{Jj} + U_{,I} \Gamma_{Iij} \right) D_{ijkl}(\mathbf{X}) \left(U_{,KL} F_{Kk} F_{Ll} + U_{,K} \Gamma_{Kkl} \right) J^{-1} \mathrm{dV} \,, \\ \mathcal{T}_0 &= \int_E \frac{1}{2} P(\mathbf{X}) \dot{U}^2 J^{-1} \mathrm{dV} \,. \end{aligned} \tag{8.3}$$

Next, they obtain the equations of motion from the stationarity conditions of the Lagrangian density $\mathcal{L}_0 = \mathcal{W}_0 - \mathcal{T}_0$:

$$\begin{aligned} \delta \mathcal{W}_0 = &\int_E \delta U \left(J^{-1} D_{ijkl} F_{Ii} F_{Jj} F_{Kk} F_{Ll} U_{,KL} \right)_{,IJ} \mathrm{dV} + \int_E \delta U \left(J^{-1} D_{ijkl} F_{Ii} F_{Jj} \Gamma_{Kkl} U_{,K} \right)_{,IJ} \mathrm{dV} \\ &- \int_E \delta U \left(J^{-1} D_{ijkl} \Gamma_{Iij} F_{Kk} F_{Ll} U_{,KL} \right)_{,I} \mathrm{dV} - \int_E \delta U \left(J^{-1} D_{ijkl} \Gamma_{Iij} \Gamma_{Kkl} U_{,K} \right)_{,I} \mathrm{dV} \,, \end{aligned} \tag{8.4}$$

that is rewritten as

$$\delta \mathcal{W}_0 = \int_E \delta U \left(-M_{IJ,IJ} - N_{IJ} U_{,IJ} + S_I U_{,I} \right) \mathrm{dV} \,. \tag{8.5}$$

Using (8.4), in turn, they determine D_{IJKL}, N_{IJ}, and S_I (their Eq. (14)).[2] Two comments are in order here: (i) Rewriting (8.4) as (8.5), one needs to take the following constraint into account

$$D_{ijkl} F_{Ii} F_{Jj} \Gamma_{Kkl} = D_{ijkl} F_{Kk} F_{Il} \Gamma_{Jij} \,, \tag{8.6}$$

which comes from the fact that the coefficient of the third gradient of U must be identical in these

[2]Note the typo in the expression for S_I in their Eq. (14).

expressions. This has been ignored in their work. We derive the constraints of this type in their general form for Kirchhoff-Love plates (see Eq. (8.151)) and for plates with both the in-plane and out-of-plane displacements (see Remark 8.4.8). (ii) They suggested using a cloaking map, for which $\Gamma = \mathbf{0}$, i.e., a cloaking map with a covariantly constant derivative map. However, as we explain in Remark 8.4.4, such mappings are not admissible for cloaking in Kirchhoff-Love plates.

In this chapter, in order to obtain the governing equations of an elastic plate, we first obtain those of a nonlinear elastic shell. This is crucial in order to properly account for the variations of the geometric objects of a surface, and thus, obtain the correct linearized governing equations of an elastic plate. An elastic shell is modeled by an orientable two-dimensional Riemannian submanifold embedded in the Euclidean space. The geometry of the shell, is thus, characterized by its first and second fundamental forms that, respectively, represent the intrinsic (in-plane) and extrinsic (out-of-plane) geometries of the surface. Utilizing a Lagrangian field theory, we derive the governing equations of motion. To account for the contribution of body forces and body moments associated with the variations of the position and the orientation (i.e., the normal vector) of the surface, we use the Lagrange–d'Alembert principle. The linearized governing equations for initially stress-free shells, and also elastic shells with non-vanishing initial stress, couple-stress, and initial body forces and moments are also derived.

Next, the transformation cloaking problem for Kirchhoff-Love plates is formulated. We start with the balance of linear momentum for the virtual plate with uniform elastic properties in the absence of initial stress (and couple-stress) and initial body forces (and moments). The governing equation of the virtual plate is then transformed to that of the physical plate up to an unknown scalar field. The physical plate is subjected to initial stress and tangential body forces that are determined using the elastic constants of the virtual plate and the cloaking map. The transformation of couple-stress under a cloaking map is then determined. It is seen that the cloaking map needs to satisfy certain conditions on the boundaries of the cloak and the hole. In particular, we show that the cloaking transformation needs to fix the outer boundary of the cloak up to the third order. Assuming a generic radial cloaking map, we show in an example that cloaking a circular hole in Kirchhoff-Love plates is not possible.

The pure bending assumption is then relaxed and the transformation cloaking problem of an elastic plate, for which both the in-plane and the out-of-plane displacements are allowed, is formulated. To our best knowledge, this has not been discussed in the literature. In this case, in addition to the flexural rigidity, one needs to consider the in-plane rigidity (stiffness), along with the tensor of elastic constants corresponding to the coupling between the in-plane and out-of-plane deformations. The physical plate is subjected to initial stress, initial body forces (normal and tangential) and moments. Also, we allow the physical plate to undergo finite in-plane deformations while remaining flat. The virtual plate is assumed to have uniform elastic parameters with vanishing pre-stress and body forces (and moments). The governing equations of the virtual plate are then mapped to those of the physical plate. There are two sets of governing equations (i.e., in-plane and out-of-plane) that need to be simultaneously transformed under a cloaking map. The pre-stress, initial body forces and moments, along with the elastic parameters of the physical plate are then determined and the balance of angular momentum in the physical problem is discussed. Calculating the stress and couple-stress in the physical problem, and determining the boundary conditions and the restrictions they impose on the cloaking map are discussed. In an example, we show that cloaking a circular hole in a plate with both the in-plane and out-of-plane displacements using a general radial cloaking map is not possible. Finally, we prove that if the tensor of elastic constants corresponding to the coupling between the in-plane and out-of-plane deformations is positive definite for the virtual plate, then cloaking is not possible for any hole covered by a cloak with an arbitrary shape.

This chapter is structured as follows: In §8.2, we tersely review some elements of the differential geometry and the kinematics of embedded hypersurfaces in three-dimensional Riemannian manifolds. The governing equations of nonlinear elastic shells are derived in §8.3. We then obtain the equations of motion of linear initially stress-free and pre-stressed elastic shells (and thus, plates) by linearizing the nonlinear shell equations. In §8.4 the transformation cloaking problem in elastic shells is formulated. We discuss how the geometry of the physical and virtual shells as well as the boundary conditions in the physical and virtual problems are related under a cloaking transformation. In §8.4.1, we formulate the transformation cloaking problem for Kirchhoff-Love plates and study the example of a circular cloak assuming a generic radial cloaking map. Next, we relax the pure bending assumption and formulate the transformation cloaking problem for an elastic plate in the presence of the in-plane and out-of-plane displacements in §8.4.2. We solve the example of a circular cloak using a radial cloaking map. Finally, we discuss the obstruction to transformation cloaking for a hole of arbitrary shape. Note that the results of this section have been previously reported in our published work [42].

8.2 Differential geometry of surfaces

In this section, we briefly review some concepts of the geometry of two-dimensional embedded surfaces in three-manifolds and the kinematics of elastic shells (see [253, 47, 66] for more detailed discussions).

8.2.1 Geometry of an embedded surface

Consider an orientable Riemannian manifold $(\mathcal{B}, \bar{\mathbf{G}})$, and let $(\mathcal{H}, \mathbf{G})$ be an orientable two-dimensional Riemannian submanifold of $(\mathcal{B}, \bar{\mathbf{G}})$ such that $\mathbf{G}$ is the induced metric on $\mathcal{H}$, i.e., $\mathbf{G} = \bar{\mathbf{G}}|_{\mathcal{H}}$. Let us denote the space of smooth vector fields on $\mathcal{H}$ and $\mathcal{B}$ by $\mathcal{X}(\mathcal{H})$ and $\mathcal{X}(\mathcal{B})$, respectively. First, we note that for any $X \in \mathcal{B}$,

$$T_X\mathcal{B} = T_X\mathcal{H} \otimes (T_X\mathcal{H})^{\perp}\,, \tag{8.7}$$

that is any vector field $\mathbf{W} \in T_X\mathcal{B}$ may be uniquely written as sum of a vector $\mathbf{W}^{\top} \in T_X\mathcal{H}$ (which is tangent to $\mathcal{H}$) and a vector $\mathbf{W}^{\perp} := \mathbf{W} - \mathbf{W}^{\top}$ (which is normal to $\mathcal{H}$, i.e., $\mathbf{W}^{\perp} \in (T_X\mathcal{H})^{\perp}$). Given the Levi-Civita connection $\bar{\nabla}$ of the Riemannian manifold $(\mathcal{B}, \bar{\mathbf{G}})$, the induced Levi-Civita connection on $(\mathcal{H}, \mathbf{G})$ is denoted by ∇ and is given by

$$\nabla_{\mathbf{X}}\mathbf{Y} = \bar{\nabla}_{\bar{\mathbf{X}}}\bar{\mathbf{Y}} - \bar{\mathbf{G}}(\bar{\nabla}_{\bar{\mathbf{X}}}\bar{\mathbf{Y}}, \bar{\mathbf{N}})\mathbf{N}\,, \quad \forall\, \mathbf{X}\,, \mathbf{Y} \in \mathcal{X}(\mathcal{H})\,, \tag{8.8}$$

where $\bar{\mathbf{X}}, \bar{\mathbf{Y}} \in \mathcal{X}(\mathcal{B})$ are arbitrary local extensions of $\mathbf{X}$ and $\mathbf{Y}$ (i.e., $\bar{\mathbf{X}}(X) = \mathbf{X}(X), \bar{\mathbf{Y}}(X) = \mathbf{Y}(X)\,, \forall X \in \mathcal{H}$), and $\mathbf{N} \in \mathcal{X}(\mathcal{H})^{\perp}$ is the smooth unit normal vector field to $\mathcal{H}$ with $\bar{\mathbf{N}}$ being its local extension. The second fundamental form of the hypersurface $\mathcal{H}$ is a bilinear and symmetric mapping

$\mathbf{B} : \mathcal{X}(\mathcal{H}) \times \mathcal{X}(\mathcal{H}) \to \mathcal{X}(\mathcal{H})^{\perp}$ given by

$$\mathbf{B}(\mathbf{X}, \mathbf{Y}) = \bar{\nabla}_{\bar{\mathbf{X}}} \bar{\mathbf{Y}} - \nabla_{\mathbf{X}} \mathbf{Y}\,, \quad \forall\, \mathbf{X}, \mathbf{Y} \in \mathcal{X}(\mathcal{H})\,. \tag{8.9}$$

The set of symmetric $\binom{0}{2}$-tensors on $\mathcal{H}$ is indicated by $\Gamma(S^2T^*\mathcal{H})$. The second fundamental form can be considered as the symmetric tensor $\mathbf{B} \in \Gamma(S^2T^*\mathcal{H})$ defined with a slight abuse of notation as[3]

$$\mathbf{B}(\mathbf{X}, \mathbf{Y}) = \bar{\mathbf{G}}(\bar{\nabla}_{\bar{\mathbf{X}}} \bar{\mathbf{Y}}, \bar{\mathbf{N}}) = -\bar{\mathbf{G}}(\bar{\nabla}_{\bar{\mathbf{X}}} \bar{\mathbf{N}}, \bar{\mathbf{Y}})\,, \quad \forall\, \mathbf{X}, \mathbf{Y} \in \mathcal{X}(\mathcal{H})\,, \tag{8.11}$$

which is known as the *Weingarten formula*. The covariant derivative extends to tensors in $S^2T^*\mathcal{H}$ in a natural way ($\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathcal{X}(\mathcal{H})$):

$$(\nabla_{\mathbf{X}} \mathbf{A})(\mathbf{Y}, \mathbf{Z}) = \mathbf{X}(\mathbf{A}(\mathbf{Y}, \mathbf{Z})) - \mathbf{A}(\nabla_{\mathbf{X}} \mathbf{Y}, \mathbf{Z}) - \mathbf{A}(\mathbf{Y}, \nabla_{\mathbf{X}} \mathbf{Z})\,, \quad \forall \mathbf{A} \in \Gamma(S^2T^*\mathcal{H})\,. \tag{8.12}$$

The curvature tensor $\bar{\boldsymbol{\mathcal{R}}}$ associated with the Riemannian manifold $(\mathcal{B}, \bar{\mathbf{G}})$ is defined as

$$\bar{\boldsymbol{\mathcal{R}}}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{T}) = \bar{\mathbf{G}}(\bar{\mathbf{R}}(\mathbf{X}, \mathbf{Y})\mathbf{Z}, \mathbf{T})\,, \quad \forall\, \mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{T} \in \mathcal{X}(\mathcal{B})\,, \tag{8.13}$$

where $\bar{\mathbf{R}}(\mathbf{X}, \mathbf{Y}) : \mathcal{X}(\mathcal{B}) \to \mathcal{X}(\mathcal{B})$, the Riemann curvature tensor, is given by

$$\bar{\mathbf{R}}(\mathbf{X}, \mathbf{Y})\mathbf{Z} = \bar{\nabla}_{\mathbf{Y}} \bar{\nabla}_{\mathbf{X}} \mathbf{Z} - \bar{\nabla}_{\mathbf{X}} \bar{\nabla}_{\mathbf{Y}} \mathbf{Z} + \bar{\nabla}_{[\mathbf{X}, \mathbf{Y}]} \mathbf{Z}\,, \quad \mathbf{Z} \in \mathcal{X}(\mathcal{B})\,, \tag{8.14}$$

[3] A linear connection is said to be compatible with a metric $\bar{\mathbf{G}}$ on the manifold provided that

$$\bar{\nabla}_{\bar{\mathbf{X}}} \langle\!\langle \bar{\mathbf{Y}}, \bar{\mathbf{Z}} \rangle\!\rangle_{\bar{\mathbf{G}}} = \langle\!\langle \bar{\nabla}_{\bar{\mathbf{X}}} \bar{\mathbf{Y}}, \bar{\mathbf{Z}} \rangle\!\rangle_{\bar{\mathbf{G}}} + \langle\!\langle \bar{\mathbf{Y}}, \bar{\nabla}_{\bar{\mathbf{X}}} \bar{\mathbf{Z}} \rangle\!\rangle_{\bar{\mathbf{G}}}\,, \tag{8.10}$$

where $\langle\!\langle ., . \rangle\!\rangle_{\bar{\mathbf{G}}}$ is the inner product induced by the metric $\bar{\mathbf{G}}$. It is straightforward to show that $\bar{\nabla}$ is compatible with $\bar{\mathbf{G}}$ if and only if $\bar{\nabla}\bar{\mathbf{G}} = \mathbf{0}$, or, in components

$$\bar{G}_{AB|C} = \frac{\partial \bar{G}_{AB}}{\partial X^C} - \bar{\Gamma}^S{}_{CA} \bar{G}_{SB} - \bar{\Gamma}^S{}_{CB} \bar{G}_{AS} = 0.$$

On any Riemannian manifold $(\mathcal{B}, \bar{\mathbf{G}})$ the Levi-Civita connection is the unique linear connection $\bar{\nabla}^{\bar{\mathbf{G}}}$ that is compatible with $\bar{\mathbf{G}}$ and is symmetric (torsion-free). Note that the metric compatibility of $\bar{\nabla}$ (see (8.10)) and the fact that $\bar{\mathbf{G}}(\bar{\mathbf{N}}, \bar{\mathbf{Y}}) = 0$, are used in deriving the second equality in (8.11).

with $[\mathbf{X},\mathbf{Y}]$ indicating the Lie brackets of $\mathbf{X}$ and $\mathbf{Y}$, i.e., in components, $[\mathbf{X},\mathbf{Y}]^a = \frac{\partial Y^a}{\partial x^b}X^b - \frac{\partial X^a}{\partial x^b}Y^b$. The Riemann curvature tensor has the following components

$$\bar{\mathcal{R}}_{ABCD} = \left(\bar{\Gamma}^K{}_{AC,B} - \bar{\Gamma}^K{}_{BC,A} + \bar{\Gamma}^L{}_{AC}\bar{\Gamma}^K{}_{BL} - \bar{\Gamma}^L{}_{BC}\bar{\Gamma}^K{}_{AL}\right)\bar{G}_{KD}\,. \tag{8.15}$$

Note that the components of the Christoffel symbols of the connection $\bar{\nabla}$ read

$$\bar{\Gamma}^A{}_{BC} = \frac{1}{2}\bar{G}^{AK}(\bar{G}_{KB,C} + \bar{G}_{KC,B} - \bar{G}_{BC,K})\,, \tag{8.16}$$

where a comma preceding a subscript denotes partial differentiation with respect to that subscript. The curvature tensor $\boldsymbol{\mathcal{R}}$ for the hyperplane is similarly defined by the induced metric $\mathbf{G}$ and connection ∇ on $\mathcal{H}$. The Gauss equation reads

$$\bar{\boldsymbol{\mathcal{R}}}(\bar{\mathbf{X}},\bar{\mathbf{Y}},\bar{\mathbf{Z}},\bar{\mathbf{T}}) = \boldsymbol{\mathcal{R}}(\mathbf{X},\mathbf{Y},\mathbf{Z},\mathbf{T}) - \mathbf{B}(\mathbf{X},\mathbf{Z})\mathbf{B}(\mathbf{Y},\mathbf{T}) + \mathbf{B}(\mathbf{X},\mathbf{T})\mathbf{B}(\mathbf{Y},\mathbf{Z})\,. \tag{8.17}$$

The Codazzi-Mainardi equation is written as

$$\bar{\boldsymbol{\mathcal{R}}}(\bar{\mathbf{X}},\bar{\mathbf{Y}},\bar{\mathbf{Z}},\mathbf{N}) = (\nabla_{\mathbf{Y}}\mathbf{B})(\mathbf{X},\mathbf{Z}) - (\nabla_{\mathbf{X}}\mathbf{B})(\mathbf{Y},\mathbf{Z})\,. \tag{8.18}$$

Consider a local coordinate chart $\{X^1, X^2, X^3\}$ for $(\mathcal{B},\bar{\mathbf{G}})$ such that $\{X^1, X^2\}$ is a local chart for $(\mathcal{H},\mathbf{G})$ and $\mathbf{N}$, the unit normal vector to $\mathcal{H}$, is in the direction $\partial/\partial X^3$ at any point of the hypersurface. The metric of $\mathcal{B}$ in this coordinate chart has the following representation

$$\bar{\mathbf{G}}(X) = \begin{bmatrix} \bar{G}_{11}(X) & \bar{G}_{12}(X) & \bar{G}_{13}(X) \\ \bar{G}_{12}(X) & \bar{G}_{22}(X) & \bar{G}_{23}(X) \\ \bar{G}_{13}(X) & \bar{G}_{23}(X) & \bar{G}_{33}(X) \end{bmatrix}. \tag{8.19}$$

The first fundamental form of the hypersurface is given by

$$\mathbf{G}(X) = \bar{\mathbf{G}}(X)|_{\mathcal{H}} = \begin{bmatrix} \bar{G}_{11}(X) & \bar{G}_{12}(X) \\ \bar{G}_{12}(X) & \bar{G}_{22}(X) \end{bmatrix}, \quad \forall\, X \in \mathcal{H}\,. \tag{8.20}$$

Therefore, the Christoffel symbols associated with the induced connection ∇ read

$$\Gamma^A{}_{BC} = \frac{1}{2} G^{AK} \left(G_{KB,C} + G_{KC,B} - G_{BC,K} \right), \quad A, B, C, K = 1, 2. \tag{8.21}$$

The second fundamental form of the hypersurface is obtained as

$$B_{AB}(X) = \bar{\Gamma}^3{}_{AB}(X)\,, \quad A, B = 1, 2\,,\ \forall\, X \in \mathcal{H}\,, \tag{8.22}$$

where $\bar{\Gamma}^A{}_{BC}$ are the Christoffel symbols of the Levi-Civita connection $\bar{\nabla}$. Thus[4]

$$B_{AB}(X) = -\frac{1}{2} \frac{\partial \bar{G}_{AB}}{\partial X^3}\bigg|_{\mathcal{H}}(X), \quad A, B = 1, 2\,, \quad \forall X \in \mathcal{H}\,. \tag{8.23}$$

The fundamental theorem of surface theory implies that the geometry of the hypersurface $\mathcal{H}$ is fully determined by its first and second fundamental forms $\mathbf{G}$ and $\mathbf{B}$, respectively.[5] The Gauss and Codazzi-Mainardi equations given in (8.17) and (8.18) in the local coordinate chart $\{X^1, X^2, X^3\}$ reduce in

[4]Note that for $X \in \mathcal{H}$, the metric (8.19) has the following representation

$$\bar{\mathbf{G}}(X) = \begin{bmatrix} \bar{G}_{11}(X) & \bar{G}_{12}(X) & 0 \\ \bar{G}_{12}(X) & \bar{G}_{22}(X) & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which using (8.16) and (8.22), implies (8.23).

[5]Note that the first and the second fundamental forms of $\mathcal{H}$ can be expressed in terms of the metric of the embedding space $\mathcal{B}$ given by $\begin{bmatrix} \bar{G}_{11} & \bar{G}_{12} \\ \bar{G}_{12} & \bar{G}_{22} \end{bmatrix}(X)$, $X \in \mathcal{B}$, which in turn, fully characterizes the geometry of $\mathcal{H}$.

components to

$$
\begin{aligned}
\bar{\mathcal{R}}_{1212} - \mathcal{R}_{1212} &= B_{12}B_{12} - B_{11}B_{22}, \\
\bar{\mathcal{R}}_{1213} &= B_{11|2} - B_{12|1}, \\
\bar{\mathcal{R}}_{2123} &= B_{22|1} - B_{12|2},
\end{aligned}
\tag{8.24}
$$

where the covariant derivatives correspond to the Levi-Civita connection ∇ of $(\mathcal{H}, \mathbf{G})$ with Christoffel symbols $\Gamma^C{}_{AB}$. Note that in components: $B_{AB|C} = B_{AB,C} - \Gamma^K{}_{AC}B_{KB} - \Gamma^K{}_{BC}B_{AK}$.

8.2.2 Kinematics of shells

A shell is a 3D body whose thickness compared to its other dimensions is very small. Thus, it can be idealized as a two-dimensional Riemannian submanifold $(\mathcal{H}, \mathbf{G}, \mathbf{B})$ of the Riemannian manifold $(\mathcal{B}, \bar{\mathbf{G}})$.[6] Let us denote the ambient space by the Riemannian manifold $(\mathcal{S}, \bar{\mathrm{g}})$, where $\bar{\mathrm{g}}$ is the standard Euclidean metric. The shell is specified by $(\mathcal{H}, \mathbf{G}, \mathbf{B})$ and $(\varphi_t(\mathcal{H}), \mathrm{g}, \boldsymbol{\theta})$ in the reference and current configurations, respectively, where $\varphi_t : \mathcal{H} \to \mathcal{S}$ is a motion (or a deformation) of $\mathcal{H}$ in $\mathcal{S}$, and the first and the second fundamental forms of the deformed shell $\varphi_t(\mathcal{H})$ are denoted by $\mathrm{g} = \bar{\mathrm{g}}|_{\varphi_t(\mathcal{H})}$ and $\boldsymbol{\theta} \in \Gamma(S^2T^*\varphi(\mathcal{H}))$, respectively.[7] Let us denote the Levi-Civita connections of g and $\bar{\mathrm{g}}$ by ∇^{g} and $\bar{\nabla}^{\bar{\mathrm{g}}}$, respectively. The smooth unit normal vector field of $\varphi(\mathcal{H})$ is denoted by $\mathbf{n} \in \mathcal{X}(\varphi(\mathcal{H}))^{\perp}$. As the ambient space $\mathcal{S}$ is flat, the Gauss and Codazzi-Mainradi equations for the Riemannian hypersuface in its current configuration $(\varphi(\mathcal{H}), \mathrm{g})$ are written as

$$
\begin{aligned}
\mathsf{R}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w}) &= \boldsymbol{\theta}(\boldsymbol{x}, \boldsymbol{z})\boldsymbol{\theta}(\boldsymbol{y}, \boldsymbol{w}) - \boldsymbol{\theta}(\boldsymbol{x}, \boldsymbol{w})\boldsymbol{\theta}(\boldsymbol{y}, \boldsymbol{z}) \,, \\
(\nabla^{\mathrm{g}}_{\boldsymbol{x}}\boldsymbol{\theta})(\boldsymbol{y}, \boldsymbol{z}) &= (\nabla^{\mathrm{g}}_{\boldsymbol{y}}\boldsymbol{\theta})(\boldsymbol{x}, \boldsymbol{z}) \,,
\end{aligned}
\tag{8.25}
$$

[6]See [205] for another equivalent way of characterizing the configuration space of a plate.

[7]Note that in coordinates (x^1, x^2, x^3), for which x^3 is the outward normal direction, the second fundamental form of the deformed shell is expressed as

$$
\theta_{ab} = -\frac{1}{2}\frac{\partial \bar{g}_{ab}}{\partial x^3}\Big|_{\varphi(\mathcal{H})}(x), \quad a, b = 1, 2, \quad \forall x \in \varphi(\mathcal{H}) \,.
$$

for any smooth vector fields $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w} \in \mathcal{X}(\varphi(\mathcal{H}))$, and R is the Riemannian curvature of the deformed surface $\varphi(\mathcal{H})$.

The deformation gradient is defined as the tangent map of $\varphi_t : \mathcal{H} \to \varphi_t(\mathcal{H})$, i.e., $\mathbf{F}(X) = T\varphi_t(X) : T_X\mathcal{H} \to T_{\varphi_t(X)}\varphi_t(\mathcal{H})$. The right Cauchy-Green deformation tensor is defined as the pull-back of the induced metric on the deformed hypersurface $\varphi_t(\mathcal{H})$ by φ_t, i.e., $\mathbf{C}^\flat = \varphi_t^*\mathbf{g}$ (in components, $C_{AB} = F^a{}_A F^b{}_B g_{ab}$, $A, B = 1, 2$). The Jacobian of the deformation J relates the deformed and undeformed Riemannian surface elements as $ds(\varphi_t(X), \mathbf{g}) = JdS(X, \mathbf{G})$, where

$$J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}} \det \mathbf{F}. \tag{8.26}$$

The material (or Lagrangian) strain tensor $\mathbf{E} \in \Gamma(S^2T^*\mathcal{H})$ is defined as $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{G})$. Alternatively, one can define the spatial strain tensor as $\mathbf{e} = \frac{1}{2}(\mathbf{g} - \mathbf{c})$, where $\mathbf{c} = \varphi_{t*}\mathbf{G}$. Note that the material and spatial strain measures are intrinsic in the sense that they only capture changes in the first fundamental form of the surface. Therefore, one needs to define the extrinsic strain measures in order to take into account variations in the second fundamental form of the surface as well. The extrinsic material strain tensor is given by

$$\mathbf{H} = \frac{1}{2}\left(\mathbf{\Theta} - \mathbf{B}\right), \tag{8.27}$$

where $\mathbf{\Theta} = \varphi_t^*\boldsymbol{\theta}$. Similarly, the spatial extrinsic strain tensor can be defined as $\mathbf{h} = \frac{1}{2}\left(\boldsymbol{\theta} - \boldsymbol{\beta}\right)$, where $\boldsymbol{\beta} = \varphi_{t*}\mathbf{B}$. It is straightforward to see that $\mathbf{h} = \varphi_{t*}\mathbf{H}$.

One can pull back the spatial Gauss and Codazzi equations (8.25) by φ and obtain the following shell compatibility equations (see [254, 66])

$$\begin{aligned} \mathsf{R}^{\mathbf{C}}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}) &= \mathbf{\Theta}(\mathbf{X}, \mathbf{Z})\mathbf{\Theta}(\mathbf{Y}, \mathbf{W}) - \mathbf{\Theta}(\mathbf{X}, \mathbf{W})\mathbf{\Theta}(\mathbf{Y}, \mathbf{Z}), \\ (\nabla^{\mathbf{C}}_{\mathbf{X}}\mathbf{\Theta})(\mathbf{Y}, \mathbf{Z}) &= (\nabla^{\mathbf{C}}_{\mathbf{Y}}\mathbf{\Theta})(\mathbf{X}, \mathbf{Z}), \end{aligned} \tag{8.28}$$

where $\nabla^{\mathbf{C}}$ and $\mathsf{R}^{\mathbf{C}}$ are the Levi-Civita connection and the Riemannian curvature of the Riemannian (convected) manifold $(\mathcal{H}, \mathbf{C})$. Note that (8.28) gives the necessary and locally sufficient conditions for the existence (and uniqueness up to isometries of $\mathcal{S} = \mathbb{R}^3$ when $\mathcal{H}$ is simply-connected) of a deformation mapping (configuration) of $\mathcal{H}$ with given deformation tensors $\mathbf{C}$ and $\mathbf{\Theta}$. In the coordinate chart $\{X^1, X^2, X^3\}$, (8.28) is written as

$$\begin{aligned}
&\mathsf{R}^{\mathbf{C}}_{1212} = \Theta_{11}\Theta_{22} - \Theta_{12}\Theta_{12}, \\
&\Theta_{11||2} = \Theta_{12||1}, \\
&\Theta_{22||1} = \Theta_{12||2},
\end{aligned} \tag{8.29}$$

where $||$ denotes the covariant derivative with respect to the Levi-Civita connection $\nabla^{\mathbf{C}}$.[8] Recall that one may write the components of $\mathbf{C}^\flat$ and $\mathbf{\Theta}^\flat$ in terms of the deformation mapping $\varphi : \mathcal{H} \to \mathbb{R}^3$ in a local chart $\{X, Y\}$ on $\mathcal{H}$ as follows

$$\begin{aligned}
&C_{AB} = \varphi_{,A} \cdot \varphi_{,B} \,, \\
&\Theta_{AB} = \varphi_{,AB} \cdot \frac{\varphi_{,X} \times \varphi_{,Y}}{\|\varphi_{,X} \times \varphi_{,Y}\|} \,,
\end{aligned} \tag{8.30}$$

where the dot product, the cross product, and the standard norm in $\mathbb{R}^3$ are denoted by "$\cdot$", "$\times$", and "$\|\cdot\|$", respectively. Note that expressing the first and the second fundamental forms in terms of the motion φ, the shell compatibility equations (8.28) are trivially satisfied.

8.2.3 Velocity and acceleration vector fields

The material velocity is defined as the mapping $\mathbf{V} : \mathcal{H} \times \mathbb{R} \to T\mathcal{S}$ such that $\mathbf{V}(X,t) = \frac{\partial \varphi_X(t)}{\partial t} \in T_{\varphi_X(t)}\mathcal{S}$, $\forall X \in \mathcal{H}$. The material velocity can be decomposed at any point $X \in \mathcal{H}$ as $\mathbf{V}_X(t) = \mathbf{V}^\top_X(t) + \mathbf{V}^\perp_X(t)$, where $\mathbf{V}^\top_X(t) \in T_{\varphi_t(X)}\varphi_t(\mathcal{H})$ is parallel and $\mathbf{V}^\perp_X(t) \in \big(T_{\varphi_t(X)}\varphi_t(\mathcal{H})\big)^\perp$ is normal to the shell in its current configuration. The spatial velocity is defined as $\mathbf{v}(x,t) = \mathbf{V}(\varphi_t^{-1}(x), t)$, which is a vector field on $\varphi_t(\mathcal{H})$ at a fixed time t. Note that at any time t the deformation mapping $\varphi_t : \mathcal{H} \to \mathcal{S}$ is a smooth embedding of the shell into the ambient space. The mapping, $\varphi : \mathcal{H} \times \mathbb{R} \to \mathcal{S}$, on the other hand, is not, in general, injective (see [66] for a detailed discussion). The tangent of φ is

[8] See [227, P.313] for a discussion on the compatibility equations of a Cosserat shell with deformable directors.

written as

$$T_{(X,t)}\varphi = \begin{bmatrix} \frac{\partial \varphi^1}{\partial X^1} & \frac{\partial \varphi^1}{\partial X^2} & \frac{\partial \varphi^1}{\partial t} \\ \frac{\partial \varphi^2}{\partial X^1} & \frac{\partial \varphi^2}{\partial X^2} & \frac{\partial \varphi^2}{\partial t} \\ \frac{\partial \varphi^3}{\partial X^1} & \frac{\partial \varphi^3}{\partial X^2} & \frac{\partial \varphi^3}{\partial t} \end{bmatrix} . \tag{8.31}$$

Notice that the first two columns in (8.31) represent the tangent map of the smooth embedding φ_t, and thus, the tangent map $T_X\varphi_t : T_X\mathcal{H} \to T_{\varphi_t(X)}\mathcal{S}$ is injective. This, in turn, implies that the first two columns are linearly independent. Also, note that the third column represents the material velocity, i.e., $\mathbf{V}(X,t) = \frac{\partial \varphi_X(t)}{\partial t}$. Therefore, $T_{(X,t)}\varphi$ is full rank if and only if $\mathbf{V}(X,t)$ is not purely tangential. The material acceleration is defined as

$$\mathbf{A}(X,t) = D^{\bar{\mathbf{g}}}_{\varphi_t}\mathbf{V}(X,t) = \bar{\nabla}^{\bar{\mathbf{g}}}_{\mathbf{V}}\mathbf{V} \, , \tag{8.32}$$

where $D^{\bar{\mathbf{g}}}_{\varphi_t}$ denotes the covariant time derivative along the curve $\varphi_X(t)$. Note that the material velocity is only defined on the surface $\varphi_t(\mathcal{H})$, but one needs to compute the covariant derivative of the velocity along the motion in the ambient space to find the normal and tangential components of the acceleration. Thus, one cannot, in general, compute $\bar{\nabla}^{\bar{\mathbf{g}}}_{\mathbf{V}}\mathbf{V}$ to find the acceleration components unless it is possible to define a local extension of $\mathbf{V}$ to an open neighborhood on $\mathcal{S}$ (see [22] for a detailed discussion). Provided that φ has a nonsingular tangent $T_{(X,t)}\varphi$ at some $(X_o, t_o) \in \mathcal{H} \times \mathbb{R}$, then by the inverse function theorem φ is a local diffeomorphism at (X_o, t_o). Therefore, one may construct a local extension vector field $\boldsymbol{\mathcal{V}}$ on $\mathcal{S}$ such that $\boldsymbol{\mathcal{V}}(\varphi(X,t)) = \mathbf{V}(X,t) = \mathbf{v}(\varphi(X,t),t)$ in some open neighborhood of (X_o, t_o).[9] Thus, we may proceed with computing the acceleration as follows

$$\mathbf{A}(X,t) = D^{\bar{\mathbf{g}}}_{\varphi_t}\mathbf{V}(X,t) := \bar{\nabla}^{\bar{\mathbf{g}}}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}(\varphi(X,t)) \, . \tag{8.33}$$

[9]Note that $T_{(X,t)}\varphi$ (cf. (8.31)) is injective, and hence, the local extension vector field always exists, unless $\mathbf{V}$ is purely tangential, i.e., $\mathbf{V}^\perp = \mathbf{0}$. In this case, however, one does not need a local extension to compute the acceleration unambiguously as $\mathbf{V} = \mathbf{V}^\top$, and hence, $\mathbf{A}(X,t) = \bar{\nabla}^{\bar{\mathbf{g}}}_{\mathbf{V}}\mathbf{V} = \bar{\nabla}^{\bar{\mathbf{g}}}_{\mathbf{V}^\top}\mathbf{V}^\top$.

Decomposing the velocity into the normal and parallel components, one obtains

$$\mathbf{A}(X,t) = \bar{\nabla}^{\bar{\mathbf{g}}}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}^{\top} + \bar{\nabla}^{\bar{\mathbf{g}}}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}^{\perp}\,. \tag{8.34}$$

Note that

$$\left[\boldsymbol{\mathcal{V}},\boldsymbol{\mathcal{V}}^{\top}\right] = \bar{\nabla}^{\bar{\mathbf{g}}}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}^{\top} - \bar{\nabla}^{\bar{\mathbf{g}}}_{\boldsymbol{\mathcal{V}}^{\top}}\boldsymbol{\mathcal{V}}\,. \tag{8.35}$$

Thus

$$\bar{\nabla}^{\bar{\mathbf{g}}}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}^{\top} = \left[\boldsymbol{\mathcal{V}},\boldsymbol{\mathcal{V}}^{\top}\right] + \bar{\nabla}^{\bar{\mathbf{g}}}_{\boldsymbol{\mathcal{V}}^{\top}}\boldsymbol{\mathcal{V}}^{\perp} + \bar{\nabla}^{\bar{\mathbf{g}}}_{\boldsymbol{\mathcal{V}}^{\top}}\boldsymbol{\mathcal{V}}^{\top}\,. \tag{8.36}$$

Using the relation (8.9) in the ambient space, one has

$$\bar{\nabla}^{\bar{\mathbf{g}}}_{\boldsymbol{\mathcal{V}}^{\top}}\boldsymbol{\mathcal{V}}^{\top} = \nabla^{\mathbf{g}}_{\boldsymbol{\mathcal{V}}^{\top}}\boldsymbol{\mathcal{V}}^{\top} + \boldsymbol{\theta}(\boldsymbol{\mathcal{V}}^{\top},\boldsymbol{\mathcal{V}}^{\top})\mathbf{n}\,. \tag{8.37}$$

Also, letting $\boldsymbol{\mathcal{V}}^{\perp} = \mathcal{V}^{\perp}\mathbf{n}$, one may write

$$\bar{\nabla}^{\bar{\mathbf{g}}}_{\boldsymbol{\mathcal{V}}^{\top}}\boldsymbol{\mathcal{V}}^{\perp} = \left(\boldsymbol{\mathcal{V}}^{\top}\left[\mathcal{V}^{\perp}\right]\right)\mathbf{n} + \mathcal{V}^{\perp}\bar{\nabla}^{\bar{\mathbf{g}}}_{\boldsymbol{\mathcal{V}}^{\top}}\mathbf{n} = \left(\mathbf{d}\mathcal{V}^{\perp}\cdot\boldsymbol{\mathcal{V}}^{\top}\right)\mathbf{n} - \mathcal{V}^{\perp}\mathbf{g}^{\sharp}\cdot\boldsymbol{\theta}\cdot\boldsymbol{\mathcal{V}}^{\top}, \tag{8.38}$$

where use was made of the relation (8.11) in the ambient space. Note that

$$\bar{\nabla}^{\bar{\mathbf{g}}}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}^{\perp} = \bar{\nabla}^{\bar{\mathbf{g}}}_{\boldsymbol{\mathcal{V}}}(\mathcal{V}^{\perp}\mathbf{n}) = \frac{d\mathcal{V}^{\perp}}{dt}\mathbf{n} + \mathcal{V}^{\perp}\bar{\nabla}^{\bar{\mathbf{g}}}_{\boldsymbol{\mathcal{V}}}\,\mathbf{n}\,. \tag{8.39}$$

Using the metric compatibility of $\bar{\mathbf{g}}$ and (8.11), one can show that[10]

$$\mathcal{V}^{\perp}\bar{\nabla}^{\bar{\mathbf{g}}}_{\boldsymbol{\mathcal{V}}}\,\mathbf{n} = -\mathcal{V}^{\perp}\mathbf{g}^{\sharp}\cdot\boldsymbol{\theta}\cdot\boldsymbol{\mathcal{V}}^{\top} - \mathcal{V}^{\perp}(d\mathcal{V}^{\perp})^{\sharp}\,. \tag{8.40}$$

[10]Let $\mathbf{W}\in\mathcal{X}(\varphi_t(\mathcal{H}))$ be an arbitrary vector field defined in a neighborhood containing (X_o,t_o). Thus, $\bar{\mathbf{g}}(\boldsymbol{\mathcal{V}}^{\perp},\mathbf{W}) = 0$, and from (8.10), $\bar{\mathbf{g}}(\bar{\nabla}^{\bar{\mathbf{g}}}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}^{\perp},\mathbf{W}) = -\bar{\mathbf{g}}(\boldsymbol{\mathcal{V}}^{\perp},\bar{\nabla}^{\bar{\mathbf{g}}}_{\boldsymbol{\mathcal{V}}}\mathbf{W})$. Note that

$$\begin{aligned}\bar{\nabla}^{\bar{\mathbf{g}}}_{\boldsymbol{\mathcal{V}}}\mathbf{W} &= [\boldsymbol{\mathcal{V}},\mathbf{W}] + \bar{\nabla}^{\bar{\mathbf{g}}}_{\mathbf{W}}\boldsymbol{\mathcal{V}} = [\boldsymbol{\mathcal{V}},\mathbf{W}] + \bar{\nabla}^{\bar{\mathbf{g}}}_{\mathbf{W}}\boldsymbol{\mathcal{V}}^{\top} + \bar{\nabla}^{\bar{\mathbf{g}}}_{\mathbf{W}}\boldsymbol{\mathcal{V}}^{\perp} \\ &= [\boldsymbol{\mathcal{V}},\mathbf{W}] + \nabla^{\mathbf{g}}_{\mathbf{W}}\boldsymbol{\mathcal{V}}^{\top} + \boldsymbol{\theta}(\boldsymbol{\mathcal{V}}^{\top},\mathbf{W})\mathbf{n} + (d\mathcal{V}^{\perp}\cdot\mathbf{W})\mathbf{n} + \mathcal{V}^{\perp}\bar{\nabla}^{\bar{\mathbf{g}}}_{\mathbf{W}}\mathbf{n}\,.\end{aligned}$$

Thus, noting that $[\boldsymbol{\mathcal{V}},\mathbf{W}]$, $\nabla^{\mathbf{g}}_{\mathbf{W}}\boldsymbol{\mathcal{V}}^{\top}$, $\mathcal{V}^{\perp}\bar{\nabla}^{\bar{\mathbf{g}}}_{\mathbf{W}}\mathbf{n}\in\mathcal{X}(\varphi_t(\mathcal{H}))$ one concludes that $\bar{\mathbf{g}}(\boldsymbol{\mathcal{V}}^{\perp},\bar{\nabla}^{\bar{\mathbf{g}}}_{\boldsymbol{\mathcal{V}}}\mathbf{W}) = \mathcal{V}^{\perp}\boldsymbol{\theta}(\boldsymbol{\mathcal{V}}^{\top},\mathbf{W}) + \mathcal{V}^{\perp}(d\mathcal{V}^{\perp}\cdot\mathbf{W})$, which by arbitrariness of $\mathbf{W}$ together with $\bar{\mathbf{g}}(\bar{\nabla}^{\bar{\mathbf{g}}}_{\boldsymbol{\mathcal{V}}}\boldsymbol{\mathcal{V}}^{\perp},\mathbf{W}) = -\bar{\mathbf{g}}(\boldsymbol{\mathcal{V}}^{\perp},\bar{\nabla}^{\bar{\mathbf{g}}}_{\boldsymbol{\mathcal{V}}}\mathbf{W})$ implies (8.40).

Hence, replacing $\boldsymbol{\mathcal{V}}$ by $\mathbf{V}(X,t) = \boldsymbol{\mathcal{V}}(\varphi(X,t))$ the parallel and normal components of the material acceleration are written as

$$\begin{aligned}\mathbf{A}^\top &= \nabla^{\mathbf{g}}_{\mathbf{V}^\top}\mathbf{V}^\top + \left[\mathbf{V},\mathbf{V}^\top\right] - 2V^\perp \mathbf{g}^\sharp\cdot\boldsymbol{\theta}\cdot\mathbf{V}^\top - V^\perp (dV^\perp)^\sharp\,,\\ \mathbf{A}^\perp &= \left(\frac{dV^\perp}{dt} + \boldsymbol{\theta}(\mathbf{V}^\top,\mathbf{V}^\top) + dV^\perp\cdot\mathbf{V}^\top\right)\mathbf{n}\,.\end{aligned} \tag{8.41}$$

8.3 The governing equations of motion of shells

In this section we use Hamilton's principle of least action to derive the governing equations of a nonlinear elastic shell. We then linearize the governing equations and obtain the equations of motion of a linear elastic shell.

The kinetic energy density per unit surface area is written as

$$T = \frac{1}{2}\rho\,\bar{\mathbf{g}}(\dot{\varphi},\dot{\varphi})\,, \tag{8.42}$$

where ρ is the material surface mass density. The elastic energy density (per unit surface area) of the shell is written as[11]

$$W = W\left(X,\mathbf{C},\boldsymbol{\Theta},\mathbf{G},\mathbf{B}\right)\,. \tag{8.43}$$

Let $\{x^1,x^2,x^3\}$ be a local coordinate chart for the ambient space such that at any point of the deformed hypersurface, $\{x^1,x^2\}$ is a local chart for $(\varphi(\mathcal{H}),\mathbf{g})$, and the vector field $\mathbf{n}$ normal to $\varphi(\mathcal{H})$ is tangent to the coordinate curve x^3. Thus, the Lagrangian density (per unit surface area) is defined in this coordinate chart as

$$\mathcal{L}(X,\dot{\varphi},\mathbf{C},\boldsymbol{\Theta},\mathbf{G},\mathbf{B}) = \frac{1}{2}\rho\, g_{ab}(\dot{\varphi}^\top)^a(\dot{\varphi}^\top)^b + \frac{1}{2}\rho(\dot{\varphi}^\perp)^2 - W\left(X,\mathbf{C},\boldsymbol{\Theta},\mathbf{G},\mathbf{B}\right)\,. \tag{8.44}$$

[11] Consider a surface embedded in the ambient space such that the embedding is given as $\varphi:\mathcal{H}\to\mathcal{S}$, where for the sake of simplicity one can assume that $\mathcal{S}=\mathbb{R}^3$. Note that $x^a = \partial\varphi^a/\partial X^A$, where $a=1,2,3$ and $A=1,2$. The fundamental theorem of surface theory proved by Bonnet [255] implies that the surface geometry (up to rigid body motions) is completely characterized by the induced first and second fundamental forms $\mathbf{C}$ and $\boldsymbol{\Theta}$. Therefore, the surface energy density must depend on $\mathbf{C}$ and $\boldsymbol{\Theta}$.

The action functional is defined as

$$S(\varphi) = \int_{t_0}^{t_1} \int_{\mathcal{H}} \mathcal{L}\Big(X, \dot{\varphi}(X,t), \mathbf{C}(X,t), \mathbf{\Theta}(X,t), \mathbf{G}(X), \mathbf{B}(X)\Big) dA(X) dt \,, \tag{8.45}$$

where $dA(X) = \sqrt{\det \mathbf{G}(X)}\, dX^1 \wedge dX^2$ is the Riemannian area element. We use the Lagrange–d'Alembert principle to take into account the contribution of non-conservative body forces and body moments associated with the variations of the position $\delta\varphi$, and orientation $\delta\boldsymbol{\mathcal{N}}$ (where $\boldsymbol{\mathcal{N}} = \mathbf{n} \circ \varphi$ is the normal vector field characterizing the orientation of the deformed surface element). The Lagrange-d'Alembert principle [256] states that the physical motion φ of $\mathcal{H}$ satisfies

$$\delta S(\varphi) + \int_{t_0}^{t_1} \int_{\mathcal{H}} \Big(\boldsymbol{\mathfrak{B}} \cdot \delta\varphi + \boldsymbol{\mathfrak{L}} \cdot \delta\boldsymbol{\mathcal{N}} \Big) \rho \, dA dt = 0 \,, \tag{8.46}$$

where $\boldsymbol{\mathfrak{B}}$ and $\boldsymbol{\mathfrak{L}}$, respectively, denote the external body forces and body moments. Note that

$$\delta S(\varphi) = \int_{t_0}^{t_1} \int_{\mathcal{H}} \left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} \cdot \delta\dot{\varphi} + \frac{\partial \mathcal{L}}{\partial \mathbf{C}} : \delta\mathbf{C} + \frac{\partial \mathcal{L}}{\partial \mathbf{\Theta}} : \delta\mathbf{\Theta} \right) dA dt \,. \tag{8.47}$$

Let φ_ϵ be a one-parameter family of motions such that $\varphi_{0,t} = \varphi_t$, where, for fixed X and t, we denote $\varphi_{\epsilon,t}(X) := \varphi_\epsilon(X,t)$. The variation of the motion is defined as

$$\delta\varphi(X,t) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \varphi_{\epsilon,t}(X) \in T_{\varphi_t(X)}\mathcal{S} \,. \tag{8.48}$$

The material velocity is given by $\dot{\varphi}_\epsilon = \frac{\partial \varphi_{\epsilon,t}(X)}{\partial t}$. Note that $\dot{\varphi}_\epsilon \in T_{\varphi_{\epsilon,t}(X)}\mathcal{S}$, i.e., for fixed time t and $X \in \mathcal{H}$, if ϵ varies, the velocity lies in different tangent spaces, and thus, a covariant derivative along the curve $\epsilon \to \varphi_{\epsilon,t}(X)$ should be used to find the variation of the material velocity. Thus

$$\delta\dot{\varphi} = \bar{\nabla}^{\bar{\mathbf{g}}}_{\frac{\partial}{\partial \epsilon}} \frac{\partial \varphi_{\epsilon,t}(X)}{\partial t}\Big|_{\epsilon=0} = \bar{\nabla}^{\bar{\mathbf{g}}}_{\frac{\partial}{\partial t}} \frac{\partial \varphi_{\epsilon,t}(X)}{\partial \epsilon}\Big|_{\epsilon=0} = \bar{\nabla}^{\bar{\mathbf{g}}}_{t} \, \delta\varphi_t(X) = D_{\varphi_X(t)} \delta\varphi \,, \tag{8.49}$$

where the symmetry lemma of Riemannian geometry was used to obtain the second equality (see [217, 257]). The variation of the right Cauchy-Green deformation tensor $\mathbf{C}^\flat_\epsilon = \varphi^*_{\epsilon,t}\mathbf{g}_\epsilon \circ \varphi_{\epsilon,t}$, where

$\mathbf{C}^\flat_\epsilon \in \Gamma(S^2T^*\mathcal{H})$, is obtained as

$$\delta\mathbf{C}^\flat = \frac{d}{d\epsilon}\mathbf{C}^\flat_\epsilon\Big|_{\epsilon=0} = \frac{d}{d\epsilon}(\varphi^*_{\epsilon,t}\mathbf{g}_\epsilon)\Big|_{\epsilon=0} = \varphi^*_t(\mathbf{L}_{\delta\varphi}\mathbf{g})\,. \tag{8.50}$$

Hence, knowing that $\mathbf{L}_{\delta\varphi}\mathbf{g} = \mathbf{L}_{\delta\varphi^\top}\mathbf{g} - 2\,\delta\varphi^\perp\,\boldsymbol{\theta}$ (see, e.g., [258, 259]), one obtains (see Appendix D.1 for the details of this derivation)

$$\delta\mathbf{C}^\flat = \varphi^*_t\,\mathbf{L}_{\delta\varphi^\top}\mathbf{g} - 2\,\delta\varphi^\perp\,\boldsymbol{\Theta}\,. \tag{8.51}$$

In components

$$\delta C_{AB} = F^a{}_A\,\delta\varphi^\top_{a|B} + F^b{}_B\,\delta\varphi^\top_{b|A} - 2\,\delta\varphi^\perp\,\Theta_{AB}\,. \tag{8.52}$$

Noting that $\boldsymbol{\Theta}^\flat_\epsilon \in \Gamma(S^2T^*\mathcal{H})$, the variation of $\boldsymbol{\Theta}^\flat_\epsilon = \varphi^*_{\epsilon,t}\boldsymbol{\theta}_\epsilon \circ \varphi_{\epsilon,t}$ is calculated as

$$\delta\boldsymbol{\Theta}^\flat = \frac{d}{d\epsilon}\boldsymbol{\Theta}^\flat_\epsilon\Big|_{\epsilon=0} = \frac{d}{d\epsilon}(\varphi^*_{\epsilon,t}\boldsymbol{\theta}_\epsilon)\Big|_{\epsilon=0} = \varphi^*_t(\mathbf{L}_{\delta\varphi}\boldsymbol{\theta})\,. \tag{8.53}$$

For a flat ambient space, the Lie derivative of the second fundamental form is expressed as (see [259])

$$\mathbf{L}_{\delta\varphi}\boldsymbol{\theta} = \mathbf{L}_{\delta\varphi^\top}\boldsymbol{\theta} - \delta\varphi^\perp\mathbf{III} + \mathrm{Hess}_{\delta\varphi^\perp}\,, \tag{8.54}$$

where $\mathbf{III}$ is the third fundamental form of the deformed hyepersurface, and $\mathrm{Hess}_{\delta\varphi^\perp}$ denotes the Hessian of $\delta\varphi^\perp$ (when viewed as a scalar-valued function on $\varphi_t(\mathcal{H})$). The third fundamental form and $\mathrm{Hess}_{\delta\varphi^\perp}$ are given for $\boldsymbol{x},\boldsymbol{y} \in \mathcal{X}(\varphi(\mathcal{H}))$ as

$$\mathbf{III}(\boldsymbol{x},\boldsymbol{y}) = \mathbf{g}\left(\bar{\nabla}^{\bar{\mathbf{g}}}_{\boldsymbol{x}}\,\mathbf{n}, \bar{\nabla}^{\bar{\mathbf{g}}}_{\boldsymbol{y}}\,\mathbf{n}\right), \tag{8.55}$$

$$\mathrm{Hess}_{\delta\varphi^\perp}(\boldsymbol{x},\boldsymbol{y}) = \mathbf{g}\left(\bar{\nabla}^{\bar{\mathbf{g}}}_{\boldsymbol{x}}\,(\mathbf{d}\,\delta\varphi^\perp)^\sharp, \boldsymbol{y}\right), \tag{8.56}$$

where $\mathbf{d}$ and $\sharp$, respectively, denote the exterior derivative and the sharp operator for raising indices.

Thus, from (8.53) and (8.54), one obtains

$$\delta\boldsymbol{\Theta}^\flat = \varphi_t^*\mathbf{L}_{\delta\varphi^\top}\boldsymbol{\theta} - \delta\varphi^\perp\,\varphi_t^*\mathbf{III} + \varphi_t^*\mathrm{Hess}_{\delta\varphi^\perp}, \tag{8.57}$$

and in components

$$\begin{aligned}\delta\Theta_{AB} =& F^a{}_A F^b{}_B\,\theta_{ab|c}\,(\delta\varphi^\top)^c + F^a{}_A\theta_{ac}(\delta\varphi^\top)^c{}_{|B} + F^b{}_B\theta_{bc}(\delta\varphi^\top)^c{}_{|A} \\ &- \delta\varphi^\perp F^a{}_A F^b{}_B\theta_{ac}\theta_{bd}\,g^{cd} + F^b{}_A\left(\frac{\partial\,\delta\varphi^\perp}{\partial\,x^b}\right)_{|B}.\end{aligned} \tag{8.58}$$

The unit normal vector field, $\boldsymbol{\mathcal{N}}_\epsilon = \mathbf{n}_\epsilon\circ\varphi_{\epsilon,t}$, $\mathbf{n}_\epsilon\in\mathcal{X}(\varphi_{\epsilon,t}(\mathcal{H}))^\perp$, lies in $T_{\varphi_{\epsilon,t}(X)}\mathcal{S}$, for fixed time t and $X\in\mathcal{H}$, and thus, its variation is given by its covariant derivative along the curve $\varphi_\epsilon(X,t)$ evaluated at $\epsilon=0$, and therefore (see Appendix D.1)[12]

$$\delta\boldsymbol{\mathcal{N}} = \frac{d}{d\epsilon}\boldsymbol{\mathcal{N}}_\epsilon\Big|_{\epsilon=0} = \bar{\nabla}^{\bar{\mathbf{g}}}_{\frac{\partial}{\partial\epsilon}}\boldsymbol{\mathcal{N}}_\epsilon\Big|_{\epsilon=0} = D_{\varphi_\epsilon(X,t)}\boldsymbol{\mathcal{N}}_\epsilon\Big|_{\epsilon=0} = \bar{\nabla}^{\bar{\mathbf{g}}}_{\delta\varphi}\boldsymbol{\mathcal{N}} = \bar{\nabla}^{\bar{\mathbf{g}}}_{\delta\varphi^\top}\boldsymbol{\mathcal{N}} - (\mathbf{d}\,\delta\varphi^\perp)^\sharp\,. \tag{8.59}$$

Using (8.11), in components one has

$$\delta\mathcal{N}^a = -(\delta\varphi^\top)^c\,\theta^a{}_c - \left(\frac{\partial\,\delta\varphi^\perp}{\partial\,x^b}\right)g^{ab}\,. \tag{8.60}$$

It is straightforward to see that the variation of the ambient space metric vanishes as $\bar{\mathbf{g}}$ is compatible with the connection, i.e., $\delta\bar{\mathbf{g}}\circ\varphi = D_{\varphi_X(t)}\bar{\mathbf{g}}\circ\varphi = \bar{\nabla}^{\bar{\mathbf{g}}}_{\delta\varphi}\,\bar{\mathbf{g}} = \mathbf{0}$, where $\varphi_X(t)=\varphi(X,t)$ for fix X.

Using Hamilton's principle (cf. (8.46)), one obtains the following Euler-Lagrange equations (see

[12] $D_{\varphi_\epsilon(X,t)}$ denotes the covariant derivative along the curve $\varphi_\epsilon(X,t)$.

Appendix D.2 for the details of the derivation)

$$\begin{aligned}
&\rho(\mathfrak{B}^\top)_a - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial(\dot{\varphi}^\top)^a}\right) - \rho\mathfrak{L}_b\theta^b{}_a - 2\left(\frac{\partial \mathcal{L}}{\partial C_{AB}}F^b{}_B g_{ab}\right)_{|A} \\
&- \left(\frac{\partial \mathcal{L}}{\partial \Theta_{AB}}F^b{}_A\right)_{|B}\theta_{ba} - \left(\frac{\partial \mathcal{L}}{\partial \Theta_{AB}}F^b{}_A\theta_{ba}\right)_{|B} = 0\,, \qquad (8.61a)\\
&\rho\mathfrak{B}^\perp - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\varphi}^\perp}\right) + \left[\rho\mathfrak{L}_b g^{ab}(F^{-1})^A{}_a\right]_{|A} - 2\frac{\partial \mathcal{L}}{\partial C_{AB}}F^a{}_A F^b{}_B\theta_{ab} \\
&- \frac{\partial \mathcal{L}}{\partial \Theta_{AB}}F^a{}_A F^b{}_B\theta_{ac}\theta_{bd}\, g^{cd} + \left[\left(\frac{\partial \mathcal{L}}{\partial \Theta_{AB}}F^b{}_A\right)_{|B}(F^{-1})^D{}_b\right]_{|D} = 0\,, \qquad (8.61b)
\end{aligned}$$

where $\mathfrak{B}^\top$ and $\mathfrak{B}^\perp$ are the tangential and normal external body forces, respectively, and $\mathfrak{L}$ is the external body moments. Note that $(8.25)_2$ was used in deriving (8.61a). The boundary conditions read

$$\left[2F^a{}_B\left(\frac{\partial \mathcal{L}}{\partial C_{AB}}g_{ac} + \frac{\partial \mathcal{L}}{\partial \Theta_{AB}}\theta_{ac}\right)\right]\mathsf{T}_A = 0\,, \qquad (8.62)$$

$$(F^{-1})^A{}_a\left[\rho\mathfrak{L}_b g^{ab} + \left(\frac{\partial \mathcal{L}}{\partial \Theta_{CB}}F^a{}_C\right)_{|B}\right]\mathsf{T}_A = 0\,, \qquad (8.63)$$

$$\frac{\partial \mathcal{L}}{\partial \Theta_{AB}}F^a{}_B\mathsf{T}_A = 0\,, \qquad (8.64)$$

where T is the outward vector field normal to $\partial\mathcal{H}$. Note that

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial(\dot{\varphi}^\top)^a} = \frac{d}{dt}\frac{\partial T}{\partial(\dot{\varphi}^\top)^a} = \rho\, g_{ac}(A^\top)^c\,, \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}^\perp} = \frac{d}{dt}\frac{\partial T}{\partial \dot{\varphi}^\perp} = \rho A^\perp\,. \qquad (8.65)$$

Remark 8.3.1. In order to prescribe non-vanishing boundary conditions on $\partial\mathcal{H}$, the Lagrange–d'Alembert principle may be modified. Let $\boldsymbol{\Upsilon}$, $\mathbf{Q}$, and $\mathscr{M}$ be the boundary surface traction, boundary shear force, and boundary moment, respectively. The Lagrange–d'Alembert's principle is modified to read

$$
\begin{aligned}
\delta S(\varphi) &+ \int_{t_0}^{t_1}\int_{\mathcal{H}} \Big(\mathfrak{B}\cdot\delta\varphi + \mathfrak{L}\cdot\delta\boldsymbol{\mathcal{N}}\Big)\rho\, dA dt \\
&+ \int_{t_0}^{t_1}\int_{\partial\mathcal{H}} J\Big(\Upsilon^a g_{ab}(\delta\varphi^\top)^b + Q\delta\varphi^\perp + \mathscr{M}^a\delta\varphi^\perp{}_{,A}(F^{-1})^A{}_a\Big) dL dt = 0\,.
\end{aligned} \tag{8.66}
$$

Therefore, in this case the boundary conditions are obtained as

$$
\left[2F^a{}_B\left(\frac{\partial\mathcal{L}}{\partial C_{AB}}g_{ac} + \frac{\partial\mathcal{L}}{\partial\Theta_{AB}}\theta_{ac}\right)\right]\mathsf{T}_A = J g_{bc}\Upsilon^b\,, \tag{8.67}
$$

$$
(F^{-1})^A{}_a\left[\rho\mathfrak{L}_b g^{ab} + \left(\frac{\partial\mathcal{L}}{\partial\Theta_{CB}}F^a{}_C\right)_{|B}\right]\mathsf{T}_A = JQ\,, \tag{8.68}
$$

$$
\frac{\partial\mathcal{L}}{\partial\Theta_{AB}}F^a{}_B\mathsf{T}_A = J\mathscr{M}^a\,. \tag{8.69}
$$

Let us introduce the following tensors

$$
\mathbf{P} = 2\mathbf{F}\frac{\partial W}{\partial\mathbf{C}}\,,\qquad \mathsf{M} = \mathbf{F}\frac{\partial W}{\partial\boldsymbol{\Theta}}\,, \tag{8.70}
$$

where $\mathbf{P}$ is the first Piola-Kirchhoff stress tensor, and M is the couple-stress tensor. Therefore, based on the symmetries of the independent objective measures of strain, i.e., the right Cauchy-Green tensor and the extrinsic deformation tensor (or, equivalently, the symmetries of the first and the second fundamental forms of the deformed shell, i.e., $\mathbf{g}$ and $\boldsymbol{\theta}$, respectively), one has the following symmetries

$$
P^{[aA}F^{b]}{}_A = 0\,,\qquad \mathsf{M}^{[aA}F^{b]}{}_A = 0\,. \tag{8.71}
$$

In terms of these tensors, the Euler-Lagrange equations (8.61) are rewritten as

$$
\left(P^{aA} + \theta^a{}_b\mathsf{M}^{bA}\right)_{|A} + \mathsf{M}^{bA}{}_{|A}\theta^a{}_b + \rho(\mathfrak{B}^\top)^a - \rho\,\theta^a{}_b\mathfrak{L}^b = \rho(A^\top)^a\,, \tag{8.72}
$$

$$
\left(P^{aA} + \theta^a{}_b\mathsf{M}^{bA}\right)F^c{}_A\theta_{ac} - \left[\mathsf{M}^{aA}{}_{|A}(F^{-1})^B{}_a\right]_{|B} + \rho\mathfrak{B}^\perp + \left[\rho\mathfrak{L}^a(F^{-1})^A{}_a\right]_{|A} = \rho A^\perp\,, \tag{8.73}
$$

and the boundary conditions (8.62) are expressed as

$$\left[g_{ab}P^{aA} + 2\theta_{ab}\mathsf{M}^{aA}\right]\mathsf{T}_A = 0\,, \tag{8.74}$$

$$(F^{-1})^A{}_a\left[\rho\mathfrak{L}^a - \mathsf{M}^{aB}{}_{|B}\right]\mathsf{T}_A = 0\,, \tag{8.75}$$

$$\mathsf{M}^{aA}\mathsf{T}_A = 0\,. \tag{8.76}$$

The tangential and normal (shear) tractions are given by

$$(T^{\top})^a = \left[P^{aA} + 2\theta_{bc}\mathsf{M}^{bA}g^{ac}\right]\mathsf{T}_A\,, \tag{8.77}$$

$$T^{\perp} = -(F^{-1})^A{}_a\left[\rho\mathfrak{L}^a - \mathsf{M}^{aB}{}_{|B}\right]\mathsf{T}_A\,. \tag{8.78}$$

Remark 8.3.2. We can regard $\mathbf{C}$ as a function of $\mathbf{F}$ and the spatial metric g, and similarly, $\boldsymbol{\Theta}$ can be regarded as a function of $\mathbf{F}$ and $\boldsymbol{\theta}$. Therefore, we may set

$$\hat{W}\left(X, \mathbf{F}, \mathbf{g}, \boldsymbol{\theta}, \mathbf{G}, \mathbf{B}\right) = W\left(X, \mathbf{C}, \boldsymbol{\Theta}, \mathbf{G}, \mathbf{B}\right). \tag{8.79}$$

Note that from (8.79)

$$\frac{\partial \hat{W}}{\partial F^a{}_C} = \frac{\partial W}{\partial C_{AB}}\frac{\partial C_{AB}}{\partial F^a{}_C} + \frac{\partial W}{\partial \Theta_{AB}}\frac{\partial \Theta_{AB}}{\partial F^a{}_C}. \tag{8.80}$$

It is straightforward to show that

$$\begin{aligned}\frac{\partial C_{AB}}{\partial F^a{}_C} &= g_{ab}\delta^C{}_A F^b{}_B + g_{ac}F^c{}_A\delta^C{}_B,\\ \frac{\partial \Theta_{AB}}{\partial F^a{}_C} &= \theta_{ab}\delta^C{}_A F^b{}_B + \theta_{ac}F^c{}_A\delta^C{}_B.\end{aligned} \tag{8.81}$$

Thus

$$\frac{\partial \hat{W}}{\partial F^c{}_A} = P^{aA}g_{ac} + 2\mathsf{M}^{aA}\theta_{ac}\,. \tag{8.82}$$

Note also that

$$F^c{}_A\frac{\partial \hat{W}}{\partial F^a{}_A} = 2\frac{\partial \hat{W}}{\partial g_{bc}}g_{ab} + 2\frac{\partial \hat{W}}{\partial \theta_{bc}}\theta_{ab}\,. \tag{8.83}$$

Note also

$$
\begin{aligned}
\frac{\partial \hat{W}}{\partial g_{ab}} &= \frac{\partial W}{\partial C_{AB}} \frac{\partial C_{AB}}{\partial g_{ab}} = \frac{\partial W}{\partial C_{AB}} F^a{}_A F^b{}_B \,, \\
\frac{\partial \hat{W}}{\partial \theta_{ab}} &= \frac{\partial W}{\partial \Theta_{AB}} \frac{\partial \Theta_{AB}}{\partial \theta_{ab}} = \frac{\partial W}{\partial \Theta_{AB}} F^a{}_A F^b{}_B \,.
\end{aligned} \tag{8.84}
$$

The Cauchy stress tensor and the spatial couple-stress tensor are accordingly defined as[13]

$$
\begin{aligned}
\sigma^{ab} &= \frac{2}{J} \frac{\partial W}{\partial C_{AB}} F^a{}_A F^b{}_B = \frac{2}{J} \frac{\partial \hat{W}}{\partial g_{ab}} \,. \\
\mathcal{M}^{ab} &= \frac{1}{J} \frac{\partial W}{\partial \Theta_{AB}} F^a{}_A F^b{}_B = \frac{1}{J} \frac{\partial \hat{W}}{\partial \theta_{ab}} \,.
\end{aligned} \tag{8.85}
$$

Note that $P^{aA} = J(F^{-1})^A{}_b \sigma^{ab}$ and $\mathsf{M}^{aA} = J(F^{-1})^A{}_b \mathcal{M}^{ab}$.

Remark 8.3.3 (Spatial covariance of the energy density and Noether's theorem)**.** Let us define the following surface Lagrangian density

$$
\mathcal{L} = \hat{\mathcal{L}} \left(X, \varphi, \dot{\varphi}, \mathbf{F}, \mathbf{g}, \boldsymbol{\theta}, \mathbf{G}, \mathbf{B} \right) . \tag{8.86}
$$

Let us consider a flow ψ_s generated by a vector field $\mathbf{w}$, i.e.,

$$
\left. \frac{d}{ds} \right|_{s=0} \psi_s \circ \varphi = \mathbf{w} \circ \varphi. \tag{8.87}
$$

Note that ψ_s is a local diffeomorphism. We assume that $\mathbf{w}$ is tangential, i.e., $\mathbf{w} \in T_{\varphi(X)} \varphi(\mathcal{H})$. Let us assume that $\mathcal{L}$ is tangentially covariant, i.e., it is invariant for local diffeomorphisms ψ_s generated by arbitrary $\mathbf{w} \in T_{\varphi(X)} \varphi(\mathcal{H})$. Thus

$$
\hat{\mathcal{L}} \left(X, \psi_s \circ \varphi, \psi_{s*} \dot{\varphi}, \psi_{s*} \mathbf{F}, \psi_{s*} \mathbf{g}, \psi_{s*} \boldsymbol{\theta}, \mathbf{G}, \mathbf{B} \right) = \hat{\mathcal{L}} \left(X, \varphi, \dot{\varphi}, \mathbf{F}, \mathbf{g}, \boldsymbol{\theta}, \mathbf{G}, \mathbf{B} \right) . \tag{8.88}
$$

[13] Note that (8.71) is equivalent to σ^{ab} and $\mathcal{M}^{ab}$ being symmetric.

Taking derivative with respect of s from both sides and evaluating at $s = 0$ one obtains[14]

$$\frac{\partial \hat{\mathcal{L}}}{\partial \varphi^a} w^a + \frac{\partial \hat{\mathcal{L}}}{\partial \dot{\varphi}^a} w^a{}_{|b} \dot{\varphi}^b + F^b{}_A \frac{\partial \hat{\mathcal{L}}}{\partial F^a{}_A} w^a{}_{|b} - 2 \frac{\partial \hat{\mathcal{L}}}{\partial g_{cb}} g_{ca} w^a{}_{|b} - 2 \frac{\partial \hat{\mathcal{L}}}{\partial \theta_{cb}} \theta_{ca} w^a{}_{|b} = 0 \,. \tag{8.89}$$

Note that

$$\frac{\partial \hat{\mathcal{L}}}{\partial \dot{\varphi}^a} \dot{\varphi}^b - 2 \frac{\partial \hat{\mathcal{L}}}{\partial g_{cb}} g_{ca} = 2 \frac{\partial \hat{W}}{\partial g_{cb}} g_{ca} \,. \tag{8.90}$$

Therefore

$$\frac{\partial \hat{\mathcal{L}}}{\partial \varphi^a} w^a + \left[-F^b{}_A \frac{\partial \hat{W}}{\partial F^a{}_A} + 2 \frac{\partial \hat{W}}{\partial g_{cb}} g_{ca} + 2 \frac{\partial \hat{W}}{\partial \theta_{cb}} \theta_{ca} \right] w^a{}_{|b} = 0 \,. \tag{8.91}$$

Knowing that $\mathbf{w}$ is arbitrary one concludes that

$$\frac{\partial \hat{\mathcal{L}}}{\partial \varphi^a} = 0, \quad F^b{}_A \frac{\partial \hat{W}}{\partial F^a{}_A} = 2 \frac{\partial \hat{W}}{\partial g_{bc}} g_{ac} + 2 \frac{\partial \hat{W}}{\partial \theta_{bc}} \theta_{ac} \,. \tag{8.92}$$

Note that $(8.92)_2$ is identical to (8.83), which can be written as

$$F^b{}_A \frac{\partial \hat{W}}{\partial F^a{}_A} = J \sigma^{bc} g_{ac} + 2 J \mathcal{M}^{bc} \theta_{ac} \,. \tag{8.93}$$

Noting that $\rho = J \varrho$, where ϱ is the spatial mass density, conservation of mass for shells is given by

$$\dot{\varrho} + \varrho \frac{\dot{J}}{J} = 0 \,. \tag{8.94}$$

[14]Note that in the local coordinate chart $\{x^1, x^2, x^3\}$, for which x^3 is the outward normal direction

$$\theta_{ab} = -\frac{1}{2} \frac{\partial \bar{g}_{ab}}{\partial x^3} \Big|_{\varphi(\mathcal{H})} \,, \quad a, b = 1, 2 \,.$$

Thus

$$(\psi_{s*}\theta)_{a'b'} = -\frac{1}{2} \frac{\partial \left((T\psi_s^{-1})^a{}_{a'} (T\psi_s^{-1})^b{}_{b'} \bar{g}_{ab} \right)}{\partial x^3} = (T\psi_s^{-1})^a{}_{a'} (T\psi_s^{-1})^b{}_{b'} \left(-\frac{1}{2} \frac{\partial \bar{g}_{ab}}{\partial x^3} \right) = (T\psi_s^{-1})^a{}_{a'} (T\psi_s^{-1})^b{}_{b'} \theta_{ab} \,, \quad a', b' = 1, 2 \,.$$

Note also that $(T\psi_s^{-1})^a{}_{a'} = -\delta^b{}_{a'} \delta^a{}_{b'} (T\psi_s)^{b'}{}_b$.

Using the identity $\frac{d}{dt}\left[\det \mathbf{K}(t)\right] = \det \mathbf{K}(t)\mathrm{tr}\left[\mathbf{K}^{-1}(t)\frac{d}{dt}\mathbf{K}(t)\right]$, one has

$$\frac{\dot{J}}{J} = \frac{1}{2}\mathrm{tr}_{\mathbf{C}^\flat}\left(\frac{d}{dt}\varphi^*\mathbf{g}\right), \tag{8.95}$$

where the trace is calculated using the metric $\mathbf{C}^\flat$. From (8.51), one obtains

$$\varphi_*\frac{d\varphi^*\mathbf{g}}{dt} = \mathbf{L}_{\mathbf{v}^\top}\mathbf{g} - 2v^\perp\boldsymbol{\theta}\,, \tag{8.96}$$

and thus

$$\mathrm{tr}_{\mathbf{C}^\flat}\left(\frac{d}{dt}\varphi^*\mathbf{g}\right) = 2\,\mathrm{div}\,\mathbf{v}^\top - 2v^\perp\mathrm{tr}\boldsymbol{\theta}\,. \tag{8.97}$$

Therefore, using (8.94), (8.95), and (8.97), one finds the spatial local form of the conservation of mass as[15]

$$\dot{\varrho} + \varrho\,\mathrm{div}\,\mathbf{v}^\top - \varrho v^\perp\mathrm{tr}\boldsymbol{\theta} = 0\,. \tag{8.98}$$

8.3.1 The linearized governing equations

Next, we linearize the balance of linear and angular momenta about a motion $\mathring{\varphi}$. We assume that the reference motion is an isometric embedding of an initially stress-free body into the Euclidean space, and thus, $\mathring{F}^a{}_A = \delta^a_A$, $\mathring{P}^{aA} = 0$, and $\mathring{\mathsf{M}}^{aA} = 0$. The tangential and normal displacement fields are defined in terms of the variation of the deformation map as

$$\mathbf{U}^\top(X,t) = \delta\varphi_t^\top, \qquad \mathbf{U}^\perp(X,t) = \delta\varphi_t^\perp. \tag{8.99}$$

Note that the deformation gradient is linearized as

$$\delta F^a{}_A = (\delta\varphi^\top)^a{}_{|A} - \mathring{\theta}^a{}_b\mathring{F}^b{}_A\delta\varphi^\perp = (U^\top)^a{}_{|A} - \mathring{\theta}^a{}_b\mathring{F}^b{}_A U^\perp = \mathring{F}^b{}_A(U^\top)^a{}_{|b} - \mathring{\theta}^a{}_b\mathring{F}^b{}_A U^\perp\,. \tag{8.100}$$

[15] Note that $\dot{\varrho} = \frac{\partial\varrho}{\partial t} + \nabla\varrho\cdot\mathbf{v}$.

Using (8.52), we have

$$\delta C_{AB} = \mathring{F}^a{}_A \, U^\top_{a|B} + \mathring{F}^b{}_B \, U^\top_{b|A} - 2\, U^\perp \, \mathring{\Theta}_{AB} \,, \tag{8.101}$$

where $\mathring{\Theta}_{AB} = \mathring{F}^a{}_A \mathring{F}^b{}_B \mathring{\theta}_{ab} = B_{AB}$. From (8.58), one writes

$$\begin{aligned}\delta\Theta_{AB} =& \mathring{F}^a{}_A \mathring{F}^b{}_B \, \mathring{\theta}_{ab|c} \, (U^\top)^c + \mathring{F}^a{}_A \mathring{\theta}_{ac} (U^\top)^c{}_{|B} + \mathring{F}^b{}_B \mathring{\theta}_{bc} (U^\top)^c{}_{|A} \\ & - U^\perp \mathring{F}^a{}_A \mathring{F}^b{}_B \mathring{\theta}_{ac} \mathring{\theta}_{bd} \, g^{cd} + \mathring{F}^b{}_A \left(\frac{\partial \, U^\perp}{\partial \, x^b} \right)_{|B} \, .\end{aligned} \tag{8.102}$$

Using (8.70), one obtains

$$\delta P^{aA} = 2 \mathring{F}^a{}_B \left[\frac{\partial^2 W}{\partial C_{AB} \partial C_{CD}} \delta C_{CD} + \frac{\partial^2 W}{\partial C_{AB} \partial \Theta_{CD}} \delta \Theta_{CD} \right] , \tag{8.103}$$

$$\delta \mathsf{M}^{aA} = \mathring{F}^a{}_B \left[\frac{\partial^2 W}{\partial \Theta_{AB} \partial \Theta_{CD}} \delta \Theta_{CD} + \frac{\partial^2 W}{\partial \Theta_{AB} \partial C_{CD}} \delta C_{CD} \right] . \tag{8.104}$$

Substituting (8.101) and (8.102) into (8.103) and (8.104) one obtains

$$\begin{aligned}\delta P^{aA} =& \, \mathbb{A}^{aAbB} \left(U^\top_{b|B} - U^\perp \, \mathring{\gamma}_{bB} \right) \\ & + \mathbb{B}^{aAbB} \left[\mathring{\theta}_{bc} (U^\top)^c{}_{|B} + \frac{1}{2} (U^\perp_{,b})_{|B} - \frac{1}{2} U^\perp \mathring{\theta}_{bc} \mathring{\gamma}_{dB} g^{cd} + \frac{1}{2} \mathring{F}^d{}_B \mathring{\theta}_{bd|c} (U^\top)^c \right] ,\end{aligned} \tag{8.105}$$

$$\begin{aligned}\delta \mathsf{M}^{aA} =& \, \mathbb{C}^{aAbB} \left[\mathring{\theta}_{bc} (U^\top)^c{}_{|B} + \frac{1}{2} (U^\perp_{,b})_{|B} - \frac{1}{2} U^\perp \mathring{\theta}_{bc} \mathring{\gamma}_{dB} g^{cd} + \frac{1}{2} \mathring{F}^d{}_B \mathring{\theta}_{bd|c} (U^\top)^c \right] \\ & + \frac{1}{2} \mathbb{B}^{bBaA} \left(U^\top_{b|B} - U^\perp \, \mathring{\gamma}_{bB} \right) \, ,\end{aligned} \tag{8.106}$$

where $\boldsymbol{\gamma}$ is a two-point tensor that in components is defined as $\gamma_{aB} = F^b{}_B \theta_{ab}$, and the shell elastic constants are defined as

$$\begin{aligned}\mathbb{A}^{aAbB} &= 4 \mathring{F}^a{}_M \mathring{F}^b{}_N \frac{\partial^2 W}{\partial C_{AM} \partial C_{BN}} \, , \quad \mathbb{B}^{aAbB} = 4 \mathring{F}^a{}_M \mathring{F}^b{}_N \frac{\partial^2 W}{\partial C_{AM} \partial \Theta_{BN}} \, , \\ \mathbb{C}^{aAbB} &= 2 \mathring{F}^a{}_M \mathring{F}^b{}_N \frac{\partial^2 W}{\partial \Theta_{AM} \partial \Theta_{BN}} \, .\end{aligned} \tag{8.107}$$

Therefore, the linearized governing equations of the shell are expressed in terms of three elasticity tensors. Note that the elastic constants satisfy the following symmetries: $\mathbb{A}^{aAbB} = \mathbb{A}^{bBaA}$ and $\mathbb{C}^{aAbB} = \mathbb{C}^{bBaA}$. The linearized normal and parallel components of the material acceleration are obtained using (8.41) as $\delta\mathbf{A}^{\top} = \ddot{\mathbf{U}}^{\top}$, and $\delta\mathbf{A}^{\perp} = \ddot{\mathbf{U}}^{\perp}$. Ignoring the body forces and body moments, from (8.72) the linearized balance of linear momentum is given by

$$\left(\delta P^{aA} + \mathring{\theta}^{a}{}_{b}\,\delta\mathsf{M}^{bA}\right)_{|A} + \delta\mathsf{M}^{bA}{}_{|A}\mathring{\theta}^{a}{}_{b} = \rho(\delta A^{\top})^{a}, \tag{8.108a}$$

$$\left(\delta P^{aA} + \mathring{\theta}^{a}{}_{b}\,\delta\mathsf{M}^{bA}\right)\mathring{\gamma}_{aA} - \left(\delta\mathsf{M}^{aA}{}_{|A}(\mathring{F}^{-1})^{B}{}_{a}\right)_{|B} = \rho\,\delta A^{\perp}\,. \tag{8.108b}$$

Similarly, the symmetry relations in (8.71) is linearized to read

$$\delta P^{[aA}\mathring{F}^{b]}{}_{A} = 0\,, \qquad \delta\mathsf{M}^{[aA}\mathring{F}^{b]}{}_{A} = 0\,. \tag{8.109}$$

Knowing that the normal and parallel components of the displacement field (and their gradients) are independent, from (8.109) one concludes that

$$\mathbb{A}^{[aAcB}\mathring{F}^{b]}{}_{A} = 0\,, \qquad \mathbb{B}^{cB[aA}\mathring{F}^{b]}{}_{A} = 0\,, \tag{8.110a}$$

$$\mathbb{B}^{[aAcB}\mathring{F}^{b]}{}_{A} = 0\,, \qquad \mathbb{C}^{[aAcB}\mathring{F}^{b]}{}_{A} = 0\,. \tag{8.110b}$$

Therefore, the elastic constants have these symmetries.

8.3.2 The linearized governing equations of pre-stressed shells

In this section, we derive the linearized governing equations of a pre-stressed elastic shell. Assume that the shell is initially stressed[16] such that the initial stress and couple-stress are, respectively, given

[16]Note that we do not explicitly specify the source of the initial stress or couple-stress. If the initial stress and couple-stress are due to elastic deformations, and the body has an energy function W with respect to its stress-free configuration, then one may express $\mathring{\mathbf{P}}$ and $\mathring{\mathsf{M}}$ as

$$\mathring{\mathbf{P}} = 2\mathring{\mathbf{F}}\frac{\partial W}{\partial \mathbf{C}}\Big|_{\mathring{\mathbf{F}}}\,, \qquad \mathring{\mathsf{M}} = \mathring{\mathbf{F}}\frac{\partial W}{\partial \boldsymbol{\Theta}}\Big|_{\mathring{\mathbf{F}}}\,.$$

by $\mathring{\mathbf{P}}$ and $\mathring{\mathsf{M}}$. Let the initial (normal and parallel) body forces and body moments be given by $\mathring{\mathfrak{B}}^\top$, $\mathring{\mathfrak{B}}^\perp$, and $\mathring{\mathfrak{L}}$, respectively. The shell must be in equilibrium in its initial configuration, i.e., the balance of linear and angular momenta must be satisfied, which read

$$\left(\mathring{P}^{aA} + \mathring{\theta}^a{}_b\mathring{\mathsf{M}}^{bA}\right)_{|A} + \mathring{\mathsf{M}}^{bA}{}_{|A}\mathring{\theta}^a{}_b + \rho(\mathring{\mathfrak{B}}^\top)^a - \rho\,\mathring{\theta}^a{}_b\mathring{\mathfrak{L}}^b = 0\,, \tag{8.111a}$$

$$\left(\mathring{P}^{aA} + \mathring{\theta}^a{}_b\mathring{\mathsf{M}}^{bA}\right)\mathring{F}^c{}_A\mathring{\theta}_{ac} - \left(\mathring{\mathsf{M}}^{aA}{}_{|A}(\mathring{F}^{-1})^B{}_a\right)_{|B} + \rho\mathring{\mathfrak{B}}^\perp + \left(\rho\mathring{\mathfrak{L}}^a(\mathring{F}^{-1})^A{}_a\right)_{|A} = 0\,, \tag{8.111b}$$

and

$$\mathring{P}^{[aA}\mathring{F}^{b]}{}_A = 0\,, \qquad \mathring{\mathsf{M}}^{[aA}\mathring{F}^{b]}{}_A = 0\,. \tag{8.112}$$

We next linearize the governing equations about the motion $\mathring{\varphi}$. The linearized balance of linear momentum reads (cf. (8.72) and (8.73))

$$\begin{aligned}&\left(\delta P^{aA} + \delta\theta^a{}_b\mathring{\mathsf{M}}^{bA} + \mathring{\theta}^a{}_b\delta\mathsf{M}^{bA}\right)_{|A} + \delta\mathsf{M}^{bA}{}_{|A}\mathring{\theta}^a{}_b + \mathring{\mathsf{M}}^{bA}{}_{|A}\delta\theta^a{}_b\\&\quad + \rho(\delta\mathfrak{B}^\top)^a - \rho\,\delta\theta^a{}_b\mathring{\mathfrak{L}}^b - \rho\,\mathring{\theta}^a{}_b\delta\mathfrak{L}^b = \rho(\ddot{U}^\top)^a\,,\end{aligned} \tag{8.113a}$$

$$\begin{aligned}&\left(\mathring{P}^{aA} + \mathring{\theta}^a{}_b\mathring{\mathsf{M}}^{bA}\right)\left(\delta F^c{}_A\mathring{\theta}_{ac} + \mathring{F}^c{}_A\delta\theta_{ac}\right) + \left(\delta P^{aA} + \delta\theta^a{}_b\mathring{\mathsf{M}}^{bA} + \mathring{\theta}^a{}_b\delta\mathsf{M}^{bA}\right)\mathring{F}^c{}_A\mathring{\theta}_{ac}\\&\quad - \left(\delta\mathsf{M}^{aA}{}_{|A}(\mathring{F}^{-1})^B{}_a\right)_{|B} - \left(\mathring{\mathsf{M}}^{aA}{}_{|A}(\delta F^{-1})^B{}_a\right)_{|B} + \rho\,\delta\mathfrak{B}^\perp + \left(\rho\,\delta\mathfrak{L}^a(\mathring{F}^{-1})^A{}_a\right)_{|A}\\&\quad + \left(\rho\,\mathring{\mathfrak{L}}^a(\delta F^{-1})^A{}_a\right)_{|A} = \rho\,\ddot{U}^\perp\,.\end{aligned} \tag{8.113b}$$

The symmetry relations in (8.71) are linearized to read

$$\delta P^{[aA}\mathring{F}^{b]}{}_A + \mathring{P}^{[aA}\delta F^{b]}{}_A = 0\,, \qquad \delta\mathsf{M}^{[aA}\mathring{F}^{b]}{}_A + \mathring{\mathsf{M}}^{[aA}\delta F^{b]}{}_A = 0\,. \tag{8.114}$$

Note that $(\delta F^{-1})^A{}_a = -(\mathring{F}^{-1})^B{}_a(\mathring{F}^{-1})^A{}_b\, \delta F^b{}_B = -(\mathring{F}^{-1})^B{}_a(\mathring{F}^{-1})^A{}_b \left[(U^\top)^b{}_{|B} - \mathring{\theta}^b{}_c \mathring{F}^c{}_B U^\perp\right]$, and using the relation $\Theta_{AB} = F^a{}_A F^b{}_B \theta_{ab}$, along with (8.102), one obtains

$$\delta\theta_{ab} = \mathring{\theta}_{ab|c}\,(U^\top)^c + U^\perp \mathring{\theta}_{ac}\mathring{\theta}_{bd}\, g^{cd} + (\mathring{F}^{-1})^B{}_b \left(\frac{\partial\, U^\perp}{\partial\, x^a}\right)_{|B} . \tag{8.115}$$

Knowing that $\mathfrak{B}^\perp = \bar{\mathrm{g}}(\mathfrak{B}, \mathcal{N})\mathcal{N}$, the normal component of the body force is linearized as (see also [82])

$$\delta\mathfrak{B}^\perp = \bar{\mathrm{g}}(\delta\mathfrak{B}, \mathcal{N})\mathcal{N} + \bar{\mathrm{g}}(\mathring{\mathfrak{B}}, \delta\mathcal{N})\mathcal{N} + \bar{\mathrm{g}}(\mathring{\mathfrak{B}}, \mathcal{N})\delta\mathcal{N}. \tag{8.116}$$

Assuming that the body force vector $\mathfrak{B}$ is fixed (dead load), i.e., $\delta\mathfrak{B} = \mathbf{0}$, we use (D.20) to obtain[17]

$$\delta\mathfrak{B}^\perp = \bar{\mathrm{g}}(\mathring{\mathfrak{B}}, \delta\mathcal{N})\mathcal{N} = -\mathbf{g}\cdot\mathring{\mathfrak{B}}^\top\cdot\mathring{\boldsymbol{\theta}}\cdot\mathbf{U}^\top - \mathbf{d}\,\mathbf{U}^\perp\cdot\mathring{\mathfrak{B}}^\top . \tag{8.117}$$

In components

$$\delta\mathfrak{B}^\perp = -g_{ab}(\mathring{\mathfrak{B}}^\top)^b(U^\top)^c\,\mathring{\theta}^a{}_c - (\mathring{\mathfrak{B}}^\top)^b \frac{\partial\, U^\perp}{\partial\, x^b} . \tag{8.118}$$

One obtains the linearized tangent component of the body force as (see also [260, P.457])

$$\delta\mathfrak{B}^\top = -\bar{\mathrm{g}}(\mathring{\mathfrak{B}}, \mathcal{N})\delta\mathcal{N} + \mathring{\mathfrak{B}}^\top\cdot\left(\nabla^{\mathbf{g}}\mathbf{U}^\top - U^\perp\mathring{\boldsymbol{\theta}}\right)\cdot\mathbf{g}\,, \tag{8.119}$$

which in components reads

$$(\delta\mathfrak{B}^\top)^a = \mathring{\mathfrak{B}}^\perp\left[(U^\top)^c\,\mathring{\theta}^a{}_c + \frac{\partial\, U^\perp}{\partial\, x^b} g^{ab}\right] + (\mathring{\mathfrak{B}}^\top)^c\left[(U^\top)_{c|b} - \mathring{\theta}_{cb}U^\perp\right] g^{ab} . \tag{8.120}$$

As $\mathring{\mathfrak{L}}$ is purely tangential, i.e., $\mathring{\mathfrak{L}}^\perp = \mathbf{0}$, the variation of the body moment is given in components by

$$\delta\mathfrak{L}^a = \mathring{\mathfrak{L}}^c\left[(U^\top)_{c|b} - \mathring{\theta}_{cb}U^\perp\right] g^{ab} . \tag{8.121}$$

[17] Note that the variation of the normal vector $\delta\mathcal{N}$ is purely tangential, and hence, so is the term $\bar{\mathrm{g}}(\mathring{\mathfrak{B}}, \mathcal{N})\delta\mathcal{N}$ in (8.116). In (8.117), with an abuse of notation we only consider the term in the normal direction.

The governing equations for pre-stressed plates. Let us reduce the governing equations of a pre-stressed shell to that of a pre-stressed plate by setting $\mathring{\theta}_{ab} = 0$. Thus, using (8.115), (8.118), (8.120), and (8.121), Eqs. (8.113) and (8.114) are simplified to read

$$\begin{aligned}&\left[\delta P^{aA} + g^{ac}(\mathring{F}^{-1})^B{}_b \left(\frac{\partial U^\perp}{\partial x^c}\right)_{|B} \mathring{\mathsf{M}}^{bA}\right]_{|A} + g^{ac}\mathring{\mathsf{M}}^{bA}{}_{|A}(\mathring{F}^{-1})^B{}_b \left(\frac{\partial U^\perp}{\partial x^c}\right)_{|B} \\ &\quad + \rho\mathring{\mathfrak{B}}^\perp g^{ab}\frac{\partial U^\perp}{\partial x^b} + \rho g^{ab}(\mathring{\mathfrak{B}}^\top)^c (U^\top)_{c|b} - \rho\, g^{ac}(\mathring{F}^{-1})^B{}_b\left(\frac{\partial U^\perp}{\partial x^c}\right)_{|B}\mathring{\mathfrak{L}}^b = \rho(\ddot{U}^\top)^a\,, \end{aligned} \tag{8.122a}$$

$$\begin{aligned}&\mathring{P}^{aA}\left(\frac{\partial U^\perp}{\partial x^a}\right)_{|A} - \left[(\mathring{F}^{-1})^B{}_a\delta\mathsf{M}^{aA}{}_{|A}\right]_{|B} + \left[\mathring{\mathsf{M}}^{aA}{}_{|A}\,(\mathring{F}^{-1})^C{}_a(\mathring{F}^{-1})^B{}_b(U^\top)^b{}_{|C}\right]_{|B} \\ &\quad - \rho(\mathring{\mathfrak{B}}^\top)^b\frac{\partial U^\perp}{\partial x^b} + \left(\rho\,\mathring{\mathfrak{L}}^c(U^\top)_{c|b}g^{ab}(\mathring{F}^{-1})^A{}_a\right)_{|A} - \left[\rho\,\mathring{\mathfrak{L}}^a\,(\mathring{F}^{-1})^B{}_a(\mathring{F}^{-1})^A{}_b(U^\top)^b{}_{|B}\right]_{|A} = \rho\,\ddot{U}^\perp\,, \end{aligned} \tag{8.122b}$$

and

$$\delta P^{[aA}\mathring{F}^{b]}{}_A + \mathring{P}^{[aA}(U^\top)^{b]}{}_{|A} = 0\,, \qquad \delta\mathsf{M}^{[aA}\mathring{F}^{b]}{}_A + \mathring{\mathsf{M}}^{[aA}(U^\top)^{b]}{}_{|A} = 0\,. \tag{8.123}$$

From (8.70), (8.105), and (8.106), one has

$$\delta P^{aA} = \left[\mathbb{A}^{aAbB} + \mathring{P}^{cA}(\mathring{F}^{-1})^B{}_c g^{ab}\right] U^\top_{b|B} + \frac{1}{2}\mathbb{B}^{aAbB}(U^\perp_{,b})_{|B}\,, \tag{8.124a}$$

$$2\,\delta\mathsf{M}^{aA} = \mathbb{C}^{aAbB}(U^\perp_{,b})_{|B} + \left[\mathbb{B}^{bBaA} + 2\mathring{\mathsf{M}}^{cA}(\mathring{F}^{-1})^B{}_c g^{ab}\right] U^\top_{b|B}\,. \tag{8.124b}$$

Recalling that the normal and parallel displacement gradients are independent from (8.123) one obtains

$$\begin{aligned}&\left(\mathbb{A}^{[aAcB} + \mathring{P}^{dA}(\mathring{F}^{-1})^B{}_d g^{[ac}\right)\mathring{F}^{b]}{}_A + \mathring{P}^{[aB}g^{cb]} = 0\,, \\ &\frac{1}{2}\left(\mathbb{B}^{cB[aA} + 2\mathring{\mathsf{M}}^{dA}(\mathring{F}^{-1})^B{}_d g^{[ac}\right)\mathring{F}^{b]}{}_A + \mathring{\mathsf{M}}^{[aB}g^{cb]} = 0\,, \\ &\mathbb{B}^{[aAcB}\mathring{F}^{b]}{}_A = 0\,, \qquad \mathbb{C}^{[aAcB}\mathring{F}^{b]}{}_A = 0\,. \end{aligned} \tag{8.125}$$

Note that if one uses the stress and the couple-stress symmetries for the finitely-deformed shell (8.112), the linearized symmetry relations in (8.125) will be simplified to (8.110).

Remark 8.3.4. Note that when $\mathbf{U}^\top = \mathbf{0}$, and in the absence of the initial body moments ($\mathring{\boldsymbol{\mathfrak{L}}} = \mathbf{0}$), the normal body forces ($\mathring{\boldsymbol{\mathfrak{B}}}^\perp = \mathbf{0}$), and the initial couple stress ($\mathring{\mathsf{M}} = \mathbf{0}$), we recover the governing equations of a classical plate discussed in [261, P. 379], [262, P. 289], and [244]. To see this, from (8.111) and (8.112), the nontrivial equilibrium equations for a classical plate in its initial configuration read

$$\mathring{P}^{aA}{}_{|A} + \rho(\mathring{\mathfrak{B}}^\top)^a = 0\,, \quad \text{and} \quad \mathring{P}^{[aA}\delta^{b]}{}_A = 0\,. \tag{8.126}$$

Also, the linearized governing equations (8.122) for a classical plate read

$$\delta P^{aA}{}_{|A} = 0\,, \quad \text{and} \quad \mathring{P}^{aA}\left(\frac{\partial U^\perp}{\partial x^a}\right)_{|A} - (\mathring{F}^{-1})^B{}_a\delta\mathsf{M}^{aA}{}_{|A|B} - \rho(\mathring{\mathfrak{B}}^\top)^b\frac{\partial U^\perp}{\partial x^b} = \rho\,\ddot{U}^\perp\,, \tag{8.127}$$

where from (8.124), $\delta P^{aA} = \frac{1}{2}\mathbb{B}^{aAbB}(U^\perp_{,b})_{|B}$, and $\delta\mathsf{M}^{aA} = \frac{1}{2}\mathbb{C}^{aAbB}(U^\perp_{,b})_{|B}$. Therefore, one obtains

$$\begin{aligned}-\frac{1}{2}(\mathring{F}^{-1})^B{}_a\Big[&\mathbb{C}^{aAbC}{}_{|A|B}(U^\perp_{,b})_{|C} + \mathbb{C}^{aAbC}{}_{|A}(U^\perp_{,b})_{|C|B} + \mathbb{C}^{aAbC}{}_{|B}(U^\perp_{,b})_{|C|A} \\ &+ \mathbb{C}^{aAbC}(U^\perp_{,b})_{|C|A|B}\Big] + \mathring{P}^{aA}\left(\frac{\partial U^\perp}{\partial x^a}\right)_{|A} - \rho(\mathring{\mathfrak{B}}^\top)^b\frac{\partial U^\perp}{\partial x^b} = \rho\,\ddot{U}^\perp\,,\end{aligned} \tag{8.128}$$

and

$$\mathbb{B}^{aAbB}{}_{|A}(U^\perp_{,b})_{|B} + \mathbb{B}^{aAbB}(U^\perp_{,b})_{|B|A} = 0\,. \tag{8.129}$$

Similarly, after some simplifications (8.123) implies that

$$\mathbb{B}^{[aAcB}\mathring{F}^{b]}{}_A = 0\,, \qquad \mathbb{C}^{[aAcB}\mathring{F}^{b]}{}_A = 0\,. \tag{8.130}$$

Note that in the case of pure bending deformations considered in the above-mentioned references, no dependence of the strain energy function on the Cauchy-Green deformation tensor is assumed, and thus, $\mathbb{B}$ vanishes. Therefore, (8.129) is trivially satisfied and (8.130) gives $\mathbb{C}^{[aAcB}\mathring{F}^{b]}{}_A = 0$, i.e., $\mathbb{C}$ must possess the minor symmetries.

8.4 Transformation cloaking in elastic plates

Let us consider an elastic plate $\mathcal{H}$ with a finite hole $\mathcal{E}$ (see Fig.8.1). In transformation cloaking one surrounds the hole with a cloaking device $\mathcal{C}$, which is an annular elastic plate such that the finite hole has negligible disturbance effects on the upcoming waves, i.e., as if the hole does not exist. The mass density and the elastic properties of the cloak are inhomogeneous and anisotropic, in general. Without loss of generality, we assume that in $\mathcal{H} \setminus \mathcal{C}$ the plate is homogeneous and isotropic. Motion of $\mathcal{H}$ is represented by a smooth map $\varphi_t : \mathcal{H} \to \mathcal{S}$. The cloaking transformation is a time-independent map $\xi : \mathring{\varphi}(\mathcal{H}) \to \mathring{\tilde{\varphi}}(\tilde{\mathcal{H}})$, which transforms the pre-stressed plate $\mathcal{H}$ (physical plate) with the initial stress $\mathring{\mathbf{P}}$ and the initial couple-stress $\mathring{\mathsf{M}}$ in its current configuration to a corresponding homogeneous and isotropic stress-free plate $\tilde{\mathcal{H}}$ (virtual plate) in its current configuration (see also [41]). The physical and virtual plates are endowed with their respective induced Euclidean metrics $\mathbf{G}$ and $\tilde{\mathbf{G}}$. The mapping ξ transforms the finite hole $\mathcal{E}$ to a very small hole $\tilde{\mathcal{E}}$ of radius ϵ and is assumed to be the identity in $\mathcal{H} \setminus \mathcal{C}$. The corresponding cloaking transformation in the reference configuration is denoted by Ξ. We also assume that the virtual plate has the same uniform and isotropic elastic properties as those of the physical plate outside the cloak ($\mathcal{H} \setminus \mathcal{C}$). Motion of the virtual plate is represented by $\tilde{\varphi}_t : \tilde{\mathcal{H}} \to \tilde{\mathcal{S}}$. The initial-boundary value problems corresponding to the motions $\varphi_t : \mathcal{H} \to \mathcal{S}$ and $\tilde{\varphi}_t : \tilde{\mathcal{H}} \to \tilde{\mathcal{S}}$ are called the physical and virtual problems, respectively. The deformation gradients corresponding to the physical and virtual problems are, respectively, denoted by $\mathbf{F} = T\varphi_t$ and $\tilde{\mathbf{F}} = T\tilde{\varphi}_t$. The tangent map of the referential and spatial cloaking transformations are, respectively, denoted by $T\Xi = \overset{\Xi}{\mathbf{F}}$ and $T\xi = \overset{\xi}{\mathbf{F}}$. The current configurations of the physical and virtual problems are required to be identical outside the cloaking region, i.e., in $\mathcal{H} \setminus \mathcal{C}$. This implies, in particular, that any elastic measurements performed in the spatial configurations of the virtual and physical plates are identical, and thus, they are indistinguishable by an observer positioned anywhere in $\mathcal{H} \setminus \mathcal{C}$.

Note that due to the structure of the governing equations of an elastic shell (and plate) (8.72) and (8.73), under a cloaking transformation, the (two-point) stress and couple-stress are not necessarily transformed using a Piola transformation[18] (unlike transformation cloaking in 3D elasticity [41]). This

[18] The Piola transformation of a vector (field) $\mathbf{w} \in T_{\varphi(X)}\mathcal{S}$ is a vector $\mathbf{W} \in T_X\mathcal{B}$ given by $\mathbf{W} = J\varphi^*\mathbf{w} = J\mathbf{F}^{-1}\mathbf{w}$. In

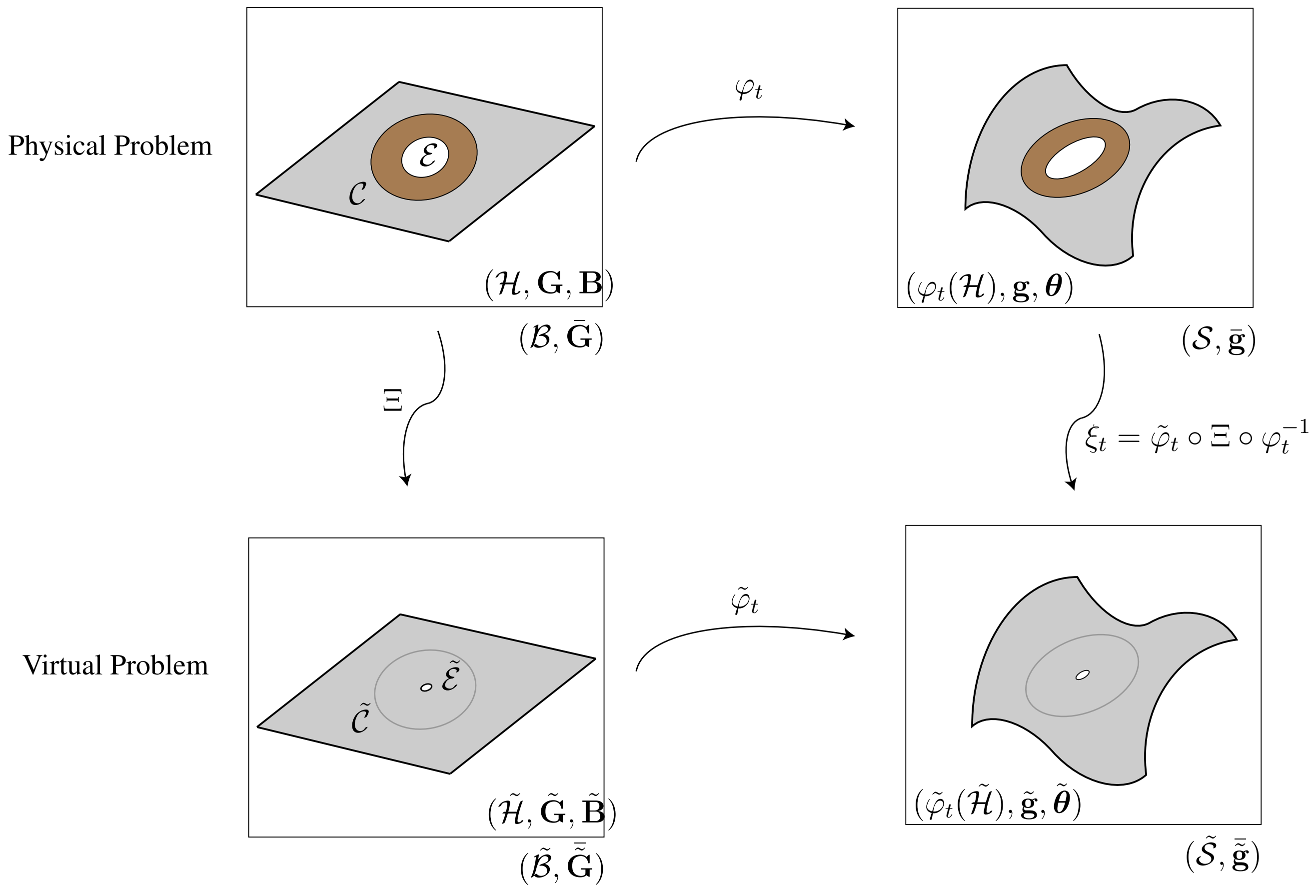


Figure 8.1: *A cloaking transformation Ξ (or ξ when the physical plate is pre-stressed) transforms a plate with a finite hole $\mathcal{E}$ to another plate with an infinitesimal hole ($\tilde{\mathcal{E}}$) that is homogeneous and isotropic. The cloaking transformation is defined to be the identity map outside the cloak $\mathcal{C}$. Note that Ξ is not a referential change of coordinates and ξ_t is not a spatial change of coordinates.*

is something that we carefully discuss in §8.4.1 and §8.4.2 in the case of elastic plates. The Jacobian of the referential and spatial *cloaking transformations*, Ξ and ξ are given by

$$J_\Xi = \sqrt{\frac{\det \tilde{\mathbf{G}} \circ \Xi}{\det \mathbf{G}}} \det \overset{\Xi}{\mathbf{F}}\,, \quad J_\xi = \sqrt{\frac{\det \tilde{\mathbf{g}} \circ \xi}{\det \mathbf{g}}} \det \overset{\xi}{\mathbf{F}}\,. \tag{8.131}$$

coordinates, one has $W^A = J(F^{-1})^A{}_b w^b$, where $J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}} \det \mathbf{F}$ is the Jacobian of φ with $\mathbf{G}$ and $\mathbf{g}$ the Riemannian metrics of $\mathcal{B}$ and $\mathcal{S}$, respectively. Note that a Piola transformation can be performed on any index of a given tensor. One can show that $\operatorname{Div} \mathbf{W} = J(\operatorname{div} \mathbf{w}) \circ \varphi$, which in coordinates is written as $W^A{}_{|A} = J w^a{}_{|a}$. This is also known as the Piola identity. Another way of writing the Piola identity is in terms of the unit normal vectors of a surface in $\mathcal{B}$ and its corresponding surface in $\mathcal{S}$, along with the area elements. It is written as $\hat{\mathbf{n}} da = J\mathbf{F}^{-\star}\hat{\mathbf{N}} dA$, or in components, one writes $n_a da = J(F^{-1})^A{}_a N_A dA$. In the literature of continuum mechanics, this is known as Nanson's formula.

Shifters in Euclidean ambient space. We assume that the reference configurations of both the physical and virtual plates are embedded in the Euclidean space. To relate vector fields in the physical problem to those in the virtual problem properly one would need to use shifters. The mapping $\mathsf{s} : T\mathcal{S} \to T\tilde{\mathcal{S}}$, defined as $\mathsf{s}(x, \mathbf{w}) = (\tilde{x}, \mathbf{w})$ is called the shifter map. The restriction of s to $x \in \mathcal{S}$ is denoted by $\mathsf{s}_x = \mathsf{s}(x) : T_x\mathcal{S} \to T_{\tilde{x}}\tilde{\mathcal{S}}$, and shifts $\mathbf{w}$ based at $x \in \mathcal{S}$ to $\mathbf{w}$ based at $\tilde{x} \in \tilde{\mathcal{S}}$. Notice that s simply parallel transports[19] vectors emanating from x to those emanating from $\tilde{x}$ utilizing the linear structure of the Euclidean ambient space. For $\mathcal{S}$ and $\tilde{\mathcal{S}}$ we choose two global colinear Cartesian coordinates $\{\tilde{z}^{\tilde{i}}\}$ and $\{z^i\}$ for the virtual and physical deformed configurations, respectively. Let us also use curvilinear coordinates $\{\tilde{x}^{\tilde{a}}\}$ and $\{x^a\}$ for these configurations. Note that $\mathsf{s}^{\tilde{i}}{}_i = \delta^{\tilde{i}}_i$. One can show that [43]

$$\mathsf{s}^{\tilde{a}}{}_a(x) = \frac{\partial \tilde{x}^{\tilde{a}}}{\partial \tilde{z}^{\tilde{i}}}(\tilde{x})\frac{\partial z^i}{\partial x^a}(x)\delta^{\tilde{i}}_i\,. \tag{8.132}$$

Note that s preserves inner products, and thus, $\mathsf{s}^{\mathsf{T}} = \mathsf{s}^{-1}$, where in components, $(\mathsf{s}^{\mathsf{T}})^a{}_{\tilde{a}} = g^{ab}\mathsf{s}^{\tilde{b}}{}_b\tilde{g}_{\tilde{a}\tilde{b}}$. Note also that

$$\mathsf{s}^{\tilde{a}}{}_{a|\tilde{b}} = \frac{\partial \mathsf{s}^{\tilde{a}}{}_a}{\partial \tilde{x}^{\tilde{b}}} + \tilde{\gamma}^{\tilde{a}}{}_{\tilde{b}\tilde{c}}\mathsf{s}^{\tilde{c}}{}_a - \frac{\partial x^b}{\partial \tilde{x}^{\tilde{b}}}\gamma^c{}_{ab}\mathsf{s}^{\tilde{a}}{}_c\,, \tag{8.133}$$

where $\gamma^a{}_{bc} = \frac{\partial x^a}{\partial z^k}\frac{\partial^2 z^k}{\partial x^b \partial x^c}$ and $\tilde{\gamma}^{\tilde{a}}{}_{\tilde{b}\tilde{c}} = \frac{\partial \tilde{x}^{\tilde{a}}}{\partial \tilde{z}^{\tilde{k}}}\frac{\partial^2 \tilde{z}^{\tilde{k}}}{\partial \tilde{x}^{\tilde{b}} \partial \tilde{x}^{\tilde{c}}}$ are the Christoffel symbols associated with $\mathcal{S}$ and $\tilde{\mathcal{S}}$ (with their induced Euclidean metrics), respectively. It is straightforward to verify that $\mathsf{s}^{\tilde{a}}{}_{a|\tilde{b}} = 0$, i.e., the shifter is covariantly constant. As the reference configurations of both the physical and virtual shells are embedded in the Euclidean space, referential shifters may also be defined similarly. As an example consider polar coordinates (r, θ) and $(\tilde{r}, \tilde{\theta})$ at $x \in \mathbb{R}^2$ and $\tilde{x} \in \mathbb{R}^2$, respectively. The shifter map has the following matrix representation with respect to these coordinates

$$\mathsf{s} = \begin{bmatrix} \cos(\tilde{\theta} - \theta) & r\sin(\tilde{\theta} - \theta) \\ -\sin(\tilde{\theta} - \theta)/\tilde{r} & r\cos(\tilde{\theta} - \theta)/\tilde{r} \end{bmatrix}. \tag{8.134}$$

In order to ensure that the (tangential and normal) components of the acceleration term remain

[19]The notion of shifter maps in a general Riemannian manifold can be defined similarly if one considers parallel translations along the curves in the two manifolds.

form invariant under the cloaking map, the tangential and normal components of the displacement field in the physical and virtual shells are related as

$$(\tilde{U}^{\top})^{\tilde{a}} = \mathsf{s}^{\tilde{a}}{}_{a}(U^{\top})^{a}\,, \quad \tilde{U}^{\perp} = U^{\perp}\,, \quad a, \tilde{a} = 1, 2, \tag{8.135}$$

where s is the shifter in the local tangent plane to plates in $\mathbb{R}^2$.[20] Let $\{X^1, X^2, X^3\}$ and $\{\tilde{X}^1, \tilde{X}^2, \tilde{X}^3\}$ be local coordinate charts for $\mathcal{B}$ and $\tilde{\mathcal{B}}$ such that $\{X^1, X^2\}$ and $\{\tilde{X}^1, \tilde{X}^2\}$ are local charts for $\mathcal{H}$ and $\tilde{\mathcal{H}}$, respectively (with $\partial/\partial X^3$ and $\partial/\partial \tilde{X}^3$ being, respectively, normal to $\mathcal{H}$ and $\tilde{\mathcal{H}}$). We assume that $\mathcal{B}$ and $\tilde{\mathcal{B}}$ are both embedded in the Euclidean space using two global collinear Cartesian coordinates $\{\tilde{Z}^{\tilde{I}}\}$ and $\{Z^I\}$, respectively. The referential shifter is similarly defined as

$$\mathsf{S}^{\tilde{A}}{}_{A}(X) = \frac{\partial \tilde{X}^{\tilde{A}}}{\partial \tilde{Z}^{\tilde{I}}}(\tilde{X}) \frac{\partial Z^I}{\partial X^A}(X) \delta^{\tilde{I}}{}_{I}\,, \quad A, \tilde{A}, I, \tilde{I} = 1, 2, 3. \tag{8.136}$$

Thus, one obtains $\bar{G}_{AB} = \mathsf{S}^{\tilde{A}}{}_{A}\mathsf{S}^{\tilde{B}}{}_{B}\tilde{\bar{G}}_{\tilde{A}\tilde{B}}$, where $A, B, \tilde{A}, \tilde{B} = 1, 2, 3$.

Boundary conditions in the physical and virtual problems. Let $\partial\mathcal{H} = \partial\mathcal{E} \cup \partial_o\mathcal{H}$, where $\partial\mathcal{E}$ is the boundary of the physical hole and $\partial_o\mathcal{H}$ is the outer boundary of $\mathcal{H}$. Let us assume that $\partial_o\mathcal{H}$ is the disjoint union of $\partial_o\mathcal{H}_d$ and $\partial_o\mathcal{H}_t$, i.e., $\partial_o\mathcal{H} = \partial_o\mathcal{H}_t \cup \partial_o\mathcal{H}_d$, such that the Neumann and Dirichlet

[20] Note that for both plates we may use two global collinear Cartesian coordinates $\{z^1, z^2, z^3\}$ and $\{\tilde{z}^1, \tilde{z}^2, \tilde{z}^3\}$ such that z^3 and $\tilde{z}^3$ are the outward normal directions to the physical and virtual plates, respectively. Therefore, $\{z^1, z^2\}$ and $\{\tilde{z}^1, \tilde{z}^2\}$ are two global collinear Cartesian coordinates for $\varphi(\mathcal{H})$ and $\tilde{\varphi}(\tilde{\mathcal{H}})$, respectively, where $\Xi : \mathcal{H} \to \tilde{\mathcal{H}}$, $\varphi : \mathcal{H} \to \varphi(\mathcal{H})$, $\tilde{\varphi} : \tilde{\mathcal{H}} \to \tilde{\varphi}(\tilde{\mathcal{H}})$, and $\xi : \varphi(\mathcal{H}) \to \tilde{\varphi}(\tilde{\mathcal{H}})$, and s is defined as

$$\mathsf{s}^{\tilde{a}}{}_{a}(x) = \frac{\partial \tilde{x}^{\tilde{a}}}{\partial \tilde{z}^{\tilde{i}}}(\tilde{x}) \frac{\partial z^i}{\partial x^a}(x) \delta^{\tilde{i}}_{i}, \quad a, \tilde{a}, i, \tilde{i} = 1, 2\,,$$

where $\{x^a\}$ and $\{\tilde{x}^{\tilde{a}}\}$ are local coordinate charts for $\varphi(\mathcal{H})$ and $\tilde{\varphi}(\tilde{\mathcal{H}})$, respectively.

boundary conditions read[21]

$$
\begin{cases}
\left[P^{aA}+2\theta_{bc}\mathsf{M}^{bA}g^{ac}\right]\mathsf{T}_A=(\bar{T}^{\top})^a\,, \\
-(F^{-1})^A{}_a\left[\rho\mathfrak{L}^a-\mathsf{M}^{aB}{}_{|B}\right]\mathsf{T}_A=\bar{T}^{\perp}\,, & \text{on} \quad \partial_o\mathcal{H}_t \\
\mathsf{M}^{aA}\mathsf{T}_A=\bar{\mathsf{m}}^a\,,
\end{cases}
\qquad
\begin{cases}
\varphi^{\top}(X,t)=\bar{\varphi}^{\top}(X,t)\,, \\
\varphi^{\perp}(X,t)=\bar{\varphi}^{\perp}(X,t)\,, & \text{on} \quad \partial_o\mathcal{H}_d \\
\frac{\partial\varphi^{\perp}(X,t)}{\partial X^A}=\frac{\partial\bar{\varphi}^{\perp}(X,t)}{\partial X^A}\,,
\end{cases}
\tag{8.137}
$$

where $\mathbf{T}$ is the unit normal one-form on $\partial_o\mathcal{H}$. Similarly, for the virtual problem, one has

$$
\begin{cases}
\left[\tilde{P}^{\tilde{a}\tilde{A}}+2\tilde{\theta}_{\tilde{b}\tilde{c}}\tilde{\mathsf{M}}^{\tilde{b}\tilde{A}}\tilde{g}^{\tilde{a}\tilde{c}}\right]\tilde{\mathsf{T}}_{\tilde{A}}=(\bar{\tilde{T}}^{\top})^{\tilde{a}}\,, \\
-(\tilde{F}^{-1})^{\tilde{A}}{}_{\tilde{a}}\left[\tilde{\rho}\tilde{\mathfrak{L}}^{\tilde{a}}-\tilde{\mathsf{M}}^{\tilde{a}\tilde{B}}{}_{|\tilde{B}}\right]\tilde{\mathsf{T}}_{\tilde{A}}=\bar{\tilde{T}}^{\perp}\,, & \text{on} \quad \partial_o\tilde{\mathcal{H}}_t \\
\tilde{\mathsf{M}}^{\tilde{a}\tilde{A}}\tilde{\mathsf{T}}_{\tilde{A}}=\bar{\tilde{\mathsf{m}}}^{\tilde{a}}\,,
\end{cases}
\qquad
\begin{cases}
\tilde{\varphi}^{\top}(\tilde{X},t)=\bar{\tilde{\varphi}}^{\top}(\tilde{X},t)\,, \\
\tilde{\varphi}^{\perp}(\tilde{X},t)=\bar{\tilde{\varphi}}^{\perp}(\tilde{X},t)\,, & \text{on} \quad \partial_o\tilde{\mathcal{H}}_d \\
\frac{\partial\tilde{\varphi}^{\perp}(\tilde{X},t)}{\partial\tilde{X}^{\tilde{A}}}=\frac{\partial\bar{\tilde{\varphi}}^{\perp}(\tilde{X},t)}{\partial\tilde{X}^{\tilde{A}}}\,.
\end{cases}
\tag{8.138}
$$

Note that the cloaking map $\Xi:\mathcal{H}\to\tilde{\mathcal{H}}$ is set to be the identity outside the cloak, i.e., in $\mathcal{H}\setminus\mathcal{C}$ (or $\tilde{\mathcal{H}}\setminus\tilde{\mathcal{C}}$), and thus, one is able to impose identical boundary conditions on the outside boundaries $\partial_o\mathcal{H}$, and $\partial_o\tilde{\mathcal{H}}$. Thus, noting that $\partial(\mathcal{H}\setminus\mathcal{C})=\partial_o\mathcal{H}\cup\partial_o\mathcal{C}$ and $\partial(\tilde{\mathcal{H}}\setminus\tilde{\mathcal{C}})=\partial_o\tilde{\mathcal{H}}\cup\partial_o\tilde{\mathcal{C}}$, in order for the two problems to have identical current configurations (and thus, elastic measurements) outside the cloak, it remains to make sure that the boundary data of $\partial_o\mathcal{C}$ and $\partial_o\tilde{\mathcal{C}}$ are identical, i.e., for $X\in\partial_o\mathcal{C}$ and

[21] See Remark. 8.3.1 for a discussion on how the boundary surface traction, boundary shear force, and boundary moment as well as their corresponding Dirichlet boundary conditions are prescribed in the boundary-value problem.

$\tilde{X} \in \partial_o \tilde{\mathcal{C}}$ (note that $\Xi|_{\partial_o \mathcal{C}} = id$), one needs to have

$$
\begin{cases}
(\tilde{T}^\top)^{\tilde{a}} = \left[\tilde{P}^{\tilde{a}\tilde{A}} + 2\tilde{\theta}_{\tilde{b}\tilde{c}}\tilde{\mathsf{M}}^{\tilde{b}\tilde{A}}\tilde{g}^{\tilde{a}\tilde{c}}\right]\tilde{\mathsf{T}}_{\tilde{A}} = \mathsf{s}^{\tilde{a}}{}_a\left[P^{aA} + 2\theta_{bc}\mathsf{M}^{bA}g^{ac}\right]\mathsf{T}_A = \mathsf{s}^{\tilde{a}}{}_a(T^\top)^a\,, \\
\quad \tilde{T}^\perp = -(\tilde{F}^{-1})^{\tilde{A}}{}_{\tilde{a}}\left[\tilde{\rho}\tilde{\mathfrak{L}}^{\tilde{a}} - \tilde{\mathsf{M}}^{\tilde{a}\tilde{B}}{}_{|\tilde{B}}\right]\tilde{\mathsf{T}}_{\tilde{A}} = -(F^{-1})^A{}_a\left[\rho\mathfrak{L}^a - \mathsf{M}^{aB}{}_{|B}\right]\mathsf{T}_A = T^\perp\,, \qquad \text{on } \partial_o\mathcal{C} \\
\quad \tilde{\mathsf{m}}^{\tilde{a}} = \tilde{\mathsf{M}}^{\tilde{a}\tilde{A}}\tilde{\mathsf{T}}_{\tilde{A}} = \mathsf{s}^{\tilde{a}}{}_a\mathsf{M}^{aA}\mathsf{T}_A = \mathsf{s}^{\tilde{a}}{}_a\mathsf{m}^a\,,
\end{cases}
$$

$$
\begin{cases}
\tilde{\varphi}^\top \circ \Xi(X,t) = \varphi^\top(X,t)\,, \\
\tilde{\varphi}^\perp \circ \Xi(X,t) = \varphi^\perp(X,t)\,, \qquad \text{on } \partial_o\mathcal{C} \\
\dfrac{\partial\tilde{\varphi}^\perp}{\partial\tilde{X}^{\tilde{A}}} \circ \Xi(X,t) = (\mathsf{S}^{-1})^A{}_{\tilde{A}}\dfrac{\partial\varphi^\perp}{\partial X^A}(X,t)\,.
\end{cases}
\tag{8.139}
$$

Moreover, the hole in the virtual shell is assumed to be traction-free, i.e.,

$$
\begin{aligned}
\left[\tilde{P}^{\tilde{a}\tilde{A}} + 2\tilde{\theta}_{\tilde{b}\tilde{c}}\tilde{\mathsf{M}}^{\tilde{b}\tilde{A}}\tilde{g}^{\tilde{a}\tilde{c}}\right]\tilde{\mathsf{T}}_{\tilde{A}} = 0\,, \quad (\tilde{F}^{-1})^{\tilde{A}}{}_{\tilde{a}}\left[\tilde{\rho}\tilde{\mathfrak{L}}^{\tilde{a}} - \tilde{\mathsf{M}}^{\tilde{a}\tilde{B}}{}_{|\tilde{B}}\right]\tilde{\mathsf{T}}_{\tilde{A}} = 0\,, \\
\tilde{\mathsf{M}}^{\tilde{a}\tilde{A}}\tilde{\mathsf{T}}_{\tilde{A}} = 0\,.
\end{aligned}
\quad \text{on} \quad \partial\tilde{\mathcal{E}}
\tag{8.140}
$$

The inner surface of the hole in the physical shell must be traction-free as well, and hence

$$
\begin{aligned}
\left[P^{aA} + 2\theta_{bc}\mathsf{M}^{bA}g^{ac}\right]\mathsf{T}_A = 0\,, \quad (F^{-1})^A{}_a\left[\rho\mathfrak{L}^a - \mathsf{M}^{aB}{}_{|B}\right]\mathsf{T}_A = 0\,, \\
\mathsf{M}^{aA}\mathsf{T}_A = 0\,.
\end{aligned}
\quad \text{on} \quad \partial\mathcal{E}
\tag{8.141}
$$

Remark 8.4.1. In the linearized setting, the condition (8.139) is written as

$$\begin{cases} (\delta\tilde{T}^{\top})^{\tilde{a}} = \left[\delta\tilde{P}^{\tilde{a}\tilde{A}} + 2\delta\tilde{\theta}_{\tilde{b}\tilde{c}}\mathring{\tilde{\mathsf{M}}}^{\tilde{b}\tilde{A}}\tilde{g}^{\tilde{a}\tilde{c}} + 2\mathring{\tilde{\theta}}_{\tilde{b}\tilde{c}}\delta\tilde{\mathsf{M}}^{\tilde{b}\tilde{A}}\tilde{g}^{\tilde{a}\tilde{c}}\right]\tilde{\mathsf{T}}_{\tilde{A}} \\ \qquad = \mathsf{s}^{\tilde{a}}{}_{a}\left[\delta P^{aA} + 2\delta\theta_{bc}\mathring{\mathsf{M}}^{bA}g^{ac} + 2\mathring{\theta}_{bc}\delta\mathsf{M}^{bA}g^{ac}\right]\mathsf{T}_{A} = \mathsf{s}^{\tilde{a}}{}_{a}(\delta T^{\top})^{a}\,, \\ \delta\tilde{T}^{\perp} = -(\mathring{\tilde{F}}^{-1})^{\tilde{A}}{}_{\tilde{a}}\left[\tilde{\rho}\delta\tilde{\mathfrak{L}}^{\tilde{a}} - \delta\tilde{\mathsf{M}}^{\tilde{a}\tilde{B}}{}_{|\tilde{B}}\right]\tilde{\mathsf{T}}_{\tilde{A}} - (\delta\tilde{F}^{-1})^{\tilde{A}}{}_{\tilde{a}}\left[\tilde{\rho}\mathring{\tilde{\mathfrak{L}}}^{\tilde{a}} - \mathring{\tilde{\mathsf{M}}}^{\tilde{a}\tilde{B}}{}_{|\tilde{B}}\right]\tilde{\mathsf{T}}_{\tilde{A}} & \text{on } \partial_o\mathcal{C} \\ \qquad = -(\mathring{F}^{-1})^{A}{}_{a}\left[\rho\delta\mathfrak{L}^{a} - \delta\mathsf{M}^{aB}{}_{|B}\right]\mathsf{T}_{A} - (\delta F^{-1})^{A}{}_{a}\left[\rho\mathring{\mathfrak{L}}^{a} - \mathring{\mathsf{M}}^{aB}{}_{|B}\right]\mathsf{T}_{A} = \delta T^{\perp}\,, \\ \delta\tilde{\mathsf{m}}^{\tilde{a}} = \delta\tilde{\mathsf{M}}^{\tilde{a}\tilde{A}}\tilde{\mathsf{T}}_{\tilde{A}} = \mathsf{s}^{\tilde{a}}{}_{a}\delta\mathsf{M}^{aA}\mathsf{T}_{A} = \mathsf{s}^{\tilde{a}}{}_{a}\delta\mathsf{m}^{a}\,, \end{cases}$$

$$\begin{cases} \tilde{\mathbf{U}}^{\top}\circ\Xi(X,t) = \mathbf{s}\mathbf{U}^{\top}(X,t)\,, \\ \tilde{\mathbf{U}}^{\perp}\circ\Xi(X,t) = \mathbf{U}^{\perp}(X,t)\,, & \text{on } \partial_o\mathcal{C} \\ \dfrac{\partial\tilde{U}^{\perp}}{\partial\tilde{X}^{\tilde{A}}}\circ\Xi(X,t) = (\mathsf{S}^{-1})^{A}{}_{\tilde{A}}\dfrac{\partial U^{\perp}}{\partial X^{A}}(X,t)\,. \end{cases} \tag{8.142}$$

Next, we discuss transformation cloaking in Kirchhoff-Love plates, for which only the out-of-plane displacement is allowed. We also examine the possibility of transformation cloaking when the pure bending assumption is relaxed and the plate is allowed to have both in-plane and out-of-plane displacements.

8.4.1 Elastodynamic transformation cloaking in Kirchhoff-Love plates

In this section, we discuss transformation cloaking in classical elastic plates in the absence of in-plane deformations (pure bending). For the sake of brevity, let us denote the normal displacement of the physical plate $U^{\perp}$ and the normal displacement of the virtual plate $\tilde{U}^{\perp}$ by W and $\widetilde{\mathsf{W}}$, respectively. For the virtual plate with uniform elastic properties and vanishing pre-stress and initial body forces, (8.128) is simplified to read

$$-\frac{1}{2}(\mathring{\tilde{F}}^{-1})^{\tilde{B}}{}_{\tilde{a}}\tilde{\mathbb{C}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{C}}(\widetilde{\mathsf{W}}_{,\tilde{b}})_{|\tilde{C}|\tilde{A}|\tilde{B}} = \tilde{\rho}\,\ddot{\widetilde{\mathsf{W}}}\,. \tag{8.143}$$

Note that in the absence of in-plane deformations ($\mathbf{U}^{\top} = \mathbf{0}$) and in the case of thin plate bending ($\mathbb{B} = \mathbf{0}$), $\delta P^{aA} = \delta\tilde{P}^{\tilde{a}\tilde{A}} = 0$, and (8.143) is the only non-trivial linearized balance of linear momentum equation. We assume a Saint Venant-Kirchhoff constitutive model[22] for the virtual plate, for which the energy density is given by

$$
\begin{aligned}
\tilde{W} =& \frac{Eh}{8(1+\nu)}\left\{\mathrm{tr}\left[\left(\tilde{\mathbf{C}}-\tilde{\mathbf{G}}\right)^2\right]+\frac{\nu}{1-\nu}\left[\mathrm{tr}\left(\tilde{\mathbf{C}}-\tilde{\mathbf{G}}\right)\right]^2\right\} \\
&+\frac{Eh^3}{24(1+\nu)}\left\{\mathrm{tr}\left[\left(\tilde{\boldsymbol{\Theta}}-\tilde{\mathbf{B}}\right)^2\right]+\frac{\nu}{1-\nu}\left[\mathrm{tr}\left(\tilde{\boldsymbol{\Theta}}-\tilde{\mathbf{B}}\right)\right]^2\right\},
\end{aligned}
\tag{8.144}
$$

where E is Young's modulus and ν is Poisson's ratio. Therefore, in the case of pure bending deformations, the flexural rigidity tensor for the virtual plate is calculated and is written with a slight abuse of notation as

$$
\tilde{\mathbb{C}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{C}}=\frac{Eh^3}{12(1+\nu)}\mathring{\tilde{F}}^{\tilde{a}}{}_{\tilde{M}}\mathring{\tilde{F}}^{\tilde{b}}{}_{\tilde{N}}\left[\tilde{G}^{\tilde{A}\tilde{N}}\tilde{G}^{\tilde{C}\tilde{M}}+\tilde{G}^{\tilde{A}\tilde{C}}\tilde{G}^{\tilde{M}\tilde{N}}+\frac{2\nu}{1-\nu}\tilde{G}^{\tilde{A}\tilde{M}}\tilde{G}^{\tilde{C}\tilde{N}}\right].
\tag{8.145}
$$

[22] See [263, 264, 265, 266] for details on the derivation of the Saint Venant-Kirchhoff shell constitutive model.

Note that (8.145) is the most general isotropic constitutive equation for a Kirchhoff-Love plate. Assuming that $\mathsf{W} = \widetilde{\mathsf{W}} \circ \Xi$, the derivatives of the normal displacement field are transformed as

$$
\begin{aligned}
\widetilde{\mathsf{W}}_{,\tilde{b}} =& (\bar{\bar{\bar{F}}}^{-1})^{b}{}_{\tilde{b}}\, \mathsf{W}_{,b}\,, \\
(\widetilde{\mathsf{W}}_{,\tilde{b}})_{|\tilde{C}} =& (\bar{\bar{\bar{F}}}^{-1})^{b}{}_{\tilde{b}|\tilde{C}}\, \mathsf{W}_{,b} + (\bar{\bar{\bar{F}}}^{-1})^{b}{}_{\tilde{b}} (\bar{\bar{\bar{F}}}^{-1})^{C}{}_{\tilde{C}}\, (\mathsf{W}_{,b})_{|C}\,, \\
(\widetilde{\mathsf{W}}_{,\tilde{b}})_{|\tilde{C}|\tilde{A}} =& (\bar{\bar{\bar{F}}}^{-1})^{b}{}_{\tilde{b}|\tilde{C}|\tilde{A}}\, \mathsf{W}_{,b} + \Big[(\bar{\bar{\bar{F}}}^{-1})^{b}{}_{\tilde{b}|\tilde{A}} (\bar{\bar{\bar{F}}}^{-1})^{C}{}_{\tilde{C}} + (\bar{\bar{\bar{F}}}^{-1})^{C}{}_{\tilde{C}|\tilde{A}} (\bar{\bar{\bar{F}}}^{-1})^{b}{}_{\tilde{b}} \\
& + (\bar{\bar{\bar{F}}}^{-1})^{b}{}_{\tilde{b}|\tilde{C}} (\bar{\bar{\bar{F}}}^{-1})^{C}{}_{\tilde{A}} \Big] (\mathsf{W}_{,b})_{|C} + (\bar{\bar{\bar{F}}}^{-1})^{b}{}_{\tilde{b}} (\bar{\bar{\bar{F}}}^{-1})^{C}{}_{\tilde{C}} (\bar{\bar{\bar{F}}}^{-1})^{A}{}_{\tilde{A}}\, (\mathsf{W}_{,b})_{|C|A}\,, \\
(\widetilde{\mathsf{W}}_{,\tilde{b}})_{|\tilde{C}|\tilde{A}|\tilde{B}} =& (\bar{\bar{\bar{F}}}^{-1})^{b}{}_{\tilde{b}|\tilde{C}|\tilde{A}|\tilde{B}}\, \mathsf{W}_{,b} + \Big[(\bar{\bar{\bar{F}}}^{-1})^{b}{}_{\tilde{b}|\tilde{A}} (\bar{\bar{\bar{F}}}^{-1})^{A}{}_{\tilde{C}|\tilde{B}} + (\bar{\bar{\bar{F}}}^{-1})^{b}{}_{\tilde{b}|\tilde{C}} (\bar{\bar{\bar{F}}}^{-1})^{A}{}_{\tilde{A}|\tilde{B}} \\
& + (\bar{\bar{\bar{F}}}^{-1})^{b}{}_{\tilde{b}|\tilde{B}} (\bar{\bar{\bar{F}}}^{-1})^{A}{}_{\tilde{C}|\tilde{A}} + (\bar{\bar{\bar{F}}}^{-1})^{b}{}_{\tilde{b}|\tilde{C}|\tilde{B}} (\bar{\bar{\bar{F}}}^{-1})^{A}{}_{\tilde{A}} + (\bar{\bar{\bar{F}}}^{-1})^{b}{}_{\tilde{b}|\tilde{A}|\tilde{B}} (\bar{\bar{\bar{F}}}^{-1})^{A}{}_{\tilde{C}} \\
& + (\bar{\bar{\bar{F}}}^{-1})^{b}{}_{\tilde{b}|\tilde{C}|\tilde{A}} (\bar{\bar{\bar{F}}}^{-1})^{A}{}_{\tilde{B}} + (\bar{\bar{\bar{F}}}^{-1})^{A}{}_{\tilde{C}|\tilde{A}|\tilde{B}} (\bar{\bar{\bar{F}}}^{-1})^{b}{}_{\tilde{b}} \Big] (\mathsf{W}_{,b})_{|A} \\
& + \Big[(\bar{\bar{\bar{F}}}^{-1})^{b}{}_{\tilde{b}|\tilde{B}} (\bar{\bar{\bar{F}}}^{-1})^{C}{}_{\tilde{C}} (\bar{\bar{\bar{F}}}^{-1})^{A}{}_{\tilde{A}} + (\bar{\bar{\bar{F}}}^{-1})^{C}{}_{\tilde{C}|\tilde{B}} (\bar{\bar{\bar{F}}}^{-1})^{b}{}_{\tilde{b}} (\bar{\bar{\bar{F}}}^{-1})^{A}{}_{\tilde{A}} \\
& + (\bar{\bar{\bar{F}}}^{-1})^{A}{}_{\tilde{A}|\tilde{B}} (\bar{\bar{\bar{F}}}^{-1})^{b}{}_{\tilde{b}} (\bar{\bar{\bar{F}}}^{-1})^{C}{}_{\tilde{C}} + (\bar{\bar{\bar{F}}}^{-1})^{b}{}_{\tilde{b}|\tilde{C}} (\bar{\bar{\bar{F}}}^{-1})^{C}{}_{\tilde{A}} (\bar{\bar{\bar{F}}}^{-1})^{A}{}_{\tilde{B}} \\
& + (\bar{\bar{\bar{F}}}^{-1})^{b}{}_{\tilde{b}|\tilde{A}} (\bar{\bar{\bar{F}}}^{-1})^{A}{}_{\tilde{B}} (\bar{\bar{\bar{F}}}^{-1})^{C}{}_{\tilde{C}} + (\bar{\bar{\bar{F}}}^{-1})^{C}{}_{\tilde{C}|\tilde{A}} (\bar{\bar{\bar{F}}}^{-1})^{A}{}_{\tilde{B}} (\bar{\bar{\bar{F}}}^{-1})^{b}{}_{\tilde{b}} \Big] (\mathsf{W}_{,b})_{|C|A} \\
& + (\bar{\bar{\bar{F}}}^{-1})^{b}{}_{\tilde{b}} (\bar{\bar{\bar{F}}}^{-1})^{C}{}_{\tilde{C}} (\bar{\bar{\bar{F}}}^{-1})^{A}{}_{\tilde{A}} (\bar{\bar{\bar{F}}}^{-1})^{B}{}_{\tilde{B}}\, (\mathsf{W}_{,b})_{|C|A|B}\,,
\end{aligned}
\tag{8.146}
$$

The equilibrium equation for the physical plate is given by (cf. (8.128))

$$
\begin{aligned}
-\frac{1}{2} (\mathring{F}^{-1})^{B}{}_{a} \Big[& \mathbb{C}^{aAbC}{}_{|A|B} (\mathsf{W}_{,b})_{|C} + \mathbb{C}^{aAbC}{}_{|A} (\mathsf{W}_{,b})_{|C|B} + \mathbb{C}^{aAbC}{}_{|B} (\mathsf{W}_{,b})_{|C|A} \\
& + \mathbb{C}^{aAbC} (\mathsf{W}_{,b})_{|C|A|B} \Big] + \mathring{P}^{aA} \left(\frac{\partial \mathsf{W}}{\partial x^{a}} \right)_{|A} - \rho (\mathring{\mathfrak{B}}^{\top})^{b} \frac{\partial \mathsf{W}}{\partial x^{b}} = \rho \ddot{\mathsf{W}}\,.
\end{aligned}
\tag{8.147}
$$

We next multiply both sides of (8.143) by some positive function $k = k(X)$ to be determined,[23] and substitute for derivatives from (8.146). Then compare the coefficients of different derivatives with those in (8.147). Comparing the coefficients of the fourth-order derivatives gives us the elastic constants of the physical plate, comparing the first-order derivatives gives the tangential body force in the finitely-deformed physical plate, and finally comparing the second-order derivatives will give the pre-stress in the physical plate. In addition, comparing the coefficients of the third-order derivatives

[23] Note that introducing the scalar field $k = k(X)$ provides an extra degree of freedom in the cloaking problem.

will result in a set of constraints on the cloaking map that we call *cloaking compatibility equations*. The flexural rigidity tensor of the physical plate is obtained as

$$\mathbb{C}^{aAbC} = k(\overset{\equiv}{F}^{-1})^{a}{}_{\tilde{a}}(\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{A}}(\overset{\equiv}{F}^{-1})^{b}{}_{\tilde{b}}(\overset{\equiv}{F}^{-1})^{C}{}_{\tilde{C}}\tilde{\mathbb{C}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{C}} \,. \tag{8.148}$$

Notice that $\mathbb{C}$ has the minor symmetries, and thus, the linearized balance of angular momentum is satisfied for the physical plate. The mass density of the physical plate is given by $\rho = k\tilde{\rho}\circ\Xi$. The tangential body force and the pre-stress in the physical plate are obtained as

$$\rho(\mathring{\mathfrak{B}}^{\top})^{b} = \frac{1}{2}k(\mathring{\tilde{F}}^{-1})^{\tilde{B}}{}_{\tilde{a}}\tilde{\mathbb{C}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{C}}(\overset{\equiv}{F}^{-1})^{b}{}_{\tilde{b}|\tilde{C}|\tilde{A}|\tilde{B}} \,, \tag{8.149}$$

$$\begin{aligned}\mathring{P}^{bA} =& \frac{1}{2}(\mathring{F}^{-1})^{B}{}_{a}\mathbb{C}^{aCbA}{}_{|C|B} - \frac{1}{2}k(\mathring{\tilde{F}}^{-1})^{\tilde{B}}{}_{\tilde{a}}\tilde{\mathbb{C}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{C}}\Big[(\overset{\equiv}{F}^{-1})^{b}{}_{\tilde{b}|\tilde{A}}(\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{C}|\tilde{B}} \\ &+ (\overset{\equiv}{F}^{-1})^{b}{}_{\tilde{b}|\tilde{C}}(\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{A}|\tilde{B}} + (\overset{\equiv}{F}^{-1})^{b}{}_{\tilde{b}|\tilde{B}}(\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{C}|\tilde{A}} + (\overset{\equiv}{F}^{-1})^{b}{}_{\tilde{b}|\tilde{C}|\tilde{B}}(\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{A}} \\ &+ (\overset{\equiv}{F}^{-1})^{b}{}_{\tilde{b}|\tilde{A}|\tilde{B}}(\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{C}} + (\overset{\equiv}{F}^{-1})^{b}{}_{\tilde{b}|\tilde{C}|\tilde{A}}(\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{B}} + (\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{C}|\tilde{A}|\tilde{B}}(\overset{\equiv}{F}^{-1})^{b}{}_{\tilde{b}}\Big] \,.\end{aligned} \tag{8.150}$$

The cloaking compatibility equations read

$$\begin{aligned}(\mathring{F}^{-1})^{B}{}_{a}\mathbb{C}^{aAbC}{}_{|B} + (\mathring{F}^{-1})^{A}{}_{a}\mathbb{C}^{aBbC}{}_{|B} = k(\mathring{\tilde{F}}^{-1})^{\tilde{B}}{}_{\tilde{a}}\tilde{\mathbb{C}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{C}}\Big[&(\overset{\equiv}{F}^{-1})^{b}{}_{\tilde{b}|\tilde{B}}(\overset{\equiv}{F}^{-1})^{C}{}_{\tilde{C}}(\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{A}} \\ &+ (\overset{\equiv}{F}^{-1})^{C}{}_{\tilde{C}|\tilde{B}}(\overset{\equiv}{F}^{-1})^{b}{}_{\tilde{b}}(\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{A}} + (\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{A}|\tilde{B}}(\overset{\equiv}{F}^{-1})^{b}{}_{\tilde{b}}(\overset{\equiv}{F}^{-1})^{C}{}_{\tilde{C}} \\ &+ (\overset{\equiv}{F}^{-1})^{b}{}_{\tilde{b}|\tilde{C}}(\overset{\equiv}{F}^{-1})^{C}{}_{\tilde{A}}(\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{B}} + (\overset{\equiv}{F}^{-1})^{b}{}_{\tilde{b}|\tilde{A}}(\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{B}}(\overset{\equiv}{F}^{-1})^{C}{}_{\tilde{C}} \\ &+ (\overset{\equiv}{F}^{-1})^{C}{}_{\tilde{C}|\tilde{A}}(\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{B}}(\overset{\equiv}{F}^{-1})^{b}{}_{\tilde{b}}\Big] \,.\end{aligned} \tag{8.151}$$

The initial body force and the pre-stress need to satisfy (8.126). Therefore

$$\begin{aligned}&k(\mathring{\tilde{F}}^{-1})^{\tilde{B}}{}_{\tilde{a}}\tilde{\mathbb{C}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{C}}(\overset{\equiv}{F}^{-1})^{b}{}_{\tilde{b}|\tilde{C}|\tilde{A}|\tilde{B}} + \Big\{(\mathring{F}^{-1})^{B}{}_{a}\mathbb{C}^{aCbA}{}_{|C|B} - k(\mathring{\tilde{F}}^{-1})^{\tilde{B}}{}_{\tilde{a}}\tilde{\mathbb{C}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{C}}\Big[(\overset{\equiv}{F}^{-1})^{b}{}_{\tilde{b}|\tilde{A}}(\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{C}|\tilde{B}} \\ &+ (\overset{\equiv}{F}^{-1})^{b}{}_{\tilde{b}|\tilde{C}}(\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{A}|\tilde{B}} + (\overset{\equiv}{F}^{-1})^{b}{}_{\tilde{b}|\tilde{B}}(\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{C}|\tilde{A}} + (\overset{\equiv}{F}^{-1})^{b}{}_{\tilde{b}|\tilde{C}|\tilde{B}}(\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{A}} \\ &+ (\overset{\equiv}{F}^{-1})^{b}{}_{\tilde{b}|\tilde{A}|\tilde{B}}(\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{C}} + (\overset{\equiv}{F}^{-1})^{b}{}_{\tilde{b}|\tilde{C}|\tilde{A}}(\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{B}} + (\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{C}|\tilde{A}|\tilde{B}}(\overset{\equiv}{F}^{-1})^{b}{}_{\tilde{b}}\Big]\Big\}_{|A} = 0 \,,\end{aligned} \tag{8.152}$$

and, with an abuse of notation

$$
\begin{aligned}
&(\mathring{F}^{-1})^{B}{}_{a}\mathbb{C}^{aC[bA]}{}_{|C|B} - k(\mathring{\tilde{F}}^{-1})^{\tilde{B}}{}_{\tilde{a}}\tilde{\mathbb{C}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{C}}\Big[(\bar{\bar{F}}^{-1})^{[b}{}_{\tilde{b}|\tilde{A}}(\bar{\bar{F}}^{-1})^{A]}{}_{\tilde{C}|\tilde{B}} \\
&+ (\bar{\bar{F}}^{-1})^{[b}{}_{\tilde{b}|\tilde{C}}(\bar{\bar{F}}^{-1})^{A]}{}_{\tilde{A}|\tilde{B}} + (\bar{\bar{F}}^{-1})^{[b}{}_{\tilde{b}|\tilde{B}}(\bar{\bar{F}}^{-1})^{A]}{}_{\tilde{C}|\tilde{A}} + (\bar{\bar{F}}^{-1})^{[b}{}_{\tilde{b}|\tilde{C}|\tilde{B}}(\bar{\bar{F}}^{-1})^{A]}{}_{\tilde{A}} \\
&+ (\bar{\bar{F}}^{-1})^{[b}{}_{\tilde{b}|\tilde{A}|\tilde{B}}(\bar{\bar{F}}^{-1})^{A]}{}_{\tilde{C}} + (\bar{\bar{F}}^{-1})^{[b}{}_{\tilde{b}|\tilde{C}|\tilde{A}}(\bar{\bar{F}}^{-1})^{A]}{}_{\tilde{B}} + (\bar{\bar{F}}^{-1})^{[b}{}_{\tilde{b}}(\bar{\bar{F}}^{-1})^{A]}{}_{\tilde{C}|\tilde{A}|\tilde{B}}\Big] = 0\,.
\end{aligned}
\tag{8.153}
$$

Note that

$$
\begin{aligned}
(\bar{\bar{F}}^{-1})^{A}{}_{\tilde{A}|\tilde{B}} =& \frac{\partial}{\partial \tilde{X}^{\tilde{B}}}\left[(\bar{\bar{F}}^{-1})^{A}{}_{\tilde{A}}\right] + \Gamma^{A}{}_{CB}(\bar{\bar{F}}^{-1})^{B}{}_{\tilde{B}}(\bar{\bar{F}}^{-1})^{C}{}_{\tilde{A}} - \tilde{\Gamma}^{\tilde{C}}{}_{\tilde{A}\tilde{B}}(\bar{\bar{F}}^{-1})^{A}{}_{\tilde{C}}\,, \\
(\bar{\bar{F}}^{-1})^{A}{}_{\tilde{A}|\tilde{B}|\tilde{C}} =& \frac{\partial}{\partial \tilde{X}^{\tilde{C}}}\left[(\bar{\bar{F}}^{-1})^{A}{}_{\tilde{A}|\tilde{B}}\right] + \Gamma^{A}{}_{CB}(\bar{\bar{F}}^{-1})^{B}{}_{\tilde{C}}(\bar{\bar{F}}^{-1})^{C}{}_{\tilde{A}|\tilde{B}} \\
&- \tilde{\Gamma}^{\tilde{D}}{}_{\tilde{A}\tilde{C}}(\bar{\bar{F}}^{-1})^{A}{}_{\tilde{D}|\tilde{B}} - \tilde{\Gamma}^{\tilde{D}}{}_{\tilde{B}\tilde{C}}(\bar{\bar{F}}^{-1})^{A}{}_{\tilde{A}|\tilde{D}}\,, \\
(\bar{\bar{F}}^{-1})^{A}{}_{\tilde{A}|\tilde{B}|\tilde{C}|\tilde{D}} =& \frac{\partial}{\partial \tilde{X}^{\tilde{D}}}\left[(\bar{\bar{F}}^{-1})^{A}{}_{\tilde{A}|\tilde{B}|\tilde{C}}\right] + \Gamma^{A}{}_{EB}(\bar{\bar{F}}^{-1})^{B}{}_{\tilde{D}}(\bar{\bar{F}}^{-1})^{E}{}_{\tilde{A}|\tilde{B}|\tilde{C}} \\
&- \tilde{\Gamma}^{\tilde{E}}{}_{\tilde{A}\tilde{D}}(\bar{\bar{F}}^{-1})^{A}{}_{\tilde{E}|\tilde{B}|\tilde{C}} - \tilde{\Gamma}^{\tilde{E}}{}_{\tilde{B}\tilde{D}}(\bar{\bar{F}}^{-1})^{A}{}_{\tilde{A}|\tilde{E}|\tilde{C}} - \tilde{\Gamma}^{\tilde{E}}{}_{\tilde{C}\tilde{D}}(\bar{\bar{F}}^{-1})^{A}{}_{\tilde{A}|\tilde{B}|\tilde{E}}\,,
\end{aligned}
\tag{8.154}
$$

where $\Gamma^{A}{}_{BC}$ and $\tilde{\Gamma}^{\tilde{A}}{}_{\tilde{B}\tilde{C}}$ are, respectively, the Christoffel symbols associated with the induced connections on the physical and virtual plates. Also[24]

$$
\begin{aligned}
\mathbb{C}^{aAbB}{}_{|C} =& \frac{\partial}{\partial X^{C}}\left[\mathbb{C}^{aAbB}\right] + \mathbb{C}^{kAbB}\gamma^{a}{}_{kl}\mathring{F}^{l}{}_{C} + \mathbb{C}^{aAkB}\gamma^{b}{}_{kl}\mathring{F}^{l}{}_{C} \\
&+ \mathbb{C}^{aKbB}\Gamma^{A}{}_{KC} + \mathbb{C}^{aAbK}\Gamma^{B}{}_{KC}\,, \\
\mathbb{C}^{aAbB}{}_{|C|D} =& \frac{\partial}{\partial X^{D}}\left[\mathbb{C}^{aAbB}{}_{|C}\right] + \mathbb{C}^{kAbB}{}_{|C}\gamma^{a}{}_{kl}\mathring{F}^{l}{}_{D} + \mathbb{C}^{aAkB}{}_{|C}\gamma^{b}{}_{kl}\mathring{F}^{l}{}_{D} \\
&+ \mathbb{C}^{aKbB}{}_{|C}\Gamma^{A}{}_{KD} + \mathbb{C}^{aAbK}{}_{|C}\Gamma^{B}{}_{KD} - \mathbb{C}^{aAbB}{}_{|K}\Gamma^{K}{}_{CD}\,.
\end{aligned}
\tag{8.155}
$$

[24] The covariant derivative of a two-point tensor $\mathbf{T}$ is given by

$$
\begin{aligned}
T^{AB\cdots F}{}_{G\cdots Q}{}^{ab\cdots f}{}_{g\cdots q|K} =& \frac{\partial}{\partial X^{K}}T^{AB\cdots F}{}_{G\cdots Q}{}^{ab\cdots f}{}_{g\cdots q} \\
&+ T^{RB\cdots F}{}_{G\cdots Q}{}^{ab\cdots f}{}_{g\cdots q}\Gamma^{A}{}_{RK} + (\text{all upper referential indices}) \\
&- T^{AB\cdots F}{}_{R\cdots Q}{}^{ab\cdots f}{}_{g\cdots q}\Gamma^{R}{}_{GK} - (\text{all lower referential indices}) \\
&+ T^{AB\cdots F}{}_{G\cdots Q}{}^{lb\cdots f}{}_{g\cdots q}\gamma^{a}{}_{lr}F^{r}{}_{K} + (\text{all upper spatial indices}) \\
&- T^{AB\cdots F}{}_{G\cdots Q}{}^{ab\cdots f}{}_{l\cdots q}\gamma^{l}{}_{gr}F^{r}{}_{K} - (\text{all lower spatial indices})\,.
\end{aligned}
$$

Using (8.148) and (8.146), the couple-stress is transformed as

$$
\begin{aligned}
\delta\mathsf{M}^{aA} &= \frac{1}{2}\mathbb{C}^{aAbB}(\mathsf{W}_{,b})_{|B} \\
&= \frac{1}{2}k\tilde{\mathbb{C}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}(\overset{\Xi}{F}{}^{-1})^{b}{}_{\tilde{b}}(\overset{\Xi}{F}{}^{-1})^{B}{}_{\tilde{B}}(\overset{\Xi}{F}{}^{-1})^{A}{}_{\tilde{A}}(\overset{\Xi}{F}{}^{-1})^{a}{}_{\tilde{a}}\left[\overset{\Xi}{F}{}^{\tilde{c}}{}_{b|B}\,\widetilde{\mathsf{W}}_{,\tilde{c}} + \overset{\Xi}{F}{}^{\tilde{c}}{}_{b}\overset{\Xi}{F}{}^{\tilde{C}}{}_{B}\,(\widetilde{\mathsf{W}}_{,\tilde{c}})_{|\tilde{C}}\right] \\
&= \frac{1}{2}k\tilde{\mathbb{C}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}(\overset{\Xi}{F}{}^{-1})^{b}{}_{\tilde{b}}(\overset{\Xi}{F}{}^{-1})^{B}{}_{\tilde{B}}(\overset{\Xi}{F}{}^{-1})^{A}{}_{\tilde{A}}(\overset{\Xi}{F}{}^{-1})^{a}{}_{\tilde{a}}\overset{\Xi}{F}{}^{\tilde{c}}{}_{b|B}\,\widetilde{\mathsf{W}}_{,\tilde{c}} + k(\overset{\Xi}{F}{}^{-1})^{A}{}_{\tilde{A}}(\overset{\Xi}{F}{}^{-1})^{a}{}_{\tilde{a}}\delta\tilde{\mathsf{M}}^{\tilde{a}\tilde{A}}\,.
\end{aligned}
\tag{8.156}
$$

From (8.142), $\widetilde{\mathsf{W}}_{,\tilde{A}} = \mathsf{W}_{,A}(\mathsf{S}^{-1})^{A}{}_{\tilde{A}}$, on $\partial_o\mathcal{C}$, whence, together with (8.146), it follows that $T\Xi|_{\partial_o\mathcal{C}} = \overset{\Xi}{\mathbf{F}}|_{\partial_o\mathcal{C}} = id$. Moreover, (8.142) also requires that $(\mathring{\tilde{F}}{}^{-1})^{\tilde{A}}{}_{\tilde{a}}\delta\tilde{\mathsf{M}}^{\tilde{a}\tilde{B}}{}_{|\tilde{B}}\tilde{\mathsf{T}}_{\tilde{A}} = (\mathring{F}{}^{-1})^{A}{}_{a}\delta\mathsf{M}^{aB}{}_{|B}\mathsf{T}_{A}$, and $\delta\tilde{\mathsf{M}}^{\tilde{a}\tilde{A}}\tilde{\mathsf{T}}_{\tilde{A}} = \mathsf{s}^{\tilde{a}}{}_{a}\delta\mathsf{M}^{aA}\mathsf{T}_{A}$, on $\partial_o\mathcal{C}$, which imply that $\overset{\Xi}{F}{}^{\tilde{A}}{}_{A|B}\Big|_{\partial_o\mathcal{C}} = 0$, $\overset{\Xi}{F}{}^{\tilde{A}}{}_{A|B|C}\Big|_{\partial_o\mathcal{C}} = 0$, $k|_{\partial_o\mathcal{C}} = 1$, and $k_{,A}|_{\partial_o\mathcal{C}} = 0$. Similarly, given that the virtual plate is not pre-stressed ($\mathring{\tilde{\mathbf{P}}} = \mathbf{0}$), (8.139) implies that the initial traction must vanish on the outer boundary of the cloak, i.e., $(\mathring{T}^{\top})^{a}|_{\partial_o\mathcal{C}} = \mathring{P}^{aA}\mathsf{T}_{A}\big|_{\partial_o\mathcal{C}} = 0$.

Knowing that the hole surface in the virtual plate is traction-free (cf. (8.140)), i.e., $(\mathring{\tilde{F}}{}^{-1})^{\tilde{A}}{}_{\tilde{a}}\delta\tilde{\mathsf{M}}^{\tilde{a}\tilde{B}}{}_{|\tilde{B}}\tilde{\mathsf{T}}_{\tilde{A}} = 0$, and $\delta\tilde{\mathsf{M}}^{\tilde{a}\tilde{A}}\tilde{\mathsf{T}}_{\tilde{A}} = 0$, on $\partial\tilde{\mathcal{E}}$, if $k_{,A}|_{\partial\mathcal{E}} = 0$, $\overset{\Xi}{F}{}^{\tilde{A}}{}_{A|B}\Big|_{\partial\mathcal{E}} = 0$, and $\overset{\Xi}{F}{}^{\tilde{A}}{}_{A|B|C}\Big|_{\partial\mathcal{E}} = 0$, then the hole inner surface $\partial\mathcal{E}$ will also be traction-free in the physical plate. Note that (8.141) requires that the initial traction vanish on the boundary of the physical hole, viz., $(\mathring{T}^{\top})^{a}|_{\partial\mathcal{E}} = \mathring{P}^{aA}\mathsf{T}_{A}\big|_{\partial\mathcal{E}} = 0$.

Remark 8.4.2. It is important to note that contrary to transformation cloaking in 3D elasticity, where stress and couple-stress are transformed using the Piola transformation under a cloaking map, for Kirchhoff-Love plates, the couple-stress is not necessarily transformed via the Piola transformation. However, this should not be surprising, as in the 3D case, divergence of stress (and couple-stress) appear in the balance of linear momentum, and one uses the Piola transformation in order to preserve the divergence terms (and thus, the governing equations) up to the Jacobian of the cloaking map.

Remark 8.4.3. The mass form is not necessarily preserved under the cloaking transformation Ξ in Kirchhoff-Love plates, unlike transformation cloaking in 3D elasticity, where the mass form is preserved under Ξ (see [41, Remark 6]). To see this let us denote the virtual and physical mass forms by $\tilde{\mathbf{m}} = \tilde{\rho}d\tilde{A}_{\tilde{\mathbf{G}}}$, and $\mathbf{m} = \rho dA_{\mathbf{G}}$, respectively. Therefore, one obtains

$$
\Xi^{*}\tilde{\mathbf{m}} = \Xi^{*}(\tilde{\rho}d\tilde{A}_{\tilde{\mathbf{G}}}) = (\tilde{\rho}\circ\Xi)\Xi^{*}d\tilde{A}_{\tilde{\mathbf{G}}} = (\tilde{\rho}\circ\Xi)J_{\Xi}dA_{\mathbf{G}} = \frac{J_{\Xi}}{k}\rho dA_{\mathbf{G}} = \frac{J_{\Xi}}{k}\mathbf{m}\,. \tag{8.157}
$$

Remark 8.4.4. It is straightforward to see that when $(\bar{\bar{F}}^{-1})^A{}_{\tilde{A}|\tilde{B}} = 0$ (i.e., $\bar{\bar{\mathbf{F}}}^{-1}$ is covariantly constant) and $k_{,A} = 0$, one has $\mathring{\mathbf{P}} = \mathbf{0}$ and $\mathring{\mathfrak{B}}^\top = \mathbf{0}$, and thus, the constraint (8.153) and the balance equations in the finitely-deformed configuration (8.152) and (8.153) are already satisfied. However, if one uses Cartesian coordinates $\{Z^I\}$ and $\{\tilde{Z}^{\tilde{I}}\}$ for $\mathcal{H}$ and $\tilde{\mathcal{H}}$, respectively, $\bar{\bar{\mathbf{F}}}$ would have constant components if it is covariantly constant. Knowing that on the outer boundary of the cloak one needs to have $T\Xi|_{\partial_o\mathcal{C}} = \bar{\bar{\mathbf{F}}}|_{\partial_o\mathcal{C}} = id$, it follows that $\bar{\bar{\mathbf{F}}}$ is the identity everywhere (so is the cloaking map Ξ), and thus, cloaking is not possible if one assumes that the tangent map of the cloaking transformation is covariantly constant.[25]

A circular cloak in a Kirchhoff-Love plate

Consider a circular hole $\mathcal{E}$ in the physical plate in its reference configuration with radius R_i that needs to be cloaked from the out-of-plane excitations using an annular cloak having inner and outer radii R_i and R_o, respectively. Let us map the reference configuration to the reference configuration of the virtual plate via a cloaking map, $\Xi : \mathcal{H} \to \tilde{\mathcal{H}}$, where for $R_i \leq R \leq R_o$, it is defined as, $(\tilde{R}, \tilde{\Theta}) = \Xi(R, \Theta) = (f(R), \Theta)$ such that $f(R_o) = R_o$ and $f(R_i) = \epsilon$, and for $R \geq R_o$ is the identity map. The physical and virtual plates are endowed with the Euclidean metrics $\mathbf{G} = \mathrm{diag}(1, R^2)$, and $\tilde{\mathbf{G}} = \mathrm{diag}(1, \tilde{R}^2)$ in the polar coordinates, respectively. Thus

$$\bar{\bar{\mathbf{F}}} = \begin{bmatrix} f'(R) & 0 \\ 0 & 1 \end{bmatrix} . \tag{8.158}$$

[25]Pomot *et al.* [252] used a linear cloaking transformation, which has a covarianlty constant tangent map. However, a linear cloaking map does not satisfy the required traction boundary condition on $\partial_o\mathcal{C}$, i.e., $T\Xi|_{\partial_o\mathcal{C}} = \bar{\bar{\mathbf{F}}}|_{\partial_o\mathcal{C}} = id$, and therefore, using a linear cloaking map is not acceptable (see [194] for another improper use of this type of mapping).

From (8.145), the flexural rigidity of the virtual plate is given by[26]

$$\hat{\tilde{\mathbb{C}}} = \left[\hat{\tilde{\mathbb{C}}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\right] = \frac{Eh^3}{12(1+\nu)} \begin{bmatrix} \begin{bmatrix} \frac{2}{1-\nu} & 0 \\ 0 & \frac{2\nu}{1-\nu} \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} \frac{2\nu}{1-\nu} & 0 \\ 0 & \frac{2}{1-\nu} \end{bmatrix} \end{bmatrix}, \tag{8.159}$$

where the first two indices identify the submatrix and the last two specify the components of that submatrix. The (surface) mass density of the physical plate is given by $\rho = k(R)\tilde{\rho}$. Using (8.148), the flexural rigidity of the cloak are determined up to the scalar $k(R)$ as follows

$$\hat{\mathbb{C}} = \left[\hat{\mathbb{C}}^{aAbB}\right] = \frac{Eh^3 k(R)}{12(1+\nu)} \begin{bmatrix} \begin{bmatrix} \frac{2}{1-\nu}\frac{1}{f'^4(R)} & 0 \\ 0 & \frac{2\nu}{1-\nu}\frac{R^2}{f^2(R)f'^2(R)} \end{bmatrix} & \begin{bmatrix} 0 & \frac{R^2}{f^2(R)f'^2(R)} \\ \frac{R^2}{f^2(R)f'^2(R)} & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & \frac{R^2}{f^2(R)f'^2(R)} \\ \frac{R^2}{f^2(R)f'^2(R)} & 0 \end{bmatrix} & \begin{bmatrix} \frac{2\nu}{1-\nu}\frac{R^2}{f^2(R)f'^2(R)} & 0 \\ 0 & \frac{2}{1-\nu}\frac{R^4}{f^4(R)} \end{bmatrix} \end{bmatrix}. \tag{8.160}$$

From (8.149), the circumferential component of the tangential body force vanishes, i.e., $(\mathring{\mathfrak{B}}^{\top})^{\theta} = 0$, and its radial component reads

$$\begin{aligned} (\mathring{\mathfrak{B}}^{\top})^{r} =& \frac{Eh^3}{12\tilde{\rho}\,(\nu^2-1)\, f^4(R) f'^7(R)} \Big[3Rf'^7(R) - 3f(R)f'^6(R) - 3f^2(R)f'^4(R)f''(R) \\ &+ 2f^3(R)f'^2(R)\left(f^{(3)}(R)f'(R) - 3f''^2(R)\right) \\ &+ f^4(R)\left(15f''^3(R) + f^{(4)}(R)f'^2(R) - 10f^{(3)}(R)f'(R)f''(R)\right)\Big]. \end{aligned} \tag{8.161}$$

[26]Note that the physical components of the flexural rigidity tensor are given by $\hat{\tilde{\mathbb{C}}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} = \sqrt{\tilde{g}_{\tilde{a}\tilde{a}}}\sqrt{\tilde{G}_{\tilde{A}\tilde{A}}}\sqrt{\tilde{g}_{\tilde{b}\tilde{b}}}\sqrt{\tilde{G}_{\tilde{B}\tilde{B}}}\tilde{\mathbb{C}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}$ (no summation).

The initial stress is given by (cf. (8.150))

$$
\begin{aligned}
\hat{\mathring{P}}^{r\Theta} =&\hat{\mathring{P}}^{\theta R} = 0\,,\\
\hat{\mathring{P}}^{rR} =&\frac{Eh^3}{12\left(\nu^2-1\right)Rf^4(R)f'^6(R)}\Big[R^3k(R)f'^6(R)+2(\nu-1)R^2f(R)k(R)f'^5(R)\\
&+Rf^2(R)f'^3(R)\left[2(\nu-1)Rk(R)f''(R)+f'(R)\left\{(1-2\nu)k(R)-(\nu-2)Rk'(R)\right\}\right]\\
&-f^4(R)\Big[5Rk(R)f''^2(R)+f'^2(R)\left(Rk''(R)+2k'(R)\right)-8f'(R)f''(R)\left(Rk'(R)+k(R)\right)\Big]\\
&-6Rf^3(R)k(R)f'^2(R)f''(R)\Big]\,,\\
\hat{\mathring{P}}^{\theta\Theta} =&\frac{Eh^3}{12\left(1-\nu^2\right)f^5(R)f'^4(R)}\Big[-R^2f(R)f'^4(R)\left(Rk'(R)-6(\nu-1)k(R)\right)+4R^3k(R)f'^5(R)\\
&-2\nu Rf^2(R)f'^2(R)\left[2f'(R)\left(Rk'(R)+2k(R)\right)-3Rk(R)f''(R)\right]\\
&+f^3(R)\Big(Rf'(R)\left[f'(R)\left(\nu Rk''(R)+(4\nu+2)k'(R)\right)-4\nu Rf''(R)k'(R)\right]\\
&+2k(R)\Big[3\nu R^2f''^2(R)+(\nu+1)f'^2(R)-\nu Rf'(R)\left\{Rf^{(3)}(R)+4f''(R)\right\}\Big]\Big)\Big]\,.
\end{aligned}
\tag{8.162}
$$

Note that $\mathring{\mathbf{P}}$ is diagonal, and thus, (8.153) is already satisfied. The constraint (8.151) gives us the following two ODEs:

$$
f''(R)=f'(R)\left[\frac{1}{R}-\frac{f'(R)}{f(R)}+\frac{k'(R)}{k(R)}\right]\,, \tag{8.163a}
$$

$$
\begin{aligned}
&(1-2\nu)Rf^2(R)k(R)f''(R)\\
&+f'(R)\left[k(R)\left(f(R)-Rf'(R)\right)\left(Rf'(R)+2(\nu-1)f(R)\right)+(\nu-1)Rf^2(R)k'(R)\right]=0\,.
\end{aligned}
\tag{8.163b}
$$

Using (8.163a) and (8.163b), one obtains the following second-order nonlinear ODE for $f(R)$:

$$
f'(R)\left[f(R)-Rf'(R)\right]\left[Rf'(R)+(\nu-1)f(R)\right]-\nu Rf^2(R)f''(R)=0\,. \tag{8.164}
$$

It is interesting to observe that the differential equation governing the gradient of the cloaking map involves the Poisson's ratio of the virtual plate. Note that for $\nu = 0$, from (8.164), the cloaking map is forced to be the identity. If $\nu \neq 0$, then (8.163a) and (8.163b) imply that

$$\frac{k'(R)}{k(R)} = -\frac{\left(f(R) - Rf'(R)\right)^2}{\nu R f^2(R)}\,, \tag{8.165}$$

and (8.164) can be rewritten as

$$f''(R) = \frac{f'(R)}{\nu R f^2(R)}\left[f(R) - Rf'(R)\right]\left[Rf'(R) + (\nu - 1)f(R)\right]\,. \tag{8.166}$$

Note, however, that (8.164) is a second-order ODE and one cannot enforce the required boundary conditions $f(R_o) = R_o$, $f(R_i) = \epsilon$, and $f'(R_o) = 1$ (i.e., $\bar{\bar{\mathbf{F}}}|_{\partial_o \mathcal{C}} = id$) simultaneously, and thus, cloaking is not possible. The finite (in-plane) balance of linear momentum (8.152), i.e., $\mathring{P}^{aA}{}_{|A} + \rho(\mathring{\mathfrak{B}}^{\top})^a = 0$, is simplified to read

$$\begin{aligned}
&6R^2k(R)f'(R)^7 - 3Rf(R)f'(R)^5\left[f'(R)\left(2Rk'(R) + 3k(R)\right) - 2Rk(R)f''(R)\right] \\
&- 2f(R)^3f'(R)^2\left[2Rf^{(3)}(R)k(R)f'(R) + 3f''(R)\left(f'(R)\left(Rk'(R) + k(R)\right) - 3Rk(R)f''(R)\right)\right] \\
&+ f(R)^2f'(R)^3\Big\{Rf'(R)\left[f'(R)\left(2Rk''(R) + 7k'(R)\right) - 6Rf''(R)k'(R)\right] \\
&+ k(R)\left[6R^2f''(R)^2 + 3f'(R)^2 - Rf'(R)\left(2Rf^{(3)}(R) + 3f''(R)\right)\right]\Big\} \\
&+ f(R)^4\Big\{45Rk(R)f''(R)^3 + f'(R)^3\left[-\left(Rk^{(3)}(R) + 3k''(R)\right)\right] \\
&- 5f'(R)f''(R)\left[4Rf^{(3)}(R)k(R) + 9f''(R)\left(Rk'(R) + k(R)\right)\right] \\
&+ f'(R)^2\Big[Rf^{(4)}(R)k(R) + 8f^{(3)}(R)k(R) + 12Rf''(R)k''(R) \\
&+ 8\left(Rf^{(3)}(R) + 3f''(R)\right)k'(R)\Big]\Big\} = 0\,.
\end{aligned} \tag{8.167}$$

One can recursively use (8.165) and (8.166) to express $k''(R)$, $k^{(3)}(R)$, $f^{(3)}(R)$, and $f^{(4)}(R)$ in terms of $f'(R)$, $f(R)$, $k'(R)$, and $k(R)$. Plugging these expressions into (8.167), one can verify that (8.167) holds. Therefore, the satisfaction of the balance of linear and angular momenta in the physical plate in its finitely deformed configuration (i.e., (8.152) and (8.153)) does not impose any additional restriction on the cloaking map; (8.151) is the only constraint on Ξ.

Remark 8.4.5. For the isotropic and homogeneous flexural rigidity (8.159) and a flat ambient space, the governing equation of the virtual plate (8.143) is expanded and written in the form of the following biharmonic equation

$$-D^{(0)}\widetilde{\nabla}^4\widetilde{\mathsf{W}} = \tilde{\rho}\,\ddot{\widetilde{\mathsf{W}}}\,. \tag{8.168}$$

We first show that the simplified governing equation (8.168) does not correspond to a unique flexural rigidity tensor of an isotropic and homogeneous plate, and thus, that of the cloak when the cloaking map is the identity. Without loss of generality, we use Cartesian coordinates, for which (8.143) is simplified to read[27]

$$\begin{aligned}-\frac{1}{2}\Big[&\tilde{\mathbb{C}}^{\tilde{X}\tilde{X}\tilde{X}\tilde{X}}\widetilde{\mathsf{W}}_{\tilde{X}\tilde{X}\tilde{X}\tilde{X}} + \tilde{\mathbb{C}}^{\tilde{Y}\tilde{Y}\tilde{Y}\tilde{Y}}\widetilde{\mathsf{W}}_{\tilde{Y}\tilde{Y}\tilde{Y}\tilde{Y}} + 4\tilde{\mathbb{C}}^{\tilde{X}\tilde{X}\tilde{X}\tilde{Y}}\widetilde{\mathsf{W}}_{\tilde{X}\tilde{X}\tilde{X}\tilde{Y}} \\ &+ 4\tilde{\mathbb{C}}^{\tilde{X}\tilde{Y}\tilde{X}\tilde{Y}}\widetilde{\mathsf{W}}_{\tilde{X}\tilde{X}\tilde{Y}\tilde{Y}} + 4\tilde{\mathbb{C}}^{\tilde{X}\tilde{Y}\tilde{Y}\tilde{Y}}\widetilde{\mathsf{W}}_{\tilde{X}\tilde{Y}\tilde{Y}\tilde{Y}} + 2\tilde{\mathbb{C}}^{\tilde{X}\tilde{X}\tilde{Y}\tilde{Y}}\widetilde{\mathsf{W}}_{\tilde{X}\tilde{X}\tilde{Y}\tilde{Y}}\Big] = \tilde{\rho}\,\ddot{\widetilde{\mathsf{W}}}\,,\end{aligned} \tag{8.169}$$

where the minor and major symmetries of the flexural rigidity tensor were used. From (8.168) we have

$$-D^{(0)}\left[\widetilde{\mathsf{W}}_{\tilde{X}\tilde{X}\tilde{X}\tilde{X}} + \widetilde{\mathsf{W}}_{\tilde{Y}\tilde{Y}\tilde{Y}\tilde{Y}} + 2\widetilde{\mathsf{W}}_{\tilde{X}\tilde{X}\tilde{Y}\tilde{Y}}\right] = \tilde{\rho}\,\ddot{\widetilde{\mathsf{W}}}\,. \tag{8.170}$$

Comparing the coefficients of different derivatives, one obtains

$$\begin{aligned}&\tilde{\mathbb{C}}^{\tilde{X}\tilde{X}\tilde{X}\tilde{X}} = 2D^{(0)}\,, \quad \tilde{\mathbb{C}}^{\tilde{Y}\tilde{Y}\tilde{Y}\tilde{Y}} = 2D^{(0)}\,, \\ &\tilde{\mathbb{C}}^{\tilde{X}\tilde{X}\tilde{X}\tilde{Y}} = 0\,, \quad \tilde{\mathbb{C}}^{\tilde{X}\tilde{Y}\tilde{Y}\tilde{Y}} = 0\,, \quad 2\tilde{\mathbb{C}}^{\tilde{X}\tilde{Y}\tilde{X}\tilde{Y}} + \tilde{\mathbb{C}}^{\tilde{X}\tilde{X}\tilde{Y}\tilde{Y}} = 2D^{(0)}\,.\end{aligned} \tag{8.171}$$

Therefore, there are infinitely many choices for $\tilde{\mathbb{C}}$; one cannot uniquely determine the flexural rigid-

[27] Note that

$$\widetilde{\mathsf{W}}_{\tilde{A}\tilde{B}\tilde{C}\tilde{D}} = \frac{\partial^4\widetilde{\mathsf{W}}}{\partial\tilde{X}^{\tilde{A}}\partial\tilde{X}^{\tilde{B}}\partial\tilde{X}^{\tilde{C}}\partial\tilde{X}^{\tilde{D}}}\,.$$

ity tensor starting from the simplified governing equation (8.168) and comparing it with the initial governing equation in its tensorial form (8.143). We next transform the biharmonic equation (8.168) under a cloaking map. In Cartesian coordinates

$$\bar{\mathbf{F}}^{-1}(X,Y)=\begin{bmatrix}\mathsf{F}^{X}{}_{\tilde{X}} & \mathsf{F}^{X}{}_{\tilde{Y}}\\ \mathsf{F}^{Y}{}_{\tilde{X}} & \mathsf{F}^{Y}{}_{\tilde{Y}}\end{bmatrix}. \tag{8.172}$$

Knowing that $\mathsf{W}=\widetilde{\mathsf{W}}\circ\Xi^{-1}$, using the chain rule, one finds

$$\begin{aligned}\frac{\partial\widetilde{\mathsf{W}}}{\partial\tilde{X}}&=\frac{\partial\mathsf{W}}{\partial X}\mathsf{F}^{X}{}_{\tilde{X}}+\frac{\partial\mathsf{W}}{\partial Y}\mathsf{F}^{Y}{}_{\tilde{X}},\\ \frac{\partial\widetilde{\mathsf{W}}}{\partial\tilde{Y}}&=\frac{\partial\mathsf{W}}{\partial X}\mathsf{F}^{X}{}_{\tilde{Y}}+\frac{\partial\mathsf{W}}{\partial Y}\mathsf{F}^{Y}{}_{\tilde{Y}}.\end{aligned} \tag{8.173}$$

One may recursively use (8.173) to find the transformed higher order derivatives of $\widetilde{\mathsf{W}}$, and eventually, obtain the transformation of the biharmonic term $\widetilde{\nabla}^4\widetilde{\mathsf{W}}=\frac{\partial^4\widetilde{\mathsf{W}}}{\partial\tilde{X}^4}+\frac{\partial^4\widetilde{\mathsf{W}}}{\partial\tilde{Y}^4}+2\frac{\partial^4\widetilde{\mathsf{W}}}{\partial\tilde{X}^2\partial\tilde{Y}^2}$.[28] Comparing the coefficients of the different derivatives in the transformed biharmonic equation with those in the governing equation of the physical plate (8.147), one determines the unknown fields. Comparing the fourth-order derivatives one finds

$$\begin{aligned}\mathbb{C}^{XXXX}=&2D^{(0)}\left[\left(\mathsf{F}^{X}{}_{\tilde{X}}\right)^2+\left(\mathsf{F}^{X}{}_{\tilde{Y}}\right)^2\right]^2,\quad \mathbb{C}^{YYYY}=2D^{(0)}\left[\left(\mathsf{F}^{Y}{}_{\tilde{X}}\right)^2+\left(\mathsf{F}^{Y}{}_{\tilde{Y}}\right)^2\right]^2,\\ \mathbb{C}^{XYYY}=&2D^{(0)}\left[\mathsf{F}^{X}{}_{\tilde{X}}\mathsf{F}^{Y}{}_{\tilde{X}}+\mathsf{F}^{X}{}_{\tilde{Y}}\mathsf{F}^{Y}{}_{\tilde{Y}}\right]\left[\left(\mathsf{F}^{Y}{}_{\tilde{X}}\right)^2+\left(\mathsf{F}^{Y}{}_{\tilde{Y}}\right)^2\right],\\ \mathbb{C}^{YXXX}=&2D^{(0)}\left[\mathsf{F}^{X}{}_{\tilde{X}}\mathsf{F}^{Y}{}_{\tilde{X}}+\mathsf{F}^{X}{}_{\tilde{Y}}\mathsf{F}^{Y}{}_{\tilde{Y}}\right]\left[\left(\mathsf{F}^{X}{}_{\tilde{X}}\right)^2+\left(\mathsf{F}^{X}{}_{\tilde{Y}}\right)^2\right],\\ 2\mathbb{C}^{XYXY}+\mathbb{C}^{XXYY}=&2D^{(0)}\Big\{4\mathsf{F}^{X}{}_{\tilde{X}}\mathsf{F}^{X}{}_{\tilde{Y}}\mathsf{F}^{Y}{}_{\tilde{X}}\mathsf{F}^{Y}{}_{\tilde{Y}}+\left(\mathsf{F}^{X}{}_{\tilde{X}}\right)^2\left[3\left(\mathsf{F}^{Y}{}_{\tilde{X}}\right)^2+\left(\mathsf{F}^{Y}{}_{\tilde{Y}}\right)^2\right]\\ &+\left(\mathsf{F}^{X}{}_{\tilde{Y}}\right)^2\left[3\left(\mathsf{F}^{Y}{}_{\tilde{Y}}\right)^2+\left(\mathsf{F}^{Y}{}_{\tilde{X}}\right)^2\right]\Big\}.\end{aligned} \tag{8.174}$$

[28]In a general curvilinear coordinate system, the biharmonic term is given by

$$\widetilde{\nabla}^4\widetilde{\mathsf{W}}=\frac{1}{\sqrt{\det\tilde{\mathbf{G}}}}\frac{\partial}{\partial\tilde{X}^{\tilde{A}}}\left[\sqrt{\det\tilde{\mathbf{G}}}\frac{\partial}{\partial\tilde{X}^{\tilde{B}}}\left(\frac{1}{\sqrt{\det\tilde{\mathbf{G}}}}\frac{\partial}{\partial\tilde{X}^{\tilde{C}}}\left[\sqrt{\det\tilde{\mathbf{G}}}\frac{\partial\widetilde{\mathsf{W}}}{\partial\tilde{X}^{\tilde{D}}}\tilde{G}^{\tilde{C}\tilde{D}}\right]\right)\tilde{G}^{\tilde{A}\tilde{B}}\right].$$

In particular, we note that transforming the biharmonic equation under the cloaking map does not fully determine the flexural rigidity tensor of the cloak. This should not be surprising as the biharmonic equation (8.168) does not correspond to a unique flexural rigidity tensor for the virtual plate (cf. (8.171)). Also, note that (8.174) is consistent with (8.148) in the sense that, if one only knows the flexural rigidity of the virtual plate $\tilde{\mathbb{C}}$ up to (8.171), then (8.148) gives us (8.174).

The work of Colquitt et al. *[244] on flexural cloaking*

Next, we show that the transformation cloaking formulation of Kirchhoff-Love plates given in [244] is, unfortunately, incorrect. They start from the biharmonic governing equation of an isotropic and homogeneous elastic plate and apply [159, Lemma 2.1] twice to transform the governing equation under the cloaking map. Using this lemma, one imposes certain constraints on the gradients of the displacements and the gradients of the Laplacian of the displacements in the virtual and physical plates. These constraints are incompatible with the way the displacement field is transformed under a cloaking map. In particular, these constraints will force the cloaking transformation to isometrically map the governing equations of the virtual plate to those of the physical plate. Therefore, the virtual and physical plates are essentially the same elastic plate and this formulation of transformation cloaking does not result in any new information. Ignoring these constraints and what restrictions they impose on the cloaking map have resulted in deriving incorrect transformed fields for the physical plate, and in particular, we show that Colquitt *et al.* [244]'s transformed flexural rigidity is incorrect. The governing equation of flexural waves in an isotropic and homogeneous thin plate with the flexural rigidity $D^{(0)} = Eh^3/12(1-\nu^2)$, mass density P, thickness h, and in the presence of time-harmonic anti-plane excitations with frequency ω reads

$$D^{(0)}\nabla^4_{\mathbf{X}}W(\mathbf{X}) - Ph\omega^2 W(\mathbf{X}) = 0\,, \quad \mathbf{X} \in \chi \subseteq \mathbb{R}^2\,. \tag{8.175}$$

Colquitt *et al.* [244] rewrite the governing equation as

$$\left(\nabla^4_{\mathbf{X}} - \frac{Ph}{D^{(0)}}\omega^2\right) W(\mathbf{X}) = 0\,, \quad \mathbf{X} \in \chi \subseteq \mathbb{R}^2\,. \tag{8.176}$$

They transform (8.176) by applying [159, Lemma 2.1] twice via an invertible map $\mathcal{F} : \chi \to \Omega$, where $\mathbf{x} = \mathcal{F}(\mathbf{X})$, $\mathbf{F} = \nabla_{\mathbf{X}}\mathbf{x}$, and $J = \det \mathbf{F}$, and obtain

$$\left(\nabla \cdot J^{-1}\mathbf{F}\mathbf{F}^{\mathsf{T}}\nabla J \nabla \cdot J^{-1}\mathbf{F}\mathbf{F}^{\mathsf{T}}\nabla - \frac{Ph}{JD^{(0)}}\omega^2\right) W(\mathbf{x}) = 0\,, \quad \mathbf{x} \in \Omega\,. \tag{8.177}$$

In particular, their transformed rigidity tensor is given by

$$D_{ijkl} = D^{(0)} J G_{ij} G_{kl}\,, \tag{8.178}$$

where $G_{ij} = J^{-1}F_{ip}F_{jp}$.

Let us discuss the implication of applying [159, Lemma 2.1] to the out-of-plane displacement field. In particular, we show that the way the fields are transformed in this lemma is incompatible with the underlying assumption of the transformation of the displacement fields under a cloaking map, i.e., $\widetilde{\mathsf{W}} \circ \Xi = \mathsf{W}$. We work in general curvilinear coordinates and distinguish between the out-of-plane displacement fields in the virtual and physical plates, i.e., $\widetilde{\mathsf{W}}$ and W, respectively. The gradients of $\widetilde{\mathsf{W}}$ and W are written in components as

$$(\widetilde{\nabla}\widetilde{\mathsf{W}})^{\tilde{A}} = \tilde{G}^{\tilde{A}\tilde{B}} \frac{\partial \widetilde{\mathsf{W}}}{\partial \tilde{X}^{\tilde{B}}}\,, \quad (\nabla \mathsf{W})^{A} = G^{AB} \frac{\partial \mathsf{W}}{\partial X^{B}}\,. \tag{8.179}$$

Note that

$$\begin{aligned} \widetilde{\nabla}^2\widetilde{\mathsf{W}} &= \widetilde{\mathrm{DIV}}(\widetilde{\nabla}\widetilde{\mathsf{W}}) = (\widetilde{\nabla}\widetilde{\mathsf{W}})^{\tilde{A}}{}_{|\tilde{A}} = \frac{1}{\sqrt{\det \tilde{\mathbf{G}}}} \frac{\partial}{\partial \tilde{X}^{\tilde{A}}} \left[\sqrt{\det \tilde{\mathbf{G}}} \frac{\partial \widetilde{\mathsf{W}}}{\partial \tilde{X}^{\tilde{B}}} \tilde{G}^{\tilde{A}\tilde{B}}\right]\,, \\ \nabla^2\mathsf{W} &= \mathrm{DIV}(\nabla \mathsf{W}) = (\nabla \mathsf{W})^{A}{}_{|A} = \frac{1}{\sqrt{\det \mathbf{G}}} \frac{\partial}{\partial X^{A}} \left[\sqrt{\det \mathbf{G}} \frac{\partial \mathsf{W}}{\partial X^{B}} G^{AB}\right]\,, \end{aligned} \tag{8.180}$$

where ∇^2 is known as the *Laplace-Beltrami* operator. Applying [159, Lemma 2.1] to the gradient of the displacement field, one assumes that $(\nabla \mathsf{W})^A = J_\Xi (\overset{\Xi}{F}{}^{-1})^A{}_{\tilde{A}} (\widetilde{\nabla}\widetilde{\mathsf{W}})^{\tilde{A}}$, or in components

$$\frac{\partial \mathsf{W}}{\partial X^B} G^{AB} = J_\Xi (\overset{\Xi}{F}{}^{-1})^A{}_{\tilde{A}} \frac{\partial \widetilde{\mathsf{W}}}{\partial \tilde{X}^{\tilde{B}}} \tilde{G}^{\tilde{A}\tilde{B}} , \tag{8.181}$$

i.e., the gradients of the out-of-plane displacements in the virtual and physical plates are related by the Piola transformation. Then, under this assumption, one obtains

$$\left[\frac{\partial \mathsf{W}}{\partial X^B} G^{AB}\right]_{|A} = \left[J_\Xi (\overset{\Xi}{F}{}^{-1})^A{}_{\tilde{A}} \frac{\partial \widetilde{\mathsf{W}}}{\partial \tilde{X}^{\tilde{B}}} \tilde{G}^{\tilde{A}\tilde{B}}\right]_{|A} = J_\Xi \left[\frac{\partial \widetilde{\mathsf{W}}}{\partial \tilde{X}^{\tilde{B}}} \tilde{G}^{\tilde{A}\tilde{B}}\right]_{|\tilde{A}} , \tag{8.182}$$

or, equivalently, $\nabla^2 \mathsf{W} = \mathrm{DIV}(\nabla \mathsf{W}) = J_\Xi \widetilde{\mathrm{DIV}}(\widetilde{\nabla}\widetilde{\mathsf{W}}) = J_\Xi \widetilde{\nabla}^2 \widetilde{\mathsf{W}}$. Applying the lemma to the Laplacian $\nabla^2 \mathsf{W}$, one finds

$$\begin{aligned}
\nabla^4 \mathsf{W} &= \left[\frac{\partial}{\partial X^B}\left(\left[\frac{\partial \mathsf{W}}{\partial X^D} G^{CD}\right]_{|C}\right) G^{AB}\right]_{|A} = \left[J_\Xi (\overset{\Xi}{F}{}^{-1})^A{}_{\tilde{A}} \frac{\partial}{\partial \tilde{X}^{\tilde{B}}}\left(\left[\frac{\partial \widetilde{\mathsf{W}}}{\partial \tilde{X}^{\tilde{D}}} \tilde{G}^{\tilde{C}\tilde{D}}\right]_{|\tilde{C}}\right) \tilde{G}^{\tilde{A}\tilde{B}}\right]_{|A} \\
&= J_\Xi \left[\frac{\partial}{\partial \tilde{X}^{\tilde{B}}}\left(\left[\frac{\partial \widetilde{\mathsf{W}}}{\partial \tilde{X}^{\tilde{D}}} \tilde{G}^{\tilde{C}\tilde{D}}\right]_{|\tilde{C}}\right) \tilde{G}^{\tilde{A}\tilde{B}}\right]_{|\tilde{A}} = J_\Xi \widetilde{\nabla}^4 \widetilde{\mathsf{W}} .
\end{aligned} \tag{8.183}$$

Using (8.182), one rewrites (8.183) as

$$\nabla^4 \mathsf{W} = \left[J_\Xi (\overset{\Xi}{F}{}^{-1})^A{}_{\tilde{A}} \frac{\partial}{\partial \tilde{X}^{\tilde{B}}}\left[J_\Xi^{-1}\left(J_\Xi (\overset{\Xi}{F}{}^{-1})^C{}_{\tilde{C}} \frac{\partial \widetilde{\mathsf{W}}}{\partial \tilde{X}^{\tilde{D}}} \tilde{G}^{\tilde{C}\tilde{D}}\right)_{|C}\right] \tilde{G}^{\tilde{A}\tilde{B}}\right]_{|A} = J_\Xi \widetilde{\nabla}^4 \widetilde{\mathsf{W}} . \tag{8.184}$$

Colquitt *et al.* [244] and many other researchers start from the biharmonic equation for the virtual plate, i.e., $D^{(0)} \widetilde{\nabla}^4 \widetilde{\mathsf{W}} - Ph\omega^2 \widetilde{\mathsf{W}} = 0$. Then, they use (8.184) and rewrite the biharmonic equation as

$$\left[J_\Xi (\overset{\Xi}{F}{}^{-1})^A{}_{\tilde{A}} \frac{\partial}{\partial \tilde{X}^{\tilde{B}}}\left[J_\Xi^{-1}\left(J_\Xi (\overset{\Xi}{F}{}^{-1})^C{}_{\tilde{C}} \frac{\partial \widetilde{\mathsf{W}}}{\partial \tilde{X}^{\tilde{D}}} \tilde{G}^{\tilde{C}\tilde{D}}\right)_{|C}\right] \tilde{G}^{\tilde{A}\tilde{B}}\right]_{|A} - J_\Xi \frac{Ph}{D^{(0)}} \omega^2 \widetilde{\mathsf{W}} = 0 . \tag{8.185}$$

They implicitly assume that the virtual and physical plates have the same displacement fields, i.e., $\mathsf{W} = \widetilde{\mathsf{W}} \circ \Xi$,[29] or, $\mathsf{W} = \widetilde{\mathsf{W}} \circ \mathcal{F}^{-1}$ (cf. (8.177)), and write (8.185) as (note that their mapping $\mathcal{F}$ corresponds to Ξ^{-1})

$$\begin{split}&\left[J_\Xi(\bar{\bar{\bar{F}}}^{-1})^A{}_{\tilde{A}}(\bar{\bar{\bar{F}}}^{-1})^B{}_{\tilde{B}}\frac{\partial}{\partial X^B}\left[J_\Xi^{-1}\left(J_\Xi(\bar{\bar{\bar{F}}}^{-1})^C{}_{\tilde{C}}(\bar{\bar{\bar{F}}}^{-1})^D{}_{\tilde{D}}\frac{\partial \mathsf{W}}{\partial X^D}\tilde{G}^{\tilde{C}\tilde{D}}\right)_{|C}\right]\tilde{G}^{\tilde{A}\tilde{B}}\right]_{|A} \\ &-J_\Xi\frac{Ph}{D^{(0)}}\omega^2\mathsf{W}=0\,.\end{split} \tag{8.186}$$

However, this formulation of the transformation cloaking problem is problematic for a number of reasons:

(i) Once one assumes $\mathsf{W} = \widetilde{\mathsf{W}} \circ \Xi$, the gradients of displacements are related by the chain rule as $\frac{\partial \widetilde{\mathsf{W}}}{\partial \tilde{X}^{\tilde{D}}} = (\bar{\bar{\bar{F}}}^{-1})^D{}_{\tilde{D}}\frac{\partial \mathsf{W}}{\partial X^D}$, and not by the Piola transformation (8.181), which is the underlying assumption of [159, Lemma 2.1]. In other words, one cannot assume that $\mathsf{W} = \widetilde{\mathsf{W}} \circ \Xi$, and at the same time use the Piola transform, which is what Colquitt *et al.* [244] did; in order to derive (8.186) from (8.185) they used the chain rule to relate the gradients of displacements as $\frac{\partial \widetilde{\mathsf{W}}}{\partial \tilde{X}^{\tilde{D}}} = (\bar{\bar{\bar{F}}}^{-1})^D{}_{\tilde{D}}\frac{\partial \mathsf{W}}{\partial X^D}$.

(ii) Applying [159, Lemma 2.1] twice requires the following extra constraint on the gradients of the Laplacian terms (this is similar to (8.181)):

$$\frac{\partial}{\partial X^B}\left[\left(\frac{\partial \mathsf{W}}{\partial X^D}G^{CD}\right)_{|C}\right]G^{AB} = J_\Xi(\bar{\bar{\bar{F}}}^{-1})^A{}_{\tilde{A}}\frac{\partial}{\partial \tilde{X}^{\tilde{B}}}\left[\left(\frac{\partial \widetilde{\mathsf{W}}}{\partial \tilde{X}^{\tilde{D}}}\tilde{G}^{\tilde{C}\tilde{D}}\right)_{|\tilde{C}}\right]\tilde{G}^{\tilde{A}\tilde{B}}\,, \tag{8.187}$$

where using (8.181), it is simplified to read

$$\bar{\bar{\bar{F}}}^{\tilde{E}}{}_B\frac{\partial}{\partial \tilde{X}^{\tilde{E}}}\left[J_\Xi\left(\frac{\partial \widetilde{\mathsf{W}}}{\partial \tilde{X}^{\tilde{D}}}\tilde{G}^{\tilde{C}\tilde{D}}\right)_{|\tilde{C}}\right]G^{AB} = J_\Xi(\bar{\bar{\bar{F}}}^{-1})^A{}_{\tilde{A}}\tilde{G}^{\tilde{A}\tilde{B}}\frac{\partial}{\partial \tilde{X}^{\tilde{B}}}\left[\left(\frac{\partial \widetilde{\mathsf{W}}}{\partial \tilde{X}^{\tilde{D}}}\tilde{G}^{\tilde{C}\tilde{D}}\right)_{|\tilde{C}}\right]\,. \tag{8.188}$$

Colquitt *et al.* [244] use the lemma twice and write their Eq. (2) without mentioning the constraint (8.188) or (8.181) and even checking if these constraints are compatible with the underlying assumption that $\mathsf{W} = \widetilde{\mathsf{W}} \circ \Xi$. Let us see what restrictions these constraints impose on the cloaking map. From

[29] This assumption ensures that the second term in (8.175) remains form invariant under the cloaking map.

(8.181) and the chain rule $\frac{\partial \widetilde{\mathsf{W}}}{\partial \tilde{X}^{\tilde{D}}} = (\bar{\bar{\bar{F}}}^{-1})^{D}{}_{\tilde{D}} \frac{\partial \mathsf{W}}{\partial X^{D}}$, one obtains

$$G^{AB} = J_{\Xi} (\bar{\bar{\bar{F}}}^{-1})^{A}{}_{\tilde{A}} (\bar{\bar{\bar{F}}}^{-1})^{B}{}_{\tilde{B}} \tilde{G}^{\tilde{A}\tilde{B}} \,. \tag{8.189}$$

Substituting (8.189) into (8.188), one finds

$$\frac{\partial}{\partial \tilde{X}^{\tilde{B}}} \left[(J_{\Xi} - 1) \left[\frac{\partial \widetilde{\mathsf{W}}}{\partial \tilde{X}^{\tilde{D}}} \tilde{G}^{\tilde{C}\tilde{D}} \right]_{|\tilde{C}} \right] = \frac{\partial}{\partial \tilde{X}^{\tilde{B}}} \left[(J_{\Xi} - 1)\, \widetilde{\nabla}^{2} \widetilde{\mathsf{W}} \right] = 0 \,. \tag{8.190}$$

Knowing that (8.190) must hold for an arbitrary displacement field $\widetilde{\mathsf{W}}$, one concludes that $J_{\Xi} = 1$,[30] and thus, $\mathbf{G} = \Xi^{*}\tilde{\mathbf{G}}$, meaning that the physical and virtual plates are isometric and are essentially the same elastic plate with the same mechanical response. To see this more clearly, using the fact that $J_{\Xi} = 1$, and $\mathbf{G} = \Xi^{*}\tilde{\mathbf{G}}$, or in components, $G^{AB} = (\bar{\bar{\bar{F}}}^{-1})^{A}{}_{\tilde{A}} (\bar{\bar{\bar{F}}}^{-1})^{B}{}_{\tilde{B}} \tilde{G}^{\tilde{A}\tilde{B}}$ and the metric compatibility of $\mathbf{G}$, i.e., $G_{AB|C} = 0$, (8.186) is simplified to read

$$D^{(0)} G^{AB} G^{CD} \left(\frac{\partial \mathsf{W}}{\partial X^{D}} \right)_{|C|B|A} - Ph\omega^{2} \mathsf{W} = 0 \,, \tag{8.191}$$

i.e., $D^{(0)} \nabla^{4} \mathsf{W} - Ph\omega^{2} \mathsf{W} = 0$, which is identical to the biharmonic equation for the virtual plate $D^{(0)} \widetilde{\nabla}^{4} \widetilde{\mathsf{W}} - Ph\omega^{2} \widetilde{\mathsf{W}} = 0$. Therefore, the physical and the virtual plates are the same elastic plate and the application of [159, Lemma 2.1] maps the biharmonic equation to itself; it does not result in any new information.

(iii) Colquitt *et al.* [244] express (8.186) in Cartesian coordinates and looking at the fourth-order derivatives of the displacement, i.e., $J_{\Xi} (\bar{\bar{\bar{F}}}^{-1})^{A}{}_{\tilde{A}} (\bar{\bar{\bar{F}}}^{-1})^{B}{}_{\tilde{A}} (\bar{\bar{\bar{F}}}^{-1})^{C}{}_{\tilde{C}} (\bar{\bar{\bar{F}}}^{-1})^{D}{}_{\tilde{C}} \frac{\partial^{4} \mathsf{W}}{\partial X^{D} \partial X^{C} \partial X^{B} \partial X^{A}}$ (summations on $\tilde{A}$ and $\tilde{C}$), incorrectly conclude that the flexural rigidity of the cloak is given by (8.178), or in our notation, $D^{ABCD} = D^{(0)} J_{\Xi} (\bar{\bar{\bar{F}}}^{-1})^{A}{}_{\tilde{A}} (\bar{\bar{\bar{F}}}^{-1})^{B}{}_{\tilde{A}} (\bar{\bar{\bar{F}}}^{-1})^{C}{}_{\tilde{C}} (\bar{\bar{\bar{F}}}^{-1})^{D}{}_{\tilde{C}}$. The reason for this mistake is that (8.186) has some hidden strong assumptions. Incorporating these assumptions one arrives at (8.191). In other words, one needs to look at (8.191) in order to calculate the transformed elastic con-

[30] Note that (8.190) implies that $(J_{\Xi} - 1)\, \widetilde{\nabla}^{2} \widetilde{\mathsf{W}} = C$, where C is a constant. Recalling that on the outer boundary of the cloak $\bar{\bar{\bar{\mathbf{F}}}}|_{\partial_{o}\mathcal{C}} = id$, and thus, $J_{\Xi}|_{\partial_{o}\mathcal{C}} = 1$, and given that $\widetilde{\mathsf{W}}$ is smooth, one concludes that $C = 0$. Therefore, $J_{\Xi} = 1$.

stants, and not (8.186). Looking at the term in (8.191) with fourth-order derivatives the transformed elasticity tensor is given by $D^{(0)}\frac{\partial^4 \mathsf{W}}{\partial X^C \partial X^C \partial X^A \partial X^A}$ in Cartesian coordinates. However, (8.191) is nothing but the governing equation of the virtual plate, i.e., the virtual and physical plates are identical. Note that when the cloaking transformation is the identity, i.e., $\mathcal{F} = id$, Colquitt *et al.* [244]'s transformed rigidity tensor (8.178) is not reduced to that of the homogeneous and isotropic plate (8.159) and is not even positive definite (see also [252]). To see this note that for the identity cloaking map (8.178) is simplified to read

$$\boldsymbol{D} = \left[D^{ijkl}\right] = \frac{Eh^3}{12(1+\nu)} \begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix}, \tag{8.192}$$

which clearly does not agree with the flexural rigidity of the isotropic and homogeneous elastic plate (8.159). It is also immediate to see that (8.192) has zero eigenvalues, and thus, is not positive definite.

(iv) The traction due to the so-called membrane forces obtained in [244] do not vanish on the boundary of the hole, and thus, the hole surface is not traction-free. This means that in the numerical simulations presented in [244], one needs to apply some forces on the boundary of the hole even if the transformation cloaking problem had been properly formulated. Furthermore, the finite traction due to the membrane forces does not vanish on the boundary of the cloak either, and their cloaking transformation does not have the identity tangent map on the outer boundary of the cloak $\partial_o\mathcal{C}$, i.e., $T\mathcal{F}|_{\partial_o\mathcal{C}} = \overset{\mathcal{F}}{\mathbf{F}}|_{\partial_o\mathcal{C}} \neq id$. Therefore, the physical and virtual plates cannot have identical current configurations outside the cloaking region.

Remark 8.4.6. In this remark we show that the work of Colquitt *et al.* [251] on the cloaking of the out-of-plane shear waves for the Helmholtz equation is also incorrect because similar to their flexural cloaking formulation [244] their use of the Piola transformation is inconsistent with their displacement transformation. Colquitt *et al.* [251] start from the Helmholtz equation for an isotropic and homogeneous medium

$$\mu\nabla_X \cdot (\nabla_X)u(X) + \varrho\,\omega^2 u(X) = 0\,, \quad X \in \chi \subset \mathbb{R}^2\,, \tag{8.193}$$

where μ and ϱ are, respectively, the shear modulus and the mass density of the isotropic and homogeneous medium, and u is the out-of-plane displacement. Applying [159, Lemma 2.1] using an invertible map $\mathcal{F} : \chi \to \Omega$ such that $\mathbf{x} = \mathcal{F}(\mathbf{X})$, $\mathbf{F} = \nabla_{\mathbf{X}}\mathbf{x}$, and $J = \det \mathbf{F}$, they transform (8.193) as

$$\left[\nabla \cdot (\boldsymbol{\mathcal{C}}(x)\nabla) + \rho(x)\omega^2\right] u(x) = 0\,, \quad x \in \Omega \subset \mathbb{R}^2\,, \tag{8.194}$$

where $\boldsymbol{\mathcal{C}}(x) = \mu/J(x)\mathbf{F}(x)\mathbf{F}^{\mathsf{T}}(x)$ is the transformed stiffness matrix and the transformed mass density is given by $\rho(x) = \varrho/J(x)$. To write this in our notation, one starts with the Helmholtz equation for the virtual medium $\tilde{\mu}\,\widetilde{\nabla}^2\widetilde{\mathsf{W}} + \tilde{\rho}\,\omega^2\widetilde{\mathsf{W}} = 0$, where $\tilde{\mu}$, $\tilde{\rho}$, and $\widetilde{\mathsf{W}}$ are, respectively, the shear modulus, the mass density, and the displacement in the virtual medium. Then, provided that (8.181) holds, one may use [159, Lemma 2.1] (see (8.182)), to obtain the transformed equation as

$$\tilde{\mu}\left[J_\Xi(\bar{\bar{\bar{F}}}^{-1})^A{}_{\tilde{A}}\frac{\partial\widetilde{\mathsf{W}}}{\partial\tilde{X}^{\tilde{B}}}\tilde{G}^{\tilde{A}\tilde{B}}\right]_{|A} + \tilde{\rho}\,J_\Xi\omega^2\widetilde{\mathsf{W}} = 0\,. \tag{8.195}$$

Next, assuming that the virtual and the physical media have identical displacements, i.e., $\mathsf{W} = \widetilde{\mathsf{W}} \circ \Xi$ (which is what Colquitt *et al.* [251] implicitly assume) and using the chain rule, one finds

$$\left[\tilde{\mu}J_\Xi(\bar{\bar{\bar{F}}}^{-1})^A{}_{\tilde{A}}(\bar{\bar{\bar{F}}}^{-1})^B{}_{\tilde{B}}\frac{\partial\mathsf{W}}{\partial X^B}\tilde{G}^{\tilde{A}\tilde{B}}\right]_{|A} + \tilde{\rho}\,J_\Xi\omega^2\mathsf{W} = 0\,. \tag{8.196}$$

This is identical to what they have in (8.194), recalling that $\mathcal{F}$ corresponds to Ξ^{-1}. As we discussed in the case of flexural transformation cloaking, (8.181) imposes a strong constraint on the cloaking map, which is given by (8.189). In fact, in Cartesian coordinates, (8.189) is simplified to read $J_\Xi(\bar{\bar{\bar{F}}}^{-1})^A{}_{\tilde{A}}(\bar{\bar{\bar{F}}}^{-1})^B{}_{\tilde{A}} = \delta^{AB}$ (summation on $\tilde{A}$). Solving this relation, one obtains the tangent of the (inverse) cloaking map as[31]

$$\bar{\bar{\mathbf{F}}}^{-1}(\tilde{X},\tilde{Y}) = \begin{bmatrix} \alpha(\tilde{X},\tilde{Y}) & \beta(\tilde{X},\tilde{Y}) \\ -\beta(\tilde{X},\tilde{Y}) & \alpha(\tilde{X},\tilde{Y}) \end{bmatrix}, \tag{8.197}$$

where $\alpha^2+\beta^2>0$. Note that the (bulk) compatibility of $\bar{\bar{\mathbf{F}}}^{-1}$ are written as $(\bar{\bar{F}}^{-1})^{A}{}_{\tilde{A}|\tilde{B}} = (\bar{\bar{F}}^{-1})^{A}{}_{\tilde{B}|\tilde{A}}$, and thus, one has[32]

$$\frac{\partial\alpha}{\partial\tilde{X}} = -\frac{\partial\beta}{\partial\tilde{Y}}, \quad \text{and} \quad \frac{\partial\alpha}{\partial\tilde{Y}} = \frac{\partial\beta}{\partial\tilde{X}}. \tag{8.198}$$

Hence, one concludes that α and β are governed by the Laplace equation, i.e.,

$$\frac{\partial^2\alpha}{\partial\tilde{X}^2} + \frac{\partial^2\alpha}{\partial\tilde{Y}^2} = 0, \quad \text{and} \quad \frac{\partial^2\beta}{\partial\tilde{X}^2} + \frac{\partial^2\beta}{\partial\tilde{Y}^2} = 0. \tag{8.199}$$

The mapping $\mathcal{F}$ that Colquitt *et al.* [251] introduce to design a square-shaped cloak by shrinking a finite rectangular cavity to a small one is not of the form (8.197).

8.4.2 Elastodynamic transformation cloaking in plates in the presence of in-plane and out-of-plane displacements

In this section, we relax the pure bending assumption and formulate the transformation cloaking problem of a classical elastic plate in the presence of both in-plane and out-of-plane displacements. In doing so, we let the physical plate undergo finite in-plane deformations while staying flat, i.e., ($\mathring{\varphi}^{\perp} = id$ and $\mathring{\boldsymbol{\theta}} = \mathbf{0}$), whereas the virtual plate does not go through any finite deformation ($\mathring{\tilde{\varphi}} = id$). For simplicity of notation, let us drop the superscripts and denote the normal and tangential displacement fields of the physical (virtual) plate $\mathbf{U}^{\perp}$ and $\mathbf{U}^{\top}$ ($\tilde{\mathbf{U}}^{\perp}$ and $\tilde{\mathbf{U}}^{\top}$) by W and U ($\widetilde{\mathsf{W}}$ and $\tilde{\mathsf{U}}$), respectively. The tangential and normal displacement fields are transformed as

$$\tilde{\mathsf{U}}^{\tilde{a}} = \mathsf{s}^{\tilde{a}}{}_{a} \circ \mathring{\varphi}\, \mathsf{U}^{a} \circ \Xi, \quad \text{and} \quad \widetilde{\mathsf{W}} = \mathsf{W} \circ \Xi^{-1}. \tag{8.200}$$

[31]Notice that (8.197) can be represented as the rotation matrix multiplied by the scalar $\left(\alpha^2+\beta^2\right)$.

[32]Note that if one defines the complex function as $f(\tilde{X}+i\tilde{Y}) = \beta(\tilde{X},\tilde{Y}) + i\alpha(\tilde{X},\tilde{Y})$, then (8.198) gives the Cauchy-Riemann equations and tells us that the complex function should be holomorphic.

The linearized balance of linear momentum for the virtual plate, which has uniform elastic properties, reads

$$\delta\tilde{P}^{\tilde{a}\tilde{A}}{}_{|\tilde{A}} = \tilde{\rho}_0\,\ddot{\tilde{\mathsf{U}}}^{\tilde{a}}\,, \quad \text{and} \quad -(\mathring{\tilde{F}}^{-1})^{\tilde{B}}{}_{\tilde{a}}\delta\tilde{\mathsf{M}}^{\tilde{a}\tilde{A}}{}_{|\tilde{A}|\tilde{B}} = \tilde{\rho}_0\,\ddot{\tilde{\mathsf{W}}}\,, \tag{8.201}$$

where

$$\delta\tilde{P}^{\tilde{a}\tilde{A}} = \tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\tilde{\mathsf{U}}_{\tilde{b}|\tilde{B}} + \frac{1}{2}\tilde{\mathbb{B}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}(\widetilde{\mathsf{W}}_{,\tilde{b}})_{|\tilde{B}}\,, \quad \text{and} \quad \delta\tilde{\mathsf{M}}^{\tilde{a}\tilde{A}} = \frac{1}{2}\tilde{\mathbb{C}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}(\widetilde{\mathsf{W}}_{,\tilde{b}})_{|\tilde{B}} + \frac{1}{2}\tilde{\mathbb{B}}^{\tilde{b}\tilde{B}\tilde{a}\tilde{A}}\tilde{\mathsf{U}}_{\tilde{b}|\tilde{B}}\,. \tag{8.202}$$

The symmetries of the elastic constants (8.110) for the virtual plate implies that

$$\tilde{\mathbb{A}}^{[\tilde{a}\tilde{A}\tilde{c}\tilde{B}}\mathring{\tilde{F}}^{\tilde{b}]}{}_{\tilde{A}} = 0\,, \qquad \tilde{\mathbb{B}}^{\tilde{c}\tilde{B}[\tilde{a}\tilde{A}}\mathring{\tilde{F}}^{\tilde{b}]}{}_{\tilde{A}} = 0\,, \tag{8.203a}$$

$$\tilde{\mathbb{B}}^{[\tilde{a}\tilde{A}\tilde{c}\tilde{B}}\mathring{\tilde{F}}^{\tilde{b}]}{}_{\tilde{A}} = 0\,, \qquad \tilde{\mathbb{C}}^{[\tilde{a}\tilde{A}\tilde{c}\tilde{B}}\mathring{\tilde{F}}^{\tilde{b}]}{}_{\tilde{A}} = 0\,. \tag{8.203b}$$

Knowing that the virtual plate is uniform, its elastic properties (constants) are (covariantly) constant. Thus, (8.201) is simplified to read

$$\begin{aligned} \tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\tilde{\mathsf{U}}_{\tilde{b}|\tilde{B}|\tilde{A}} + \frac{1}{2}\tilde{\mathbb{B}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}(\widetilde{\mathsf{W}}_{,\tilde{b}})_{|\tilde{B}|\tilde{A}} &= \tilde{\rho}_0\,\ddot{\tilde{\mathsf{U}}}^{\tilde{a}}\,, \\ -\frac{1}{2}(\mathring{\tilde{F}}^{-1})^{\tilde{B}}{}_{\tilde{a}}\left(\tilde{\mathbb{C}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{C}}(\widetilde{\mathsf{W}}_{,\tilde{b}})_{|\tilde{C}|\tilde{A}|\tilde{B}} + \tilde{\mathbb{B}}^{\tilde{b}\tilde{C}\tilde{a}\tilde{A}}\tilde{\mathsf{U}}_{\tilde{b}|\tilde{C}|\tilde{A}|\tilde{B}}\right) &= \tilde{\rho}_0\,\ddot{\tilde{\mathsf{W}}}\,. \end{aligned} \tag{8.204}$$

The balance of linear momentum for the physical plate in the absence of initial couple-stress ($\mathring{\mathbf{M}} = \mathbf{0}$) reads (cf. (8.122))

$$\delta P^{aA}{}_{|A} + \rho_0\mathring{\mathfrak{B}}^{\perp}g^{ab}\mathsf{W}_{,b} + \rho_0 g^{ab}(\mathring{\mathfrak{B}}^{\top})^c\mathsf{U}_{c|b} - \rho_0\,g^{ac}(\mathring{F}^{-1})^B{}_b\,(\mathsf{W}_{,c})_{|B}\,\mathring{\mathfrak{L}}^b = \rho_0\ddot{\mathsf{U}}^a\,, \tag{8.205a}$$

$$\begin{aligned} &\mathring{P}^{aA}\,(\mathsf{W}_{,a})_{|A} - \left[(\mathring{F}^{-1})^B{}_a\delta\mathsf{M}^{aA}{}_{|A}\right]_{|B} - \rho_0(\mathring{\mathfrak{B}}^{\top})^b\mathsf{W}_{,b} + \left(\rho_0\,\mathring{\mathfrak{L}}^c\mathsf{U}_{c|b}g^{ab}(\mathring{F}^{-1})^A{}_a\right)_{|A} \\ &\qquad - \left[\rho_0\,\mathring{\mathfrak{L}}^a\,(\mathring{F}^{-1})^B{}_a(\mathring{F}^{-1})^A{}_b\mathsf{U}^b{}_{|B}\right]_{|A} = \rho_0\ddot{\mathsf{W}}\,, \end{aligned} \tag{8.205b}$$

where using (8.124)

$$
\begin{aligned}
\delta P^{aA} &= \left[\mathbb{A}^{aAbB} + \mathring{P}^{cA}(\mathring{F}^{-1})^{B}{}_{c}\, g^{ab}\right] \mathsf{U}_{b|B} + \frac{1}{2}\mathbb{B}^{aAbB}(\mathsf{W}_{,b})_{|B}\,, \\
\delta \mathsf{M}^{aA} &= \frac{1}{2}\mathbb{C}^{aAbB}(\mathsf{W}_{,b})_{|B} + \frac{1}{2}\mathbb{B}^{bBaA}\mathsf{U}_{b|B}\,.
\end{aligned}
\tag{8.206}
$$

Note that

$$
\begin{aligned}
\tilde{\mathsf{U}}_{\tilde{b}|\tilde{C}} =& (\mathsf{s}^{-1})^{b}{}_{\tilde{b}}(\overset{\equiv}{F}{}^{-1})^{C}{}_{\tilde{C}}\mathsf{U}_{b|C}\,, \\
\tilde{\mathsf{U}}_{\tilde{b}|\tilde{C}|\tilde{A}} =& (\mathsf{s}^{-1})^{b}{}_{\tilde{b}}\left[(\overset{\equiv}{F}{}^{-1})^{C}{}_{\tilde{C}|\tilde{A}}\mathsf{U}_{b|C} + (\overset{\equiv}{F}{}^{-1})^{C}{}_{\tilde{C}}(\overset{\equiv}{F}{}^{-1})^{A}{}_{\tilde{A}}\mathsf{U}_{b|C|A}\right]\,, \\
\tilde{\mathsf{U}}_{\tilde{b}|\tilde{C}|\tilde{A}|\tilde{B}} =& (\mathsf{s}^{-1})^{b}{}_{\tilde{b}}\Big[(\overset{\equiv}{F}{}^{-1})^{C}{}_{\tilde{C}|\tilde{A}|\tilde{B}}\mathsf{U}_{b|C} + \Big((\overset{\equiv}{F}{}^{-1})^{C}{}_{\tilde{C}|\tilde{A}}(\overset{\equiv}{F}{}^{-1})^{A}{}_{\tilde{B}} + (\overset{\equiv}{F}{}^{-1})^{C}{}_{\tilde{C}|\tilde{B}}(\overset{\equiv}{F}{}^{-1})^{A}{}_{\tilde{A}} \\
&+ (\overset{\equiv}{F}{}^{-1})^{C}{}_{\tilde{C}}(\overset{\equiv}{F}{}^{-1})^{A}{}_{\tilde{A}|\tilde{B}}\Big)\mathsf{U}_{b|C|A} + (\overset{\equiv}{F}{}^{-1})^{C}{}_{\tilde{C}}(\overset{\equiv}{F}{}^{-1})^{A}{}_{\tilde{A}}(\overset{\equiv}{F}{}^{-1})^{B}{}_{\tilde{B}}\mathsf{U}_{b|C|A|B}\Big]\,,
\end{aligned}
\tag{8.207}
$$

$$
\begin{aligned}
\widetilde{\mathsf{W}}_{,\tilde{b}} =& (\overset{\xi}{F}{}^{-1})^{b}{}_{\tilde{b}}\,\mathsf{W}_{,b}\,, \\
(\widetilde{\mathsf{W}}_{,\tilde{b}})_{|\tilde{C}} =& (\overset{\xi}{F}{}^{-1})^{b}{}_{\tilde{b}|\tilde{C}}\,\mathsf{W}_{,b} + (\overset{\xi}{F}{}^{-1})^{b}{}_{\tilde{b}}(\overset{\equiv}{F}{}^{-1})^{C}{}_{\tilde{C}}\,(\mathsf{W}_{,b})_{|C}\,, \\
(\widetilde{\mathsf{W}}_{,\tilde{b}})_{|\tilde{C}|\tilde{A}} =& (\overset{\xi}{F}{}^{-1})^{b}{}_{\tilde{b}|\tilde{C}|\tilde{A}}\,\mathsf{W}_{,b} + \Big[(\overset{\xi}{F}{}^{-1})^{b}{}_{\tilde{b}|\tilde{A}}(\overset{\equiv}{F}{}^{-1})^{C}{}_{\tilde{C}} + (\overset{\equiv}{F}{}^{-1})^{C}{}_{\tilde{C}|\tilde{A}}(\overset{\xi}{F}{}^{-1})^{b}{}_{\tilde{b}} \\
&+ (\overset{\xi}{F}{}^{-1})^{b}{}_{\tilde{b}|\tilde{C}}(\overset{\equiv}{F}{}^{-1})^{C}{}_{\tilde{A}}\Big](\mathsf{W}_{,b})_{|C} + (\overset{\xi}{F}{}^{-1})^{b}{}_{\tilde{b}}(\overset{\equiv}{F}{}^{-1})^{C}{}_{\tilde{C}}(\overset{\equiv}{F}{}^{-1})^{A}{}_{\tilde{A}}\,(\mathsf{W}_{,b})_{|C|A}\,, \\
(\widetilde{\mathsf{W}}_{,\tilde{b}})_{|\tilde{C}|\tilde{A}|\tilde{B}} =& (\overset{\xi}{F}{}^{-1})^{b}{}_{\tilde{b}|\tilde{C}|\tilde{A}|\tilde{B}}\,\mathsf{W}_{,b} + \Big[(\overset{\xi}{F}{}^{-1})^{b}{}_{\tilde{b}|\tilde{A}}(\overset{\equiv}{F}{}^{-1})^{A}{}_{\tilde{C}|\tilde{B}} + (\overset{\xi}{F}{}^{-1})^{b}{}_{\tilde{b}|\tilde{C}}(\overset{\equiv}{F}{}^{-1})^{A}{}_{\tilde{A}|\tilde{B}} \\
&+ (\overset{\xi}{F}{}^{-1})^{b}{}_{\tilde{b}|\tilde{B}}(\overset{\equiv}{F}{}^{-1})^{A}{}_{\tilde{C}|\tilde{A}} + (\overset{\xi}{F}{}^{-1})^{b}{}_{\tilde{b}|\tilde{C}|\tilde{B}}(\overset{\equiv}{F}{}^{-1})^{A}{}_{\tilde{A}} + (\overset{\xi}{F}{}^{-1})^{b}{}_{\tilde{b}|\tilde{A}|\tilde{B}}(\overset{\equiv}{F}{}^{-1})^{A}{}_{\tilde{C}} \\
&+ (\overset{\xi}{F}{}^{-1})^{b}{}_{\tilde{b}|\tilde{C}|\tilde{A}}(\overset{\equiv}{F}{}^{-1})^{A}{}_{\tilde{B}} + (\overset{\equiv}{F}{}^{-1})^{A}{}_{\tilde{C}|\tilde{A}|\tilde{B}}(\overset{\xi}{F}{}^{-1})^{b}{}_{\tilde{b}}\Big](\mathsf{W}_{,b})_{|A} \\
&+ \Big[(\overset{\xi}{F}{}^{-1})^{b}{}_{\tilde{b}|\tilde{B}}(\overset{\equiv}{F}{}^{-1})^{C}{}_{\tilde{C}}(\overset{\equiv}{F}{}^{-1})^{A}{}_{\tilde{A}} + (\overset{\equiv}{F}{}^{-1})^{C}{}_{\tilde{C}|\tilde{B}}(\overset{\xi}{F}{}^{-1})^{b}{}_{\tilde{b}}(\overset{\equiv}{F}{}^{-1})^{A}{}_{\tilde{A}} \\
&+ (\overset{\equiv}{F}{}^{-1})^{A}{}_{\tilde{A}|\tilde{B}}(\overset{\xi}{F}{}^{-1})^{b}{}_{\tilde{b}}(\overset{\equiv}{F}{}^{-1})^{C}{}_{\tilde{C}} + (\overset{\xi}{F}{}^{-1})^{b}{}_{\tilde{b}|\tilde{C}}(\overset{\equiv}{F}{}^{-1})^{C}{}_{\tilde{A}}(\overset{\equiv}{F}{}^{-1})^{A}{}_{\tilde{B}} \\
&+ (\overset{\xi}{F}{}^{-1})^{b}{}_{\tilde{b}|\tilde{A}}(\overset{\equiv}{F}{}^{-1})^{A}{}_{\tilde{B}}(\overset{\equiv}{F}{}^{-1})^{C}{}_{\tilde{C}} + (\overset{\equiv}{F}{}^{-1})^{C}{}_{\tilde{C}|\tilde{A}}(\overset{\equiv}{F}{}^{-1})^{A}{}_{\tilde{B}}(\overset{\xi}{F}{}^{-1})^{b}{}_{\tilde{b}}\Big](\mathsf{W}_{,b})_{|C|A} \\
&+ (\overset{\xi}{F}{}^{-1})^{b}{}_{\tilde{b}}(\overset{\equiv}{F}{}^{-1})^{C}{}_{\tilde{C}}(\overset{\equiv}{F}{}^{-1})^{A}{}_{\tilde{A}}(\overset{\equiv}{F}{}^{-1})^{B}{}_{\tilde{B}}\,(\mathsf{W}_{,b})_{|C|A|B}\,,
\end{aligned}
\tag{8.208}
$$

where $\overset{\xi}{\mathbf{F}} = \mathring{\tilde{\mathbf{F}}}\overset{\Xi}{\mathbf{F}}\mathring{\mathbf{F}}^{-1}$.[33] Under the cloaking transformation $\Xi : \mathcal{H} \to \tilde{\mathcal{H}}$ and using (8.207) and (8.208), the divergence term in $(8.201)_1$ is transformed via the Piola transformation as

$$\begin{aligned}
&\left[\tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\tilde{\mathsf{U}}_{\tilde{b}|\tilde{B}} + \frac{1}{2}\tilde{\mathbb{B}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}(\widetilde{\mathsf{W}}_{,\tilde{b}})_{|\tilde{B}}\right]_{|\tilde{A}} \\
&= J_{\Xi}^{-1}\left[J_{\Xi}(\overset{\Xi}{F}{}^{-1})^{A}{}_{\tilde{A}}\tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\tilde{\mathsf{U}}_{\tilde{b}|\tilde{B}} + \frac{1}{2}J_{\Xi}(\overset{\Xi}{F}{}^{-1})^{A}{}_{\tilde{A}}\tilde{\mathbb{B}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}(\widetilde{\mathsf{W}}_{,\tilde{b}})_{|\tilde{B}}\right]_{|A} \\
&= J_{\Xi}^{-1}\Bigg[J_{\Xi}(\overset{\Xi}{F}{}^{-1})^{A}{}_{\tilde{A}}(\overset{\Xi}{F}{}^{-1})^{B}{}_{\tilde{B}}(\mathsf{s}^{-1})^{b}{}_{\tilde{b}}\tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\mathsf{U}_{b|B} + \frac{1}{2}J_{\Xi}(\overset{\Xi}{F}{}^{-1})^{A}{}_{\tilde{A}}(\overset{\xi}{F}{}^{-1})^{b}{}_{\tilde{b}}(\overset{\Xi}{F}{}^{-1})^{B}{}_{\tilde{B}}\tilde{\mathbb{B}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}(\mathsf{W}_{,b})_{|B} \\
&\quad + \frac{1}{2}J_{\Xi}(\overset{\Xi}{F}{}^{-1})^{A}{}_{\tilde{A}}(\overset{\xi}{F}{}^{-1})^{b}{}_{\tilde{b}|\tilde{B}}\tilde{\mathbb{B}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\mathsf{W}_{,b}\Bigg]_{|A} .
\end{aligned} \tag{8.209}$$

Using the shifter map, we try to write (8.209) as the in-plane governing equation of the physical plate (8.205a). Therefore, the referential mass density of the physical plate is given by $\rho_0 = J_{\Xi}\tilde{\rho}_0$,[34] and recalling that the shifter map is covariantly constant, one obtains

$$\mathbb{A}^{aAbB} = J_{\Xi}(\mathsf{s}^{-1})^{a}{}_{\tilde{a}}(\overset{\Xi}{F}{}^{-1})^{A}{}_{\tilde{A}}(\mathsf{s}^{-1})^{b}{}_{\tilde{b}}(\overset{\Xi}{F}{}^{-1})^{B}{}_{\tilde{B}}\tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} - \mathring{P}^{cA}(\mathring{F}^{-1})^{B}{}_{c}g^{ab}\,, \tag{8.210a}$$

$$\mathbb{B}^{aAbB} = J_{\Xi}(\mathsf{s}^{-1})^{a}{}_{\tilde{a}}(\overset{\Xi}{F}{}^{-1})^{A}{}_{\tilde{A}}(\overset{\xi}{F}{}^{-1})^{b}{}_{\tilde{b}}(\overset{\Xi}{F}{}^{-1})^{B}{}_{\tilde{B}}\tilde{\mathbb{B}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\,, \tag{8.210b}$$

$$\rho_0\mathring{\mathfrak{B}}^{\perp} = \frac{1}{2}J_{\Xi}(\mathsf{s}^{-1})^{a}{}_{\tilde{a}}(\overset{\xi}{F}{}^{-1})^{b}{}_{\tilde{b}|\tilde{B}|\tilde{A}}\tilde{\mathbb{B}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}g_{ab}\,, \tag{8.210c}$$

$$\rho_0\mathring{\mathfrak{L}}^{b} = -\frac{1}{2}J_{\Xi}(\mathsf{s}^{-1})^{a}{}_{\tilde{a}}(\overset{\xi}{F}{}^{-1})^{b}{}_{\tilde{c}}\mathring{\tilde{F}}^{\tilde{c}}{}_{\tilde{A}}(\overset{\xi}{F}{}^{-1})^{c}{}_{\tilde{b}|\tilde{B}}\tilde{\mathbb{B}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}g_{ac}\,, \tag{8.210d}$$

[33]Note that $\mathring{\tilde{\mathbf{F}}} = id$, while $\mathring{\mathbf{F}}$ is not the identity, in general. However, for thin plates (due to the inextensibility constraint) $\mathring{\mathbf{F}} = id$, and thus, the mappings ξ and Ξ are identical, whence (8.208) reduces to (8.146) with a slight abuse of notation.

[34]Conservation of mass for the physical and virtual plates implies that $\rho_0 = \varrho\mathring{J}$ and $\tilde{\rho}_0 = \tilde{\varrho}\mathring{\tilde{J}}$. Noting that $\mathring{\tilde{\varphi}} = id$ and $J_{\xi} = \mathring{\tilde{J}}J_{\Xi}\mathring{J}^{-1}$, the spatial mass density of the cloak is given by $\varrho = J_{\xi}\tilde{\rho}_0$.

$$(\mathring{\mathfrak{B}}^{\top})^b = 0\,, \tag{8.210e}$$

where in deriving (8.210c), the Piola identity $\left[J_{\Xi}(\overset{\Xi}{F}{}^{-1})^{A}{}_{\tilde{A}}\right]_{|A} = 0$, and the fact that the virtual plate has uniform elastic parameters were used.

Remark 8.4.7. Note that in this case we have two sets of governing equations that need to be simultaneously transformed under a cloaking map: the in-plane and the out-of-plane governing equations (unlike the previous case where the out-of-plane equilibrium equation was the only non-trivial governing equation). Note also that similar to 3D elasticity, the in-plane governing equations are transformed using a Piola transformation via the cloaking map. This, in turn, implies that the density is transformed as $\rho_0 = J_{\Xi}\tilde{\rho}_0$. Now because density must be transformed the same way for the in-plane and the out-of-plane governing equations, the scalar field k introduced in §8.4.1 is is equal to the Jacobian of the cloaking map J_{Ξ}, i.e., $k(X) = J_{\Xi}(X)$.

Similarly, we write $(8.204)_2$ as the out-of-plane governing equation of the physical plate (8.205b) up to the Jacobian of the (referential) cloaking map J_{Ξ}. Hence, one finds the initial pre-stress, the tangential body force, and the flexural rigidity of the physical plate as[35]

$$\mathbb{C}^{aAbB} = J_{\Xi}(\overset{\xi}{F}{}^{-1})^{a}{}_{\tilde{a}}(\overset{\Xi}{F}{}^{-1})^{A}{}_{\tilde{A}}(\overset{\xi}{F}{}^{-1})^{b}{}_{\tilde{b}}(\overset{\Xi}{F}{}^{-1})^{B}{}_{\tilde{B}}\tilde{\mathbb{C}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\,, \tag{8.211a}$$

$$\rho_0(\mathring{\mathfrak{B}}^{\top})^b = \frac{1}{2}J_{\Xi}(\overset{\circ}{\tilde{F}}{}^{-1})^{\tilde{B}}{}_{\tilde{a}}(\overset{\xi}{F}{}^{-1})^{b}{}_{\tilde{b}|\tilde{C}|\tilde{A}|\tilde{B}}\tilde{\mathbb{C}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{C}}\,, \tag{8.211b}$$

[35]Note that

$$\begin{aligned}
&(\mathring{F}^{-1})^{A}{}_{a|B} = \frac{\partial}{\partial X^B}\left[(\mathring{F}^{-1})^{A}{}_{a}\right] + \Gamma^{A}{}_{CB}(\mathring{F}^{-1})^{C}{}_{a} - \gamma^{b}{}_{ac}(\mathring{F}^{-1})^{A}{}_{b}\mathring{F}^{c}{}_{B}\,. \\
&(\overset{\xi}{F}{}^{-1})^{a}{}_{\tilde{a}|\tilde{B}} = (\overset{\xi}{F}{}^{-1})^{a}{}_{\tilde{a}|\tilde{b}}\overset{\circ}{\tilde{F}}{}^{\tilde{b}}{}_{\tilde{B}} = \overset{\circ}{\tilde{F}}{}^{\tilde{b}}{}_{\tilde{B}}\left(\frac{\partial}{\partial\tilde{x}^{\tilde{b}}}\left[(\overset{\xi}{F}{}^{-1})^{a}{}_{\tilde{a}}\right] + \gamma^{a}{}_{cb}(\overset{\xi}{F}{}^{-1})^{b}{}_{\tilde{b}}(\overset{\xi}{F}{}^{-1})^{c}{}_{\tilde{a}} - \tilde{\gamma}^{\tilde{c}}{}_{\tilde{a}\tilde{b}}(\overset{\xi}{F}{}^{-1})^{a}{}_{\tilde{c}}\right)\,,
\end{aligned}$$

where $\tilde{\gamma}^{\tilde{c}}{}_{\tilde{a}\tilde{b}}$ are the (induced) Christoffel symbols corresponding to the virtual plate in its current configuration.

$$
\begin{aligned}
\mathring{P}^{bA} =& \frac{1}{2}\Big[(\mathring{F}^{-1})^{B}{}_{a|B}\mathbb{C}^{aCbA}{}_{|C} + (\mathring{F}^{-1})^{B}{}_{a}\mathbb{C}^{aCbA}{}_{|C|B}\Big] \\
& - \frac{1}{2}J_{\Xi}(\tilde{\mathring{F}}^{-1})^{\tilde{B}}{}_{\tilde{a}}\tilde{\mathbb{C}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{C}}\Big[(\overset{\xi}{F}^{-1})^{b}{}_{\tilde{b}|\tilde{A}}(\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{C}|\tilde{B}} + (\overset{\xi}{F}^{-1})^{b}{}_{\tilde{b}|\tilde{C}}(\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{A}|\tilde{B}} \\
& + (\overset{\xi}{F}^{-1})^{b}{}_{\tilde{b}|\tilde{B}}(\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{C}|\tilde{A}} + (\overset{\xi}{F}^{-1})^{b}{}_{\tilde{b}|\tilde{C}|\tilde{B}}(\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{A}} + (\overset{\xi}{F}^{-1})^{b}{}_{\tilde{b}|\tilde{A}|\tilde{B}}(\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{C}} \\
& + (\overset{\xi}{F}^{-1})^{b}{}_{\tilde{b}|\tilde{C}|\tilde{A}}(\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{B}} + (\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{C}|\tilde{A}|\tilde{B}}(\overset{\xi}{F}^{-1})^{b}{}_{\tilde{b}}\Big]\,,
\end{aligned}
\tag{8.211c}
$$

along with the following cloaking compatibility equations[36]

$$
\begin{aligned}
& (\mathring{F}^{-1})^{B}{}_{a|B}\mathbb{C}^{aAbC} + (\mathring{F}^{-1})^{B}{}_{a}\mathbb{C}^{aAbC}{}_{|B} + (\mathring{F}^{-1})^{A}{}_{a}\mathbb{C}^{aBbC}{}_{|B} \\
&\qquad = J_{\Xi}(\tilde{\mathring{F}}^{-1})^{\tilde{B}}{}_{\tilde{a}}\tilde{\mathbb{C}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{C}}\Big[(\overset{\xi}{F}^{-1})^{b}{}_{\tilde{b}|\tilde{B}}(\overset{\equiv}{F}^{-1})^{C}{}_{\tilde{C}}(\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{A}} + (\overset{\equiv}{F}^{-1})^{C}{}_{\tilde{C}|\tilde{B}}(\overset{\xi}{F}^{-1})^{b}{}_{\tilde{b}}(\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{A}} \\
&\qquad + (\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{A}|\tilde{B}}(\overset{\xi}{F}^{-1})^{b}{}_{\tilde{b}}(\overset{\equiv}{F}^{-1})^{C}{}_{\tilde{C}} + (\overset{\xi}{F}^{-1})^{b}{}_{\tilde{b}|\tilde{C}}(\overset{\equiv}{F}^{-1})^{C}{}_{\tilde{A}}(\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{B}} \\
&\qquad + (\overset{\xi}{F}^{-1})^{b}{}_{\tilde{b}|\tilde{A}}(\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{B}}(\overset{\equiv}{F}^{-1})^{C}{}_{\tilde{C}} + (\overset{\equiv}{F}^{-1})^{C}{}_{\tilde{C}|\tilde{A}}(\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{B}}(\overset{\xi}{F}^{-1})^{b}{}_{\tilde{b}}\Big]\,,
\end{aligned}
\tag{8.212a}
$$

$$
\begin{aligned}
& (\mathring{F}^{-1})^{B}{}_{a|B}\mathbb{B}^{bCaA}{}_{|A} + (\mathring{F}^{-1})^{B}{}_{a}\mathbb{B}^{bCaA}{}_{|A|B} + 2\left[\rho_0\left(\mathring{\mathfrak{L}}^{a}g^{bc} - \mathring{\mathfrak{L}}^{b}g^{ac}\right)(\mathring{F}^{-1})^{C}{}_{a}(\mathring{F}^{-1})^{A}{}_{c}\right]_{|A} \\
&\qquad = J_{\Xi}(\tilde{\mathring{F}}^{-1})^{\tilde{B}}{}_{\tilde{a}}(\mathsf{s}^{-1})^{b}{}_{\tilde{b}}\tilde{\mathbb{B}}^{\tilde{b}\tilde{C}\tilde{a}\tilde{A}}(\overset{\equiv}{F}^{-1})^{C}{}_{\tilde{C}|\tilde{A}|\tilde{B}}\,,
\end{aligned}
\tag{8.212b}
$$

$$
\begin{aligned}
& (\mathring{F}^{-1})^{B}{}_{a|B}\mathbb{B}^{bCaA} + (\mathring{F}^{-1})^{A}{}_{a}\mathbb{B}^{bCaB}{}_{|B} + (\mathring{F}^{-1})^{B}{}_{a}\mathbb{B}^{bCaA}{}_{|B} + 2\rho_0\left(\mathring{\mathfrak{L}}^{a}g^{bc} - \mathring{\mathfrak{L}}^{b}g^{ac}\right)(\mathring{F}^{-1})^{C}{}_{a}(\mathring{F}^{-1})^{A}{}_{c} \\
&\qquad = J_{\Xi}(\tilde{\mathring{F}}^{-1})^{\tilde{B}}{}_{\tilde{a}}(\mathsf{s}^{-1})^{b}{}_{\tilde{b}}\tilde{\mathbb{B}}^{\tilde{b}\tilde{C}\tilde{a}\tilde{A}}\Big((\overset{\equiv}{F}^{-1})^{C}{}_{\tilde{C}|\tilde{A}}(\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{B}} + (\overset{\equiv}{F}^{-1})^{C}{}_{\tilde{C}|\tilde{B}}(\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{A}} \\
&\qquad + (\overset{\equiv}{F}^{-1})^{C}{}_{\tilde{C}}(\overset{\equiv}{F}^{-1})^{A}{}_{\tilde{A}|\tilde{B}}\Big)\,,
\end{aligned}
\tag{8.212c}
$$

[36]One starts from the governing equations of the virtual plate and substitutes the derivatives with their corresponding transformed derivatives and compares the coefficients of the different derivatives in the transformed governing equations with those in the physical plate. This overdetermined system of equations gives all the transformed fields, and a set of constraints on the cloaking map.

and the ones given by (8.210e) and (8.211b). Notice that $\mathbb{C}$ already possesses the major symmetries. The linearized symmetry relations in the presence of pre-stress are given by (8.110). Recalling that the elastic parameters of the virtual plate satisfy (8.203), the relations $\mathbb{C}^{[aAcB}\mathring{F}^{b]}{}_A = 0$, and $\mathbb{B}^{cB[aA}\mathring{F}^{b]}{}_A = 0$, already hold, i.e.,

$$
\begin{aligned}
\mathbb{B}^{cB[aA}\mathring{F}^{b]}{}_A &= J_\Xi (\mathsf{s}^{-1})^c{}_{\tilde c} (\overset{\Xi}{F}{}^{-1})^B{}_{\tilde B} (\overset{\xi}{F}{}^{-1})^{[a}{}_{\tilde a} (\overset{\Xi}{F}{}^{-1})^A{}_{\tilde A} \mathring{F}^{b]}{}_A \tilde{\mathbb{B}}^{\tilde c \tilde B \tilde a \tilde A} \\
&= J_\Xi (\mathsf{s}^{-1})^c{}_{\tilde c} (\overset{\Xi}{F}{}^{-1})^B{}_{\tilde B} (\overset{\xi}{F}{}^{-1})^{[a}{}_{\tilde a} (\overset{\xi}{F}{}^{-1})^{b]}{}_{\tilde b} \mathring{\tilde F}{}^{\tilde b}{}_{\tilde A} \tilde{\mathbb{B}}^{\tilde c \tilde B \tilde a \tilde A} \\
&= J_\Xi (\mathsf{s}^{-1})^c{}_{\tilde c} (\overset{\Xi}{F}{}^{-1})^B{}_{\tilde B} (\overset{\xi}{F}{}^{-1})^{[b}{}_{\tilde a} (\overset{\xi}{F}{}^{-1})^{a]}{}_{\tilde b} \mathring{\tilde F}{}^{\tilde b}{}_{\tilde A} \tilde{\mathbb{B}}^{\tilde c \tilde B \tilde a \tilde A} = 0\,,
\end{aligned}
\tag{8.213a}
$$

$$
\begin{aligned}
\mathbb{C}^{[aAcB}\mathring{F}^{b]}{}_A &= J_\Xi (\overset{\xi}{F}{}^{-1})^c{}_{\tilde c} (\overset{\Xi}{F}{}^{-1})^B{}_{\tilde B} (\overset{\xi}{F}{}^{-1})^{[a}{}_{\tilde a} (\overset{\Xi}{F}{}^{-1})^A{}_{\tilde A} \mathring{F}^{b]}{}_A \tilde{\mathbb{C}}^{\tilde a \tilde A \tilde c \tilde B} \\
&= J_\Xi (\overset{\xi}{F}{}^{-1})^c{}_{\tilde c} (\overset{\Xi}{F}{}^{-1})^B{}_{\tilde B} (\overset{\xi}{F}{}^{-1})^{[a}{}_{\tilde a} (\overset{\xi}{F}{}^{-1})^{b]}{}_{\tilde b} \mathring{\tilde F}{}^{\tilde b}{}_{\tilde A} \tilde{\mathbb{C}}^{\tilde a \tilde A \tilde c \tilde B} \\
&= J_\Xi (\overset{\xi}{F}{}^{-1})^c{}_{\tilde c} (\overset{\Xi}{F}{}^{-1})^B{}_{\tilde B} (\overset{\xi}{F}{}^{-1})^{[b}{}_{\tilde a} (\overset{\xi}{F}{}^{-1})^{a]}{}_{\tilde b} \mathring{\tilde F}{}^{\tilde b}{}_{\tilde A} \tilde{\mathbb{C}}^{\tilde a \tilde A \tilde c \tilde B} = 0\,.
\end{aligned}
\tag{8.213b}
$$

On the other hand, $\mathbb{A}^{[aAcB}\mathring{F}^{b]}{}_A = 0$, and $\mathbb{B}^{[aAcB}\mathring{F}^{b]}{}_A = 0$, respectively, imply that

$$
\begin{aligned}
\mathbb{A}^{[aAcB}\mathring{F}^{b]}{}_A &= J_\Xi (\mathsf{s}^{-1})^c{}_{\tilde c} (\overset{\Xi}{F}{}^{-1})^B{}_{\tilde B} (\mathsf{s}^{-1})^{[a}{}_{\tilde a} (\overset{\Xi}{F}{}^{-1})^A{}_{\tilde A} \mathring{F}^{b]}{}_A \tilde{\mathbb{A}}^{\tilde a \tilde A \tilde c \tilde B} - \mathring{P}^{dA} (\mathring{F}^{-1})^B{}_d g^{[ac} \mathring{F}^{b]}{}_A \\
&= J_\Xi (\mathsf{s}^{-1})^c{}_{\tilde c} (\overset{\Xi}{F}{}^{-1})^B{}_{\tilde B} (\mathsf{s}^{-1})^{[a}{}_{\tilde a} (\overset{\xi}{F}{}^{-1})^{b]}{}_{\tilde b} \mathring{\tilde F}{}^{\tilde b}{}_{\tilde A} \tilde{\mathbb{A}}^{\tilde a \tilde A \tilde c \tilde B} - \mathring{P}^{dA} (\mathring{F}^{-1})^B{}_d g^{[ac} \mathring{F}^{b]}{}_A = 0\,,
\end{aligned}
\tag{8.214a}
$$

$$
\begin{aligned}
\mathbb{B}^{[aAcB}\mathring{F}^{b]}{}_A &= J_\Xi (\overset{\xi}{F}{}^{-1})^c{}_{\tilde c} (\overset{\Xi}{F}{}^{-1})^B{}_{\tilde B} (\mathsf{s}^{-1})^{[a}{}_{\tilde a} (\overset{\Xi}{F}{}^{-1})^A{}_{\tilde A} \mathring{F}^{b]}{}_A \tilde{\mathbb{B}}^{\tilde a \tilde A \tilde c \tilde B} \\
&= J_\Xi (\overset{\xi}{F}{}^{-1})^c{}_{\tilde c} (\overset{\Xi}{F}{}^{-1})^B{}_{\tilde B} (\mathsf{s}^{-1})^{[a}{}_{\tilde a} (\overset{\xi}{F}{}^{-1})^{b]}{}_{\tilde b} \mathring{\tilde F}{}^{\tilde b}{}_{\tilde A} \tilde{\mathbb{B}}^{\tilde a \tilde A \tilde c \tilde B} = 0\,.
\end{aligned}
\tag{8.214b}
$$

Pushing forward theses expressions to the current configuration, one obtains

$$
\begin{aligned}
\mathbb{C}^{[ab]cd} &= \frac{1}{\mathring{J}}\mathbb{A}^{[aAcB}\mathring{F}^{b]}{}_{A}\mathring{F}^{d}{}_{B} \\
&= \frac{J_{\Xi}}{\mathring{J}}(\mathsf{s}^{-1})^{c}{}_{\tilde{c}}(\overset{\Xi}{F}{}^{-1})^{B}{}_{\tilde{B}}\mathring{F}^{d}{}_{B}(\mathsf{s}^{-1})^{[a}{}_{\tilde{a}}(\overset{\xi}{F}{}^{-1})^{b]}{}_{\tilde{b}}\tilde{\mathring{F}}^{\tilde{b}}{}_{\tilde{A}}\tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{c}\tilde{B}} \\
&\quad - \frac{1}{\mathring{J}}\mathring{P}^{eA}(\mathring{F}^{-1})^{B}{}_{e}\mathring{F}^{d}{}_{B}g^{[ac}\mathring{F}^{b]}{}_{A} \\
&= J_{\xi}(\mathsf{s}^{-1})^{c}{}_{\tilde{c}}(\overset{\xi}{F}{}^{-1})^{d}{}_{\tilde{d}}(\mathsf{s}^{-1})^{[a}{}_{\tilde{a}}(\overset{\xi}{F}{}^{-1})^{b]}{}_{\tilde{b}}\frac{1}{\tilde{\mathring{J}}}\tilde{\mathring{F}}^{\tilde{b}}{}_{\tilde{A}}\tilde{\mathring{F}}^{\tilde{d}}{}_{\tilde{B}}\tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{c}\tilde{B}} \\
&\quad - \frac{1}{\mathring{J}}\mathring{P}^{eA}(\mathring{F}^{-1})^{B}{}_{e}\mathring{F}^{d}{}_{B}g^{[ac}\mathring{F}^{b]}{}_{A} \\
&= J_{\xi}(\mathsf{s}^{-1})^{c}{}_{\tilde{c}}(\overset{\xi}{F}{}^{-1})^{d}{}_{\tilde{d}}(\mathsf{s}^{-1})^{[a}{}_{\tilde{a}}(\overset{\xi}{F}{}^{-1})^{b]}{}_{\tilde{b}}\tilde{\mathbb{C}}^{\tilde{a}\tilde{b}\tilde{c}\tilde{d}} - \mathring{\sigma}^{[bd}g^{ca]} = 0\,,
\end{aligned}
\tag{8.215a}
$$

$$
\begin{aligned}
\mathbb{b}^{[ab]cd} &= \frac{1}{\mathring{J}}\mathbb{B}^{[aAcB}\mathring{F}^{b]}{}_{A}\mathring{F}^{d}{}_{B} \\
&= \frac{J_{\Xi}}{\mathring{J}}(\overset{\xi}{F}{}^{-1})^{c}{}_{\tilde{c}}(\overset{\Xi}{F}{}^{-1})^{B}{}_{\tilde{B}}\mathring{F}^{d}{}_{B}(\mathsf{s}^{-1})^{[a}{}_{\tilde{a}}(\overset{\xi}{F}{}^{-1})^{b]}{}_{\tilde{b}}\tilde{\mathring{F}}^{\tilde{b}}{}_{\tilde{A}}\tilde{\mathbb{B}}^{\tilde{a}\tilde{A}\tilde{c}\tilde{B}} \\
&= J_{\xi}(\overset{\xi}{F}{}^{-1})^{c}{}_{\tilde{c}}(\overset{\xi}{F}{}^{-1})^{d}{}_{\tilde{d}}(\mathsf{s}^{-1})^{[a}{}_{\tilde{a}}(\overset{\xi}{F}{}^{-1})^{b]}{}_{\tilde{b}}\frac{1}{\tilde{\mathring{J}}}\tilde{\mathring{F}}^{\tilde{b}}{}_{\tilde{A}}\tilde{\mathring{F}}^{\tilde{d}}{}_{\tilde{B}}\tilde{\mathbb{B}}^{\tilde{a}\tilde{A}\tilde{c}\tilde{B}} \\
&= J_{\xi}(\overset{\xi}{F}{}^{-1})^{c}{}_{\tilde{c}}(\overset{\xi}{F}{}^{-1})^{d}{}_{\tilde{d}}(\mathsf{s}^{-1})^{[a}{}_{\tilde{a}}(\overset{\xi}{F}{}^{-1})^{b]}{}_{\tilde{b}}\tilde{\mathbb{b}}^{\tilde{a}\tilde{b}\tilde{c}\tilde{d}} = 0\,.
\end{aligned}
\tag{8.215b}
$$

Additionally, the initial body forces and the pre-stress need to satisfy the following balance of linear and angular momentum in the finitely deformed configuration

$$
\mathring{P}^{aA}{}_{|A} + \rho_0(\mathring{\mathfrak{B}}^{\top})^{a} = 0\,,
\tag{8.216a}
$$

$$
\rho_0\mathring{\mathfrak{B}}^{\perp} + \Big[\rho_0\mathring{\mathfrak{L}}^{a}(\mathring{F}^{-1})^{A}{}_{a}\Big]_{|A} = 0\,,
\tag{8.216b}
$$

$$
\mathring{P}^{[aA}\mathring{F}^{b]}{}_{A} = 0\,.
\tag{8.216c}
$$

Notice that if (8.216c) holds (or, equivalently, $\mathring{\boldsymbol{\sigma}}$ is symmetric), then $\mathbb{A}$ possesses the major symmetries (cf. (8.210a)). The stress and couple-stress are transformed as

$$\begin{aligned}\delta P^{aA} =& J_\Xi(\mathsf{s}^{-1})^a{}_{\tilde{a}}(\overset{\Xi}{F}{}^{-1})^A{}_{\tilde{A}}\delta\tilde{P}^{\tilde{a}\tilde{A}} \\ &+\frac{1}{2}J_\Xi(\mathsf{s}^{-1})^a{}_{\tilde{a}}(\overset{\Xi}{F}{}^{-1})^A{}_{\tilde{A}}(\overset{\xi}{F}{}^{-1})^b{}_{\tilde{b}}(\overset{\Xi}{F}{}^{-1})^B{}_{\tilde{B}}\overset{\xi}{F}{}^{\tilde{c}}{}_{b|B}\tilde{\mathbb{B}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\widetilde{\mathsf{W}}_{,\tilde{c}}\,, \\ \delta\mathsf{M}^{aA} =& J_\Xi(\overset{\xi}{F}{}^{-1})^a{}_{\tilde{a}}(\overset{\Xi}{F}{}^{-1})^A{}_{\tilde{A}}\delta\tilde{\mathsf{M}}^{\tilde{a}\tilde{A}} \\ &+\frac{1}{2}J_\Xi(\overset{\xi}{F}{}^{-1})^a{}_{\tilde{a}}(\overset{\Xi}{F}{}^{-1})^A{}_{\tilde{A}}(\overset{\xi}{F}{}^{-1})^b{}_{\tilde{b}}(\overset{\Xi}{F}{}^{-1})^B{}_{\tilde{B}}\overset{\xi}{F}{}^{\tilde{c}}{}_{b|B}\tilde{\mathbb{C}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\widetilde{\mathsf{W}}_{,\tilde{c}}\,.\end{aligned} \tag{8.217}$$

Using (8.142), one needs to have $\widetilde{\mathsf{W}}_{,\tilde{a}} = \mathsf{W}_{,a}(\mathsf{s}^{-1})^a{}_{\tilde{a}}$ on the boundary of the cloak $\partial_o\mathcal{C}$, which implies that $T\xi|_{\partial_o\mathcal{C}} = \overset{\xi}{\mathbf{F}}|_{\partial_o\mathcal{C}} = id$. Additionally, the boundary surface traction and moment in the physical and virtual plates need to be identical on the boundary of the cloak, i.e., $(\delta\tilde{T}^\top)^{\tilde{a}} = \mathsf{s}^{\tilde{a}}{}_a(\delta T^\top)^a$, $\delta\tilde{T}^\perp = \delta T^\perp$, and $\delta\tilde{\mathsf{m}}^{\tilde{a}} = \mathsf{s}^{\tilde{a}}{}_a\delta\mathsf{m}^a$, on $\partial_o\mathcal{C}$ (see (8.142)). To ensure this one needs to impose the following constraints on the outer boundary of the cloak: $\overset{\xi}{F}{}^{\tilde{a}}{}_{a|A}\Big|_{\partial_o\mathcal{C}} = 0$, $\overset{\xi}{F}{}^{\tilde{a}}{}_{a|A|B}\Big|_{\partial_o\mathcal{C}} = 0$, $T\Xi|_{\partial_o\mathcal{C}} = \overset{\Xi}{\mathbf{F}}|_{\partial_o\mathcal{C}} = id$ (and thus, $\mathring{\mathbf{F}}|_{\partial_o\mathcal{C}} = id$, given that $\overset{\xi}{\mathbf{F}}|_{\partial_o\mathcal{C}} = id$ and $\overset{\xi}{\mathbf{F}} = \mathring{\tilde{\mathbf{F}}}\overset{\Xi}{\mathbf{F}}\mathring{\mathbf{F}}^{-1}$). Similarly, knowing that the hole surface $\partial\tilde{\mathcal{E}}$ in the virtual plate is traction-free, the hole inner surface in the physical plate $\partial\mathcal{E}$ will be traction-free as well if one requires that $\overset{\xi}{F}{}^{\tilde{a}}{}_{a|A}\Big|_{\partial\mathcal{E}} = 0$, and $\overset{\xi}{F}{}^{\tilde{a}}{}_{a|A|B}\Big|_{\partial\mathcal{E}} = 0$.[37] Furthermore, the hole must be traction-free in the physical plate in its finitely-deformed configuration, i.e., from (8.141), one needs to have $(\mathring{T}^\top)^a|_{\partial\mathcal{E}} = \left(\mathring{P}^{aA}\mathsf{T}_A\right)\Big|_{\partial\mathcal{E}} = 0$, and $\mathring{T}^\perp|_{\partial\mathcal{E}} = -\left[(\mathring{F}^{-1})^A{}_a\rho\mathring{\mathfrak{L}}^a\mathsf{T}_A\right]\Big|_{\partial\mathcal{E}} = 0$.

Remark 8.4.8. Upon using (8.211a) and the Piola identity, the constraint (8.212a) is simplified to read

$$\tilde{\mathbb{C}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{C}}(\overset{\xi}{F}{}^{-1})^b{}_{\tilde{b}}(\overset{\Xi}{F}{}^{-1})^C{}_{\tilde{C}}(\mathring{F}^{-1})^A{}_a(\overset{\xi}{F}{}^{-1})^a{}_{\tilde{a}|\tilde{A}} = \tilde{\mathbb{C}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{C}}(\overset{\xi}{F}{}^{-1})^c{}_{\tilde{a}}(\overset{\Xi}{F}{}^{-1})^C{}_{\tilde{A}}(\mathring{F}^{-1})^A{}_c(\overset{\xi}{F}{}^{-1})^b{}_{\tilde{b}|\tilde{C}}\,. \tag{8.218}$$

Remark 8.4.9. Note that for an isotropic (virtual) plate, the elastic constant $\tilde{\mathbb{B}}$ is given (with a slight

[37] Note that $\overset{\xi}{F}{}^{\tilde{a}}{}_{a|A}\Big|_{\partial_o\mathcal{C}} = 0$, implies that $\mathring{\mathfrak{L}}|_{\partial_o\mathcal{C}} = \mathbf{0}$, and $\delta\mathfrak{L}|_{\partial_o\mathcal{C}} = \mathbf{0}$, (see (8.142) and (8.210d)), and thus

$$\begin{aligned}\delta T^\perp &= -(\mathring{F}^{-1})^A{}_a\delta\mathsf{M}^{aB}{}_{|B}\mathsf{T}_A = -\left[J_\Xi(\overset{\Xi}{F}{}^{-1})^B{}_{\tilde{B}}(\overset{\xi}{F}{}^{-1})^a{}_{\tilde{a}}\delta\tilde{\mathsf{M}}^{\tilde{a}\tilde{B}}\right]_{|B}(\mathring{F}^{-1})^A{}_a\mathsf{T}_A \\ &= -J_\Xi(\overset{\Xi}{F}{}^{-1})^B{}_{\tilde{B}}(\overset{\xi}{F}{}^{-1})^a{}_{\tilde{a}}\delta\tilde{\mathsf{M}}^{\tilde{a}\tilde{B}}{}_{|B}(\mathring{F}^{-1})^A{}_a\mathsf{T}_A = -(\mathring{\tilde{F}}{}^{-1})^{\tilde{A}}{}_{\tilde{a}}\delta\mathsf{M}^{\tilde{a}\tilde{B}}{}_{|\tilde{B}}\tilde{\mathsf{T}}_{\tilde{A}} = \delta\tilde{T}^\perp\,,\end{aligned} \quad \text{on} \quad \partial_o\mathcal{C}\,,$$

where $\left[J_\Xi(\overset{\Xi}{F}{}^{-1})^B{}_{\tilde{B}}\right]_{|B} = 0$, and the fact that Ξ (and ξ) fixes the boundary of the cloak $\partial_o\mathcal{C}$ to the third order were used.

abuse of notation) by

$$\tilde{\mathbb{B}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} = \tilde{\mathsf{b}}_1 \tilde{G}^{\tilde{a}\tilde{A}}\tilde{G}^{\tilde{b}\tilde{B}} + \tilde{\mathsf{b}}_2(\tilde{G}^{\tilde{a}\tilde{b}}\tilde{G}^{\tilde{A}\tilde{B}} + \tilde{G}^{\tilde{a}\tilde{B}}\tilde{G}^{\tilde{b}\tilde{A}})\,, \tag{8.219}$$

for some scalars $\tilde{\mathsf{b}}_1$ and $\tilde{\mathsf{b}}_2$. This is a consequence of the fact that the most general form of a fourth-order isotropic tensor is given by $a_1\delta_{ij}\delta_{kl} + a_2\delta_{ik}\delta_{jl} + a_3\delta_{il}\delta_{jk}$ (for some scalars a_i, $i = 1, 2, 3$) and the minor symmetries of $\tilde{\mathbb{B}}$ dictated by (8.203).

Remark 8.4.10. When restricting to the in-plane deformations, we recover our result [41, §4.4] for the elastodynamic cloaking of a cylindrical hole in the context of the small-on-large theory of (classical) elasticity in 3D. For in-plane deformations, $\mathsf{W} = 0$, and for a classical solid $\mathbb{B}$ (and $\mathbb{C}$) vanishes. Therefore, the out-of-plane equilibrium equation (8.205b) is trivially satisfied, and the in-plane equilibrium equation (8.205a) would be the only non-trivial governing equation that needs to be studied (and transformed) under a cloaking transformation. Note that in this case, the pre-stress is determined such that the (only non-trivial) balance of angular momentum (8.215a) is satisfied. We should emphasis that in [41], the variation of the body force is assumed to be independent of that of the motion. Therefore, the linearized equations involve the load increment $\delta\mathfrak{B}$ independently of the infinitesimal deformation $\delta\varphi$ (see [43, p.237]).

Remark 8.4.11. Note that (8.216b) is already satisfied. To see this, using (8.210c) and (8.210d), the expression in (8.216b) is simplified to read

$$\begin{aligned}
&\frac{1}{2}J_\Xi(\mathsf{s}^{-1})^a{}_{\tilde{a}}(\overset{\xi}{F}{}^{-1})^b{}_{\tilde{b}|\tilde{B}|\tilde{A}}\tilde{\mathbb{B}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}g_{ab} - \frac{1}{2}\tilde{\mathbb{B}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}g_{ac}(\mathsf{s}^{-1})^a{}_{\tilde{a}}\overset{\circ}{\tilde{F}}{}^{\tilde{c}}{}_{\tilde{A}}\left[J_\Xi(\overset{\xi}{F}{}^{-1})^b{}_{\tilde{c}}(\overset{\xi}{F}{}^{-1})^c{}_{\tilde{b}|\tilde{B}}(\overset{\circ}{F}{}^{-1})^A{}_b\right]_{|A}\\
&\quad = -\frac{1}{2}\tilde{\mathbb{B}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}g_{ac}(\mathsf{s}^{-1})^a{}_{\tilde{a}}\overset{\circ}{\tilde{F}}{}^{\tilde{c}}{}_{\tilde{A}}(\overset{\xi}{F}{}^{-1})^c{}_{\tilde{b}|\tilde{B}}\left[J_\Xi(\overset{\xi}{F}{}^{-1})^b{}_{\tilde{c}}(\overset{\circ}{F}{}^{-1})^A{}_b\right]_{|A}\\
&\quad = -\frac{1}{2}\tilde{\mathbb{B}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}g_{ac}(\mathsf{s}^{-1})^a{}_{\tilde{a}}(\overset{\xi}{F}{}^{-1})^c{}_{\tilde{b}|\tilde{B}}\left[J_\Xi(\overset{\Xi}{F}{}^{-1})^A{}_{\tilde{A}}\right]_{|A} = 0\,,
\end{aligned} \tag{8.220}$$

where the relation $\overset{\xi}{\mathbf{F}} = \overset{\circ}{\tilde{\mathbf{F}}}\overset{\Xi}{\mathbf{F}}\overset{\circ}{\mathbf{F}}{}^{-1}$, and the Piola identity were used.

Remark 8.4.12. If $\tilde{\mathbb{B}}$ vanishes for the virtual plate, which already does if one assumes the Saint

Venant-Kirchhoff energy function (cf. (8.144)) for the virtual plate, then so does the tensor $\mathbb{B}$ for the physical plate, i.e., $\mathbb{B} = \mathbf{0}$. Moreover, if $\tilde{\mathbb{B}} = \mathbf{0}$, then $\mathring{\boldsymbol{\mathfrak{B}}}^{\perp} = \mathbf{0}$, $\mathring{\boldsymbol{\mathfrak{L}}} = \mathbf{0}$, and the constraints (8.212b) and (8.212c) are trivially satisfied.

Remark 8.4.13. The second-order change (variation) in the energy (density) of the physical plate is given by

$$\delta^2 W = \frac{1}{2}\frac{\partial^2 W}{\partial C_{AB}\partial C_{CD}}\delta C_{AB}\delta C_{CD} + \frac{\partial^2 W}{\partial C_{AB}\partial \Theta_{CD}}\delta C_{AB}\delta\Theta_{CD} + \frac{1}{2}\frac{\partial^2 W}{\partial \Theta_{AB}\partial \Theta_{CD}}\delta\Theta_{AB}\delta\Theta_{CD}\,. \tag{8.221}$$

Therefore, one obtains

$$\delta^2 W = \frac{1}{2}\mathbb{A}^{aAbB}\mathsf{U}_{a|A}\mathsf{U}_{b|B} + \frac{1}{2}\mathbb{B}^{aAbB}\mathsf{U}_{a|A}(\mathsf{W}_{,b})_{|B} + \frac{1}{4}\mathbb{C}^{aAbB}(\mathsf{W}_{,a})_{|A}(\mathsf{W}_{,b})_{|B}\,. \tag{8.222}$$

Similarly, we note that for the virtual plate

$$\delta^2 \tilde{W} = \frac{1}{2}\tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\tilde{\mathsf{U}}_{\tilde{a}|\tilde{A}}\tilde{\mathsf{U}}_{\tilde{b}|\tilde{B}} + \frac{1}{2}\tilde{\mathbb{B}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\tilde{\mathsf{U}}_{\tilde{a}|\tilde{A}}(\widetilde{\mathsf{W}}_{,\tilde{b}})_{|\tilde{B}} + \frac{1}{4}\tilde{\mathbb{C}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}(\widetilde{\mathsf{W}}_{,\tilde{a}})_{|\tilde{A}}(\widetilde{\mathsf{W}}_{,\tilde{b}})_{|\tilde{B}}\,. \tag{8.223}$$

Using (8.207), (8.208), (8.210a), (8.210b), (8.211a), and (8.223), one rewrites (8.222) as

$$\begin{aligned}\delta^2 W =& J_{\Xi}\delta^2\tilde{W} - \frac{1}{2}\mathring{P}^{cA}(\mathring{F}^{-1})^{B}{}_{c}g^{ab}\mathsf{s}^{\tilde{a}}{}_{a}\mathsf{s}^{\tilde{b}}{}_{b}\overset{\Xi}{F}{}^{\tilde{A}}{}_{A}\overset{\Xi}{F}{}^{\tilde{B}}{}_{B}\tilde{\mathsf{U}}_{\tilde{a}|\tilde{A}}\tilde{\mathsf{U}}_{\tilde{b}|\tilde{B}}\\ &+\frac{1}{2}J_{\Xi}(\overset{\xi}{F}{}^{-1})^{b}{}_{\tilde{b}}(\overset{\Xi}{F}{}^{-1})^{B}{}_{\tilde{B}}\overset{\xi}{F}{}^{\tilde{c}}{}_{b|B}\tilde{\mathbb{B}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\tilde{\mathsf{U}}_{\tilde{a}|\tilde{A}}\widetilde{\mathsf{W}}_{,\tilde{c}}\\ &+\frac{1}{2}J_{\Xi}(\overset{\xi}{F}{}^{-1})^{a}{}_{\tilde{a}}(\overset{\Xi}{F}{}^{-1})^{A}{}_{\tilde{A}}\overset{\xi}{F}{}^{\tilde{c}}{}_{a|A}\tilde{\mathbb{C}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\widetilde{\mathsf{W}}_{,\tilde{c}}(\widetilde{\mathsf{W}}_{,\tilde{b}})_{|\tilde{B}}\\ &+\frac{1}{4}J_{\Xi}(\overset{\xi}{F}{}^{-1})^{a}{}_{\tilde{a}}(\overset{\Xi}{F}{}^{-1})^{A}{}_{\tilde{A}}(\overset{\xi}{F}{}^{-1})^{b}{}_{\tilde{b}}(\overset{\Xi}{F}{}^{-1})^{B}{}_{\tilde{B}}\overset{\xi}{F}{}^{\tilde{c}}{}_{a|A}\overset{\xi}{F}{}^{\tilde{d}}{}_{b|B}\tilde{\mathbb{C}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}}\widetilde{\mathsf{W}}_{,\tilde{c}}\widetilde{\mathsf{W}}_{,\tilde{d}}\,.\end{aligned} \tag{8.224}$$

Hence, the positive-definiteness of the second order variation of the energy of the physical plate involves that of the virtual plate (i.e., $\delta^2\tilde{W}$), along with the elastic parameters and (in-plane and out-of-plane) displacements of the virtual plate, the first and the second derivatives of the cloaking map, and the pre-stress. This is in contrast to transformation cloaking in classical (and generalized Cosserat) 3D elasticity, where $\delta^2 W = J_{\Xi}\delta^2\tilde{W}$ (see [41]), and thus, one would simply conclude that (the second order variation of) the energy density is positive-definite in the physical problem if and only if it is positive-definite in the virtual problem.

A circular cloak in the presence of in-plane and out-of-plane displacements

Let us consider the cylindrical cloak example in the presence of the initial stress $\mathring{\mathbf{P}}$, the initial body force $\mathring{\mathfrak{B}}$, and the initial body moment $\mathring{\mathfrak{L}}$. The cloaking transformation ξ maps a pre-stressed cylindrical annulus (in the physical plate) with inner and outer radii r_i and r_o, respectively, to a cylindrical annulus (in the virtual plate) with inner and outer radii ϵ and r_o, respectively. Let, in polar coordinates, $(\tilde{r}, \tilde{\theta}) = \xi(r, \theta) = (f(r), \theta)$. Therefore

$$\overset{\xi}{\mathbf{F}} = \begin{bmatrix} f'(r) & 0 \\ 0 & 1 \end{bmatrix} , \tag{8.225}$$

where $f(r_o) = r_o$ and $f(r_i) = \epsilon$. Let $(\tilde{R}, \tilde{\Theta}) = \Xi(R, \Theta)$, $(r, \theta) = \varphi(R, \Theta)$, and $(\tilde{r}, \tilde{\theta}) = \tilde{\varphi}(\tilde{R}, \tilde{\Theta})$ such that $\mathring{\tilde{\varphi}} = id$ (and thus, $\mathring{\tilde{\mathbf{F}}} = id$). We assume that the physical plate is finitely deformed such that $(r, \theta) = \varphi(R, \Theta) = (\psi(R), \Theta)$. Therefore

$$\mathring{\mathbf{F}} = \begin{bmatrix} \psi'(R) & 0 \\ 0 & 1 \end{bmatrix} . \tag{8.226}$$

Using $\overset{\xi}{\mathbf{F}} = \mathring{\tilde{\mathbf{F}}} \overset{\Xi}{\mathbf{F}} \mathring{\mathbf{F}}^{-1}$, one has

$$\overset{\Xi}{\mathbf{F}} = \begin{bmatrix} \psi'(R) f'(\psi(R)) & 0 \\ 0 & 1 \end{bmatrix} , \tag{8.227}$$

and thus, $J_\Xi = \psi'(R) f(\psi(R)) f'(\psi(R))/R$. The referential mass density of the cloak is given by

$$\rho_0(R) = \psi'(R) f'(\psi(R)) \frac{f(\psi(R))}{R} \tilde{\rho}_0 , \quad R_i \leq R \leq R_o . \tag{8.228}$$

Note that $\mathbf{s} = \operatorname{diag}(1, r/f(r))$. Assuming the Saint Venant-Kirchhoff constitutive equation (8.144) for the virtual plate, $\tilde{\mathbb{C}}$ is given by (8.159), $\tilde{\mathbb{B}} = \mathbf{0}$ (and thus, $\mathbb{B} = \mathbf{0}$, see (8.210b), which using (8.210c) and (8.210d) implies that $\mathring{\mathfrak{B}}^{\perp} = \mathbf{0}$, and $\mathring{\mathfrak{L}} = \mathbf{0}$), and $\tilde{\mathbb{A}}$ is obtained as[38]

$$\tilde{\mathbb{A}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{C}} = \frac{Eh}{2(1+\nu)} \mathring{\tilde{F}}^{\tilde{a}}{}_{\tilde{M}} \mathring{\tilde{F}}^{\tilde{b}}{}_{\tilde{N}} \left(\tilde{G}^{\tilde{A}\tilde{N}} \tilde{G}^{\tilde{C}\tilde{M}} + \tilde{G}^{\tilde{A}\tilde{C}} \tilde{G}^{\tilde{M}\tilde{N}} + \frac{2\nu}{1-\nu} \tilde{G}^{\tilde{A}\tilde{M}} \tilde{G}^{\tilde{C}\tilde{N}} \right) . \tag{8.229}$$

Thus

$$\hat{\tilde{\mathbb{A}}} = \left[\hat{\tilde{\mathbb{A}}}^{\tilde{a}\tilde{A}\tilde{b}\tilde{B}} \right] = \frac{Eh}{2(1+\nu)} \begin{bmatrix} \begin{bmatrix} \frac{2}{1-\nu} & 0 \\ 0 & \frac{2\nu}{1-\nu} \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} \frac{2\nu}{1-\nu} & 0 \\ 0 & \frac{2}{1-\nu} \end{bmatrix} \end{bmatrix} . \tag{8.230}$$

Using (8.211a), the flexural rigidity tensor of the physical plate is given by

$$\begin{aligned} \hat{\mathbb{C}} &= \left[\hat{\mathbb{C}}^{aAbB} \right] \\ &= \frac{Eh^3}{12(1+\nu)} \begin{bmatrix} \begin{bmatrix} \frac{2}{1-\nu} \frac{f(\psi(R))}{R\psi'(R) f'^3(\psi(R))} & 0 \\ 0 & \frac{2\nu}{1-\nu} \frac{\psi(R)}{f(\psi(R)) f'(\psi(R))} \end{bmatrix} & \begin{bmatrix} 0 & \frac{R\psi'(R)}{f(\psi(R)) f'(\psi(R))} \\ \frac{\psi(R)}{f(\psi(R)) f'(\psi(R))} & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & \frac{\psi(R)}{f(\psi(R)) f'(\psi(R))} \\ \frac{\psi^2(R)}{R f(\psi(R)) f'(\psi(R)) \psi'(R)} & 0 \end{bmatrix} & \begin{bmatrix} \frac{2\nu}{1-\nu} \frac{\psi(R)}{f(\psi(R)) f'(\psi(R))} & 0 \\ 0 & \frac{2}{1-\nu} \frac{R f'(\psi(R)) \psi'(R) \psi^2(R)}{f^3(\psi(R))} \end{bmatrix} \end{bmatrix} . \end{aligned} \tag{8.231}$$

[38]Note that in the case of a general isotropic energy function for the virtual plate $\mathbb{B}$, $\mathring{\mathfrak{L}}$, and $\mathring{\mathfrak{B}}^{\perp}$ do not vanish and their expressions are given in Remark. 8.4.14.

From (8.210a), the first elasticity tensor of the physical plate is obtained as

$$
\begin{aligned}
\hat{\mathbf{A}} = \left[\hat{\mathbb{A}}^{aAbB}\right] =& \frac{Eh}{2(1+\nu)} \begin{bmatrix} \begin{bmatrix} \frac{2}{1-\nu}\frac{f(\psi(R))}{Rf'(\psi(R))\psi'(R)} & 0 \\ 0 & \frac{2\nu}{1-\nu} \end{bmatrix} & \begin{bmatrix} 0 & \frac{Rf'(\psi(R))\psi'(R)}{f(\psi(R))} \\ 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 \\ \frac{f(\psi(R))}{Rf'(\psi(R))\psi'(R)} & 0 \end{bmatrix} & \begin{bmatrix} \frac{2\nu}{1-\nu} & 0 \\ 0 & \frac{2}{1-\nu}\frac{Rf'(\psi(R))\psi'(R)}{f(\psi(R))} \end{bmatrix} \end{bmatrix} \\
&- \begin{bmatrix} \begin{bmatrix} \frac{1}{\psi'(R)}\hat{\mathring{P}}^{rR} & \frac{R}{\psi(R)}\hat{\mathring{P}}^{\theta R} \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} \frac{1}{\psi'(R)}\hat{\mathring{P}}^{r\Theta} & \frac{R}{\psi(R)}\hat{\mathring{P}}^{\theta\Theta} \\ 0 & 0 \end{bmatrix} \\ \\ \begin{bmatrix} 0 & 0 \\ \frac{1}{\psi'(R)}\hat{\mathring{P}}^{rR} & \frac{R}{\psi(R)}\hat{\mathring{P}}^{\theta R} \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ \frac{1}{\psi'(R)}\hat{\mathring{P}}^{r\Theta} & \frac{R}{\psi(R)}\hat{\mathring{P}}^{\theta\Theta} \end{bmatrix} \end{bmatrix},
\end{aligned}
\tag{8.232}
$$

where utilizing (8.211c), the pre-stress is calculated as

$$
\begin{aligned}
\hat{\mathring{P}}^{r\Theta} =& \hat{\mathring{P}}^{\theta R} = 0\,, \\
\hat{\mathring{P}}^{rR} =& \frac{Eh^3}{12\left(\nu^2-1\right)R\psi(R)f^3(\psi(R))f'^5(\psi(R))}\Big\{\psi(R)f(\psi(R))f'^5(\psi(R))\left[(\nu-2)R\psi'(R)+2\psi(R)\right] \\
&+ R\psi^2(R)\psi'(R)f'^6(\psi(R)) + f^2(\psi(R))f'^3(\psi(R))\Big[\nu R\psi(R)\psi'(R)f''(\psi(R)) \\
&+ f'(\psi(R))\left\{-(\nu-1)R\psi'(R)-2\psi(R)\right\}\Big] - \psi(R)f^3(\psi(R))f'^2(\psi(R))f''(\psi(R)) \\
&+ \psi(R)f^4(\psi(R))\left[3f''^2(\psi(R)) - f^{(3)}(\psi(R))f'(\psi(R))\right]\Big\}\,, \\
\hat{\mathring{P}}^{\theta\Theta} =& \frac{Eh^3}{12\left(\nu^2-1\right)R\psi(R)f^4(\psi(R))f'^3(\psi(R))}\Big\{-3R\psi^3(R)\psi'(R)f'^5(\psi(R)) \\
&+ \psi^2(R)f(\psi(R))f'^3(\psi(R))\left[R\psi(R)\psi'(R)f''(\psi(R)) + f'(\psi(R))\left\{\psi(R)-2(\nu-2)R\psi'(R)\right\}\right] \\
&+ \psi(R)f^2(\psi(R))f'^2(\psi(R))\left[f'(\psi(R))\left\{\nu R\psi'(R)+(\nu-2)\psi(R)\right\} - \nu R\psi(R)\psi'(R)f''(\psi(R))\right] \\
&+ f^3(\psi(R))\Big[\nu R\psi^2(R)f^{(3)}(\psi(R))\psi'(R)f'(\psi(R)) + \Big(\left(\psi(R)-R\psi'(R)\right)f'(\psi(R)) \\
&- 2R\psi(R)\psi'(R)f''(\psi(R))\Big)\left\{\nu\psi(R)f''(\psi(R)) + (1-\nu)f'(\psi(R))\right\}\Big]\Big\}\,.
\end{aligned}
\tag{8.233}
$$

Using (8.211b), one finds the components of the tangential body force as $(\mathring{\mathfrak{B}}^\top)^\theta = 0$, and

$$\begin{aligned}(\mathring{\mathfrak{B}}^\top)^r =& \frac{Eh^3}{12\tilde{\rho}_0\,(\nu^2-1)\,f^4(\psi(R))f'^7(\psi(R))}\Big\{3\psi(R)f'^7(\psi(R)) - 3f(\psi(R))f'^6(\psi(R))\\ &- 3f^2(\psi(R))f'^4(\psi(R))f''(\psi(R))\\ &+ 2f^3(\psi(R))f'^2(\psi(R))\left[f^{(3)}(\psi(R))f'(\psi(R)) - 3f''^2(\psi(R))\right]\\ &+ f^4(\psi(R))\left[15f''^3(\psi(R)) + f^{(4)}(\psi(R))f'^2(\psi(R)) - 10f^{(3)}(\psi(R))f'(\psi(R))f''(\psi(R))\right]\Big\}.\end{aligned} \tag{8.234}$$

However, note that (8.210e) implies that $(\mathring{\mathfrak{B}}^\top)^r = 0$, and thus, (8.234) can be viewed as a constraint. Notice that (8.216c) is already satisfied. The constraint (8.212a) gives the following ODE

$$\begin{aligned}&f'(\psi(R))\,[f(\psi(R)) - \psi(R)f'(\psi(R))]\,[\psi(R)f'(\psi(R)) + (\nu-1)f(\psi(R))]\\ &\qquad - \nu\psi(R)f^2(\psi(R))f''(\psi(R)) = 0\,.\end{aligned} \tag{8.235}$$

Recalling that $r = \psi(R)$, we may rewrite (8.235) as

$$f'(r)\,[f(r) - rf'(r)]\,[rf'(r) + (\nu-1)f(r)] - \nu rf^2(r)f''(r) = 0\,. \tag{8.236}$$

Noting that (8.236) is a second-order ODE and the cloaking transformation ξ needs to satisfy $f(r_o) = r_o$, $f(r_i) = \epsilon$, and $f'(r_o) = 1$, one concludes that cloaking is not possible. The balance of linear momentum in the finitely deformed configuration (8.216a) is simplified to read

$$
\begin{aligned}
&3R\psi(R)^3\psi'(R)\left(\psi(R)-R\psi'(R)\right)f'(\psi(R))^5+\psi(R)^2 f(\psi(R))f'(\psi(R))^3\Big[R^2\psi(R)\psi''(R)f'(\psi(R))\\
&\quad+\left(R\psi'(R)-\psi(R)\right)\left\{R\psi(R)\psi'(R)f''(\psi(R))+f'(\psi(R))\left[(5-2\nu)R\psi'(R)+\psi(R)\right]\right\}\Big]\\
&\quad+\psi(R)f(\psi(R))^2f'(\psi(R))^2\Big[(\nu-2)R^2\psi(R)\psi''(R)f'(\psi(R))\\
&\quad+\left(\psi(R)-R\psi'(R)\right)\left\{\nu R\psi(R)\psi'(R)f''(\psi(R))+f'(\psi(R))\left[(2-\nu)\psi(R)-(\nu-1)R\psi'(R)\right]\right\}\Big]\\
&\quad+f(\psi(R))^3\Bigg((\nu-1)R^2\psi'(R)^2f'(\psi(R))^2-(\nu-1)R\psi(R)f'(\psi(R))\Big[f'(\psi(R))\left(R\psi''(R)+2\psi'(R)\right)\\
&\quad-R\psi'(R)^2f''(\psi(R))\Big]+\psi(R)^2\Big[-2\nu R^2\psi'(R)^2f''(\psi(R))^2+(\nu-1)f'(\psi(R))^2\\
&\quad+Rf'(\psi(R))\left(\nu Rf^{(3)}(\psi(R))\psi'(R)^2+f''(\psi(R))\left[\nu R\psi''(R)+\psi'(R)\right]\right)\Big]\\
&\quad-\nu\psi(R)^3\left(f'(\psi(R))\left(Rf^{(3)}(\psi(R))\psi'(R)+f''(\psi(R))\right)-2R\psi'(R)f''(\psi(R))^2\right)\Bigg)=0\,.
\end{aligned}
\tag{8.237}
$$

Provided that $\nu \neq 0$, one may use (8.236) to obtain expressions for $f''(\psi(R))$ and $f^{(3)}(\psi(R))$ in terms of $\psi(R)$, $\psi'(R)$, $f(\psi(R))$, and $f'(\psi(R))$. Plugging these expressions into (8.237), it is straightforward to verify that (8.237) already holds. Therefore, the balance of linear and angular momenta (cf. (8.216)) for the physical plate in its finitely deformed configuration are already satisfied as long as the cloaking map satisfies the constraint given by (8.236).

Remark 8.4.14. Assuming that $\tilde{\mathbb{B}}$ is given by (8.219), one obtains $\mathbb{B}$, the initial normal body force $\mathring{\mathfrak{B}}^{\perp}$, and the initial body moment $\mathring{\mathfrak{L}}$ of the physical plate using (8.210b), (8.210c), and (8.210d), respectively, as

$$
\hat{\mathbb{B}}=\left[\hat{\mathbb{B}}^{aAbB}\right]=\begin{bmatrix}\begin{bmatrix}\frac{\left(\tilde{\mathsf{b}}_1+2\tilde{\mathsf{b}}_2\right)f(\psi(R))}{Rf'^2(\psi(R))\psi'(R)} & 0\\ 0 & \frac{\tilde{\mathsf{b}}_1\psi(R)}{f(\psi(R))}\end{bmatrix} & \begin{bmatrix}0 & \frac{\tilde{\mathsf{b}}_2R\psi'(R)}{f(\psi(R))}\\ \frac{\tilde{\mathsf{b}}_2\psi(R)}{f(\psi(R))} & 0\end{bmatrix}\\ \begin{bmatrix}0 & \frac{\tilde{\mathsf{b}}_2}{f'(\psi(R))}\\ \frac{\tilde{\mathsf{b}}_2\psi(R)}{Rf'(\psi(R))\psi'(R)} & 0\end{bmatrix} & \begin{bmatrix}\frac{\tilde{\mathsf{b}}_1}{f'(\psi(R))} & 0\\ 0 & \frac{\left(\tilde{\mathsf{b}}_1+2\tilde{\mathsf{b}}_2\right)R\psi(R)f'(\psi(R))\psi'(R)}{f^2(\psi(R))}\end{bmatrix}\end{bmatrix}, \tag{8.238}
$$

$$\mathring{\mathfrak{B}}^{\perp} = -\frac{(\tilde{\mathsf{b}}_1 + 2\tilde{\mathsf{b}}_2)}{2\tilde{\rho}_0 f^3(\psi(R)) f'^5(\psi(R))} \Big[-\psi(R) f'^5(\psi(R)) + f(\psi(R)) f'^4(\psi(R)) \\ + 2f^2(\psi(R)) f'^2(\psi(R)) f''(\psi(R)) + f^3(\psi(R)) \Big(f^{(3)}(\psi(R)) f'(\psi(R)) - 3f''^2(\psi(R)) \Big) \Big] , \tag{8.239}$$

$$\mathring{\mathfrak{L}}^{r} = \frac{(\tilde{\mathsf{b}}_1 + 2\tilde{\mathsf{b}}_2)}{2\tilde{\rho}_0 f^2(\psi(R)) f'^4(\psi(R))} \Big[f^2(\psi(R)) f''(\psi(R)) + \psi(R) f'^3(\psi(R)) - f(\psi(R)) f'^2(\psi(R)) \Big] , \tag{8.240}$$

and the circumferential component of the body moment vanishes, i.e., $\mathring{\mathfrak{L}}^{\theta} = 0$.

Remark 8.4.15. General cloaking transformations: Next we show that if $\tilde{\mathsf{b}}_2 > 0$ and $\tilde{\mathsf{b}}_1 + \tilde{\mathsf{b}}_2 > 0$, i.e., the tensor $\tilde{\mathbb{B}}$ for the virtual plate is positive definite, then transformation cloaking would not be realizable even if one uses a general cloaking map ξ for an arbitrary hole surrounded by a cloak (with an arbitrary shape). Without loss of generality, we use Cartesian coordinates, where the shifters and the metrics have trivial representations. Let us consider an arbitrary cloaking map ξ such that

$$\overset{\xi}{\mathbf{F}}{}^{-1} = \begin{bmatrix} \mathsf{F}_{11}(x,y) & \mathsf{F}_{12}(x,y) \\ \mathsf{F}_{21}(x,y) & \mathsf{F}_{22}(x,y) \end{bmatrix} . \tag{8.241}$$

Therefore, $J_{\xi} = [\mathsf{F}_{11}\mathsf{F}_{22} - \mathsf{F}_{12}\mathsf{F}_{21}]^{-1}$, and (8.215b) is simplified to read

$$\tilde{\mathsf{b}}_1 \mathsf{F}_{21} \left(\mathsf{F}_{11}^2 + \mathsf{F}_{12}^2 \right) + 2\tilde{\mathsf{b}}_2 \mathsf{F}_{11} \left(\mathsf{F}_{11}\mathsf{F}_{21} + \mathsf{F}_{12}\mathsf{F}_{22} \right) - \mathsf{F}_{12} \left(\tilde{\mathsf{b}}_1 + 2\tilde{\mathsf{b}}_2 \right) \left(\mathsf{F}_{11}^2 + \mathsf{F}_{12}^2 \right) = 0 , \tag{8.242a}$$

$$\mathsf{F}_{21} \left(\mathsf{F}_{21}^2 + \mathsf{F}_{22}^2 \right) \left(\tilde{\mathsf{b}}_1 + 2\tilde{\mathsf{b}}_2 \right) - 2\mathsf{F}_{22}\tilde{\mathsf{b}}_2 \left(\mathsf{F}_{11}\mathsf{F}_{21} + \mathsf{F}_{12}\mathsf{F}_{22} \right) - \mathsf{F}_{12}\tilde{\mathsf{b}}_1 \left(\mathsf{F}_{21}^2 + \mathsf{F}_{22}^2 \right) = 0 , \tag{8.242b}$$

$$\mathsf{F}_{11} \Big(\mathsf{F}_{22}^2 \tilde{\mathsf{b}}_2 - \mathsf{F}_{12}\mathsf{F}_{21} (\tilde{\mathsf{b}}_1 + \tilde{\mathsf{b}}_2) + \mathsf{F}_{21}^2 (\tilde{\mathsf{b}}_1 + 2\tilde{\mathsf{b}}_2) \Big) \\ + \mathsf{F}_{12}\mathsf{F}_{22} \Big(\mathsf{F}_{21} (\tilde{\mathsf{b}}_1 + \tilde{\mathsf{b}}_2) - \mathsf{F}_{12} (\tilde{\mathsf{b}}_1 + 2\tilde{\mathsf{b}}_2) \Big) - \mathsf{F}_{11}^2 \mathsf{F}_{22} \tilde{\mathsf{b}}_2 = 0 . \tag{8.242c}$$

Provided that $\mathsf{F}_{12}^2\tilde{\mathsf{b}}_1 + \mathsf{F}_{11}^2(\tilde{\mathsf{b}}_1 + 2\tilde{\mathsf{b}}_2) \neq 0$, from (8.242a) one obtains

$$\mathsf{F}_{21} = \mathsf{F}_{12}\frac{(\mathsf{F}_{11}^2 + \mathsf{F}_{12}^2)(\tilde{\mathsf{b}}_1 + 2\tilde{\mathsf{b}}_2) - 2\mathsf{F}_{11}\mathsf{F}_{22}\tilde{\mathsf{b}}_2}{\mathsf{F}_{12}^2\tilde{\mathsf{b}}_1 + \mathsf{F}_{11}^2(\tilde{\mathsf{b}}_1 + 2\tilde{\mathsf{b}}_2)}\,. \tag{8.243}$$

Substituting for F_{21} into (8.242c), one has

$$\begin{aligned}\frac{(\mathsf{F}_{11}^2 + \mathsf{F}_{12}^2)(\mathsf{F}_{12}^2 - \mathsf{F}_{11}\mathsf{F}_{22})\tilde{\mathsf{b}}_2(\tilde{\mathsf{b}}_1 + 2\tilde{\mathsf{b}}_2)}{\left[\mathsf{F}_{12}^2\tilde{\mathsf{b}}_1 + \mathsf{F}_{11}^2(\tilde{\mathsf{b}}_1 + 2\tilde{\mathsf{b}}_2)\right]^2}&\Big(\mathsf{F}_{12}^2\mathsf{F}_{22}\tilde{\mathsf{b}}_1 + \mathsf{F}_{11}^2(\mathsf{F}_{11} - \mathsf{F}_{22})(\tilde{\mathsf{b}}_1 + 2\tilde{\mathsf{b}}_2) \\ &+ \mathsf{F}_{11}\mathsf{F}_{12}^2(3\tilde{\mathsf{b}}_1 + 4\tilde{\mathsf{b}}_2)\Big) = 0\,.\end{aligned} \tag{8.244}$$

Using (8.243), the Jacobian of the cloaking map is simplified to read

$$J_\xi = \frac{\mathsf{F}_{12}^2\tilde{\mathsf{b}}_1 + \mathsf{F}_{11}^2(\tilde{\mathsf{b}}_1 + 2\tilde{\mathsf{b}}_2)}{(\mathsf{F}_{11}^2 + \mathsf{F}_{12}^2)(\mathsf{F}_{11}\mathsf{F}_{22} - \mathsf{F}_{12}^2)(\tilde{\mathsf{b}}_1 + 2\tilde{\mathsf{b}}_2)}\,. \tag{8.245}$$

Knowing that $\tilde{\mathsf{b}}_2 > 0$, and the Jacobian of the cloaking map cannot be singular, (8.244) implies that

$$\mathsf{F}_{12}^2\mathsf{F}_{22}\tilde{\mathsf{b}}_1 + \mathsf{F}_{11}^2(\mathsf{F}_{11} - \mathsf{F}_{22})(\tilde{\mathsf{b}}_1 + 2\tilde{\mathsf{b}}_2) + \mathsf{F}_{11}\mathsf{F}_{12}^2(3\tilde{\mathsf{b}}_1 + 4\tilde{\mathsf{b}}_2) = 0\,, \tag{8.246}$$

where as long as $\mathsf{F}_{12}^2\tilde{\mathsf{b}}_1 - \mathsf{F}_{11}^2(\tilde{\mathsf{b}}_1 + 2\tilde{\mathsf{b}}_2) \neq 0$, gives

$$\mathsf{F}_{22} = \mathsf{F}_{11}\frac{\mathsf{F}_{11}^2(\tilde{\mathsf{b}}_1 + 2\tilde{\mathsf{b}}_2) + \mathsf{F}_{12}^2(3\tilde{\mathsf{b}}_1 + 4\tilde{\mathsf{b}}_2)}{\mathsf{F}_{11}^2(\tilde{\mathsf{b}}_1 + 2\tilde{\mathsf{b}}_2) - \mathsf{F}_{12}^2\tilde{\mathsf{b}}_1}\,. \tag{8.247}$$

Plugging (8.243) and (8.247) into (8.242b), one obtains

$$\frac{\mathsf{F}_{12}(\mathsf{F}_{11}^2 + \mathsf{F}_{12}^2)^4\tilde{\mathsf{b}}_2(\tilde{\mathsf{b}}_1 + \tilde{\mathsf{b}}_2)(\tilde{\mathsf{b}}_1 + 2\tilde{\mathsf{b}}_2)^2}{\left(\mathsf{F}_{11}^2(\tilde{\mathsf{b}}_1 + 2\tilde{\mathsf{b}}_2) - \mathsf{F}_{12}^2\tilde{\mathsf{b}}_1\right)^3} = 0\,, \tag{8.248}$$

where recalling that $\tilde{\mathsf{b}}_1 + \tilde{\mathsf{b}}_2 > 0$, implies that $\mathsf{F}_{12} = 0$, and thus, $\mathsf{F}_{21} = 0$ (cf. (8.243)), and from (8.247), $\mathsf{F}_{11} = \mathsf{F}_{22}$. Given that $\overset{\xi}{\mathbf{F}}|_{\partial_o\mathcal{C}} = id$, one concludes that ξ must be the identity, i.e., cloaking is not possible.

Let us consider the case where $\mathsf{F}_{12}^2\tilde{\mathsf{b}}_1 + \mathsf{F}_{11}^2(\tilde{\mathsf{b}}_1 + 2\tilde{\mathsf{b}}_2) = 0$, and thus

$$\tilde{\mathsf{b}}_1 = -\frac{2\mathsf{F}_{11}^2}{\mathsf{F}_{11}^2 + \mathsf{F}_{12}^2}\tilde{\mathsf{b}}_2 \,. \tag{8.249}$$

Therefore, (8.242a) is simplified to read $2\tilde{\mathsf{b}}_2\mathsf{F}_{12}(\mathsf{F}_{12}^2 - \mathsf{F}_{11}\mathsf{F}_{22}) = 0$, implying that $\mathsf{F}_{12}^2 = \mathsf{F}_{11}\mathsf{F}_{22}$ (or $\mathsf{F}_{22} = \mathsf{F}_{12}^2/\mathsf{F}_{11}$).[39] Substituting for $\tilde{\mathsf{b}}_1$ and F_{22} into (8.242b) and (8.242c), one obtains

$$2\tilde{\mathsf{b}}_2\frac{\mathsf{F}_{12}(\mathsf{F}_{12} - \mathsf{F}_{21})}{\mathsf{F}_{11}^2(\mathsf{F}_{11}^2 + \mathsf{F}_{12}^2)}\left[\mathsf{F}_{12}^5 + \mathsf{F}_{11}^2\mathsf{F}_{21}(\mathsf{F}_{11}^2 + \mathsf{F}_{12}^2) + \mathsf{F}_{11}^2\mathsf{F}_{21}^2\mathsf{F}_{12}\right] = 0\,, \tag{8.250a}$$

$$\tilde{\mathsf{b}}_2\frac{\mathsf{F}_{12}(\mathsf{F}_{21} - \mathsf{F}_{12})}{\mathsf{F}_{11}(\mathsf{F}_{11}^2 + \mathsf{F}_{12}^2)}\left[\mathsf{F}_{11}^4 + \mathsf{F}_{12}^4 + 2\mathsf{F}_{11}^2\mathsf{F}_{12}\mathsf{F}_{21}\right] = 0\,. \tag{8.250b}$$

Note that $J_\xi = [\mathsf{F}_{11}\mathsf{F}_{22} - \mathsf{F}_{12}\mathsf{F}_{21}]^{-1} = [\mathsf{F}_{12}(\mathsf{F}_{12} - \mathsf{F}_{21})]^{-1}$, and thus, (8.250) implies that

$$\mathsf{F}_{12}^5 + \mathsf{F}_{11}^2\mathsf{F}_{21}(\mathsf{F}_{11}^2 + \mathsf{F}_{12}^2) + \mathsf{F}_{11}^2\mathsf{F}_{21}^2\mathsf{F}_{12} = 0\,, \tag{8.251a}$$

$$\mathsf{F}_{11}^4 + \mathsf{F}_{12}^4 + 2\mathsf{F}_{11}^2\mathsf{F}_{12}\mathsf{F}_{21} = 0\,, \tag{8.251b}$$

from which, one obtains $\mathsf{F}_{12} = -\mathsf{F}_{21}$ and $\mathsf{F}_{11}^2 = -\mathsf{F}_{12}\mathsf{F}_{21}$. Thus, $\mathsf{F}_{11}^2 = \mathsf{F}_{12}^2$, and using (8.249), one concludes that $\tilde{\mathsf{b}}_1 + \tilde{\mathsf{b}}_2 = 0$, i.e., $\tilde{\mathbb{B}}$ is not positive definite, which is a contradiction. Similarly, it is straightforward to show that assuming $\mathsf{F}_{12}^2\tilde{\mathsf{b}}_1 - \mathsf{F}_{11}^2(\tilde{\mathsf{b}}_1 + 2\tilde{\mathsf{b}}_2) = 0$, the Jacobian of the cloaking map is forced to be singular.

[39]Note that $\mathsf{F}_{12} \neq 0$, because otherwise, $\tilde{\mathsf{b}}_1 = -2\tilde{\mathsf{b}}_2$ (from (8.249)), contradicting the positive definiteness of $\tilde{\mathbb{B}}$.

CHAPTER 9

CONCLUSION

Utilizing the transformation properties of nonlinear and linearized elasticity, in this PhD thesis, we present theoretical frameworks to study some elastic and anelastic problems in solids.

In Chapter 2, we briefly review some fundamental elements of the geometric theory of nonlinear elasticity and anelasticity for isotropic and anisotropic solids.

In Chapter 3, we studied the residual stress field generated by a circumferentially-symmetric distribution of finite eigenstrains in an incompressible, isotropic elastic wedge. Using a semi-inverse method by assuming a specific class of deformations, we solved for the deformation and stress fields in the wedge for an arbitrary circumferentially-symmetric distribution of finite eigenstrains. We solved two examples. In the first one, we considered an inclusion with uniform eigenstrains in a neo-Hookean wedge with traction-free lateral boundaries and obtained exact solutions for the residual stress and deformation fields. We observed that if the eigenstrain distribution is purely circumferential, the pressure field remains continuous at the inclusion-matrix interface and the stress tensor is zero everywhere. Moreover, we observed that the deformation of the wedge fails to be unidirectional for an inclusion with a negative radial ($\omega_1 < 0$) and positive circumferential ($\omega_2 > 0$) eigenstrians even for large negative values of the radial eigenstrain. Furthermore, we found that the total wedge angle is reduced for any value of pure dilatational eigenstrains. In the second example, we considered a neo-Hookean wedge with clamped lateral boundaries having a symmetric Mooney-Rivlin inhomogeneity with uniform eigenstrains. We examined several cases of eigenstrain distributions for different relative stiffnesses of the inhomogeneity and the matrix. We observed that the circumferential and radial deformations are more pronounced in wedges containing inhomogeneities with only radial and only circumferential eigenstrains. In addition, we noticed that for a pure radial eigenstrain distribution, $\hat{\sigma}^{rr}$ and $\hat{\sigma}^{\theta\theta}$ are almost uniform in the inhomogeneity, and $\hat{\sigma}^{rr}$ has a jump at the inhomogeneity-matrix interface. In contrast, for a pure circumferential eigenstrain distribution $\hat{\sigma}^{rr}$ and $\hat{\sigma}^{\theta\theta}$ are nonuniform in the inhomogeneity, with $\hat{\sigma}^{rr}$ being continuous at the inhomogeneity-matrix interface.

In Chapter 4 we studied the residual stress and deformation fields of a solid torus containing a toroidal inclusion with finite eigenstrains that is concentric with the solid torus. We used a perturbation analysis and obtained the stress and displacement fields to the first order in the thinness ratio. We showed that the stress field in the toroidal inclusion is nonuniform, unlike cylindrical and spherical inclusions in infinitely-long and finite circular cylindrical bars and spherical balls, respectively, in which the stress field inside the inclusion is uniform. We presented some numerical results for a neo-Hookean solid torus having an inclusion with a uniform pure dilatational eigenstrain distribution. In particular, we observed that all the first-order stress components in the inclusion have a linear dependence on the referential radial coordinate. Moreover, the maximum shear stress in the torus is first increasing, then decreasing as the relative size of the inclusion increases from zero. We observed shear stress concentration regions across the inclusion-matrix interface for a torus with a negative pure dilatational eigenstrain distribution. Interestingly, the torus exhibits different responses for positive and negative eigenstrain values. It was observed that for the positive eigenstrians, $\frac{b}{B}$ monotonically increases as the eigenstrain Ω_o increases, and the increase is more rapid for inclusions with larger relative sizes. For negative eigesntrains, nonetheless, $\frac{b}{B}$ reaches a minimum, the value of which decreases as the relative size of the inclusion becomes larger. We noticed that the deformed shape of the outer boundaries of the matrix and the inclusion are eccentric circles with the corresponding zero-order radii in the first-order approximation with respect to the thinness ratio. Finally, we proved that the stress field inside a toroidal inclusion with nonzero uniform pure dilatational (infinitesimal) eigenstrains in an isotropic incompressible linear elastic solid torus is always nonuniform for any size of the solid torus.

To this date the study of anisotropic inclusion problems in the literature has been restricted to linear elasticity. In Chapter 5, we considered finite eigenstrains in transversely isotropic spherical balls and orthotropic cylindrical bars made of both compressible and incompressible solids. We identified the conditions under which the stress field in the spherical and cylindrical inclusions with a uniform distribution of dilatational eigenstrains is uniform. We showed that the stress in a spherical inclusion with uniform eigenstrains contained in an incompressible transversely isotropic spherical ball with the material preferred direction being radial is uniform and hydrostatic if the radial and circumferential eigenstrains are equal. A similar result holds for cylindrical inclusions in incompressible orthotropic cylindrical bars when orthotropic axes are in the radial, circumferential, and axial directions, provided that the axial stretch is equal to a value determined by the longitudinal eigenstrain. Except for some special cases for which the energy function is constrained depending on the eigenstrains, in the case of incompressible solids a stress singularity emerges as a result of a mismatch between radial and circumferential eigenstrains at the center of a ball or on the axis of a cylindrical bar.

We generalized the results of Yavari and Goriely [24] to any compressible isotropic material. Specifically, we showed that for compressible isotropic spherical balls and cylindrical bars with spherical and cylindrical inclusions with uniform eigenstrains, respectively, if the radial and circumferential eigenstrains are equal the stress in the inclusion is uniform (and hydrostatic for the spherical inclusion).

We observed that for compressible transversely isotropic and orthotropic solids the stress field in the inclusion with uniform dilatational eigenstrains is not necessarily uniform. We showed, however, that there are some energy functions for which for a given applied pressure on the outer boundary, the ratio R_i/R_o is determined if a uniform stress field is to be maintained in the inclusion. Similarly, for such special energy functions, fixing R_i/R_o uniquely determines the pressure that must be applied on the outer boundary to maintain a uniform stress field inside the inclusion. Moreover, material parameters must satisfy certain conditions depending on the eigenstrains (and the axial stretch in the case of cylindrical bars). To explore these special cases, we assumed some specific energy functions, namely compressible Mooney-Rivlin and Blatz-Ko reinforced models and found analytical expressions for the stress field in the inclusion.

Despite the crucial role that anisotropy plays in the overall response of materials in the presence of large strains, the study of defects in nonlinear solids has been overwhelmingly restricted to isotropic materials to this date. In Chapter 6, we presented a few analytical solutions for the stress fields induced by distributed line and point defects in nonlinear anisotropic solids. We considered a parallel cylindrically-symmetric distribution of screw dislocations in infinite orthotropic and monoclinic media, and also, a cylindrically-symmetric distribution of parallel wedge disclinations in an orthotropic medium. Because the material manifold is endowed with a nontrivial Riemannian metric that explicitly depends on the defect distribution, the material preferred directions, and hence, the class of anisotropy of the defective body are, in general, different in the reference and current configurations. We observed, in particular, that for a cylindrically-symmetric distribution of screw dislocations, assuming that the body is orthotropic in its reference (current) configuration, it is monoclinic in its current (reference) configuration. We found that for an arbitrary cylindrically-symmetric distribution of parallel screw dislocations, and for a uniform wedge disclination distribution, the stress field is logarithmically singular on the dislocation and the disclination axes unless the axial deformation is suppressed. These stress singularities are inherently due to the anisotropy effects, e.g., the radial fiber-reinforcement, and do not, in particular, arise in isotropic materials. This observation demonstrates the significance of taking material anisotropy into consideration in the analysis of solids with distributed defects. For a single screw dislocation, we employed the standard reinforcing model and discussed the conditions that guarantee that the energy per unit length and the resultant axial force are finite for a fiber-reinforced material as long as the isotropic base material has a finite energy per unit length and a finite axial force. For a distribution of edge dislocations the resulting stresses are calculated when the medium is orthotropic. Finally, we studied a spherically-symmetric distribution of point defects in a transversely isotropic spherical ball. We showed that for an incompressible transversely isotropic ball with the radial material preferred direction, a uniform point defect distribution results in a uniform hydrostatic stress field inside the spherical region the distribution is supported in.

In Chapter 7, we investigated the problem of hiding an object from elastic waves in elastic solids. We started by discussing the invariance of the governing equations of elastodynamics under time-dependent spatial changes of frame and arbitrary time-independent referential changes of frame. We presented a mathematically coherent formulation of the elastodynamic transformation cloaking problem. We note that the cloaking transformation is the mapping that transforms the boundary-value problem of an anisotropic and nonuniform elastic body with a hole reinforced by a cloak to that of a homogeneous and isotropic body with an infinitesimal hole with negligible scattering effects on the waves. The mechanical properties of the cloak are independent of the frequency of the incident wave. In particular, we investigated the transformation cloaking problem in the classical elasticity; in the context of the small-on-large theory of elasticity, and in solids with microstructure, namely the gradient and (generalized) Cosserat solids.

Finally, in Chapter 8, we formulated the problem of elastodynamic transformation cloaking in plates starting from nonlinear shell theory. In particular, we considered transformation cloaking in Kirchhoff-Love plates as well as elastic plates with both the in-plane and out-of-plane displacements. Using a Lagrangian field theory, the governing equations of nonlinear (and linearized) elastic shells (and plates) were derived by characterizing the geometry of a shell as an embedded hypersuface in the Euclidean space using the first and the second fundamental forms. The body forces and body moments were taken into consideration in the boundary-value problem of an elastic plate using the Lagrange–d'Alembert principle. A cloaking map transforms the boundary-value problem of an isotropic and homogeneous elastic plate (virtual problem) to that of an anisotropic and inhomogeneous elastic plate with a finite hole covered by a cloak (physical problem) that is designed such that the response of the virtual plate is mimicked outside the cloak.

Cloaking in Kirchhoff-Love plates involves transforming the (out-of-plane) governing equation of the virtual plate to that of the physical plate up to an unknown scalar field via a cloaking map. In doing so, one obtains a set of constraints involving the cloaking transformation, the scalar field, and the elastic parameters of the virtual plate. In addition, there are some conditions that the cloaking transformation and the scalar field need to satisfy on the boundary of the cloak and the hole. In particular, the cloaking map needs to fix the outer boundary of the cloak up to the third order and the scalar field needs to be the identity up to the first order on the outer boundary of the cloak. In the example of a circular hole, we show that cloaking a circular hole in Kirchhoff-Love plates is not possible for a generic radial cloaking map; the obstruction to transformation cloaking are the constraints and the boundary conditions that the cloaking map needs to satisfy.

In the case of a hole with an arbitrary shape, the constraints are a system of second-order nonlinear

PDEs, and the balance of linear and angular momenta for the (physical) plate in its finitely-deformed configuration lead to fourth-order and third-order nonlinear PDEs. The complexity of this system of nonlinear PDEs makes studying the obstruction to cloaking for an arbitrary cloaking map very complicated. This is in contrast to 3D elastodynamics and elastic plates with both in-plane and out-of-plane deformations, where the linearized balance of angular is the obstruction to cloaking. The nature of the linearized balance of angular momentum usually allows one to analyze these equations for an arbitrary cloaking map and be able to prove obstruction to cloaking. Note that Kirchoff-Love plates can only have bending deformations, and in this case, the linearized balance of angular momentum only implies that the flexural rigidity tensor must have the minor symmetries. As the flexural rigidity tensor preserves its minor (and major) symmetries under the cloaking map, the only non-trivial linearized balance of angular momentum is already satisfied. Therefore, one only needs to analyze the constraints and the balance of linear and angular momenta in the finitely-deformed configuration (of the physical plate) to study obstruction to transformation cloaking.

Next, we relaxed the pure bending assumption and formulated the transformation cloaking problem for an elastic plate in the presence of in-plane and out-of-plane displacements. The physical plate is initially stressed and is subjected to (in-plane and out-of-plane) body forces and moments in its finitely-deformed configuration. This problem involves transforming the in-plane governing equations using the Piola transformation given by the cloaking map as well as transforming the out-of-plane governing equation up to the Jacobian of the cloaking map. Assuming a general radial cloaking map, we showed that cloaking in the presence of in-plane and out-of-plane excitations is not possible for a circular hole; the constraints and the boundary conditions that the cloaking map needs to satisfy obstruct transformation cloaking, similar to the case of Kirchhoff-Love plates. We also showed that if for the virtual plate the elasticity tensor pertaining to the coupling between the in-plane and out-of-plane displacements is positive-definite, then cloaking is not possible even if one uses a general cloaking map for a hole and a cloak with arbitrary shapes; the balance of angular momentum is the obstruction to cloaking similar to transformation cloaking in 3D elastodynamics.

As a sequel to this work, we see many paths for future research that we would like to explore.

Transformation anelasticity can be utilized to study many anelastic phenomena in the context of nonlinear and linear elasticity. One line of investigation is to study the nonlinear anisotropic inclusion problem for other types of material anisotropy and other geometries such as non-simply connected bodies in order to understand how anisotropy and geometry affect the induced stress distribution of nonlinear inclusions in anisotropic solids. Further developments are needed to investigate the role that anisotropy plays in the dynamics, stability, and interactions of defects at finite strains, and in particular, how anisotropy affects stress singularity for a nonlinear anisotropic solid with a given distribution of defects. Transformation elasticity can be used to extend our formulations of the transformation cloaking problem. One extension of this work would be a further generalization of the developed theory of transformation cloaking in elastic plates such that one can study cloaking in Cosserat shells. This is done by assigning a set of deformable directors at each point of an elastic shell in its reference and current configurations. This will lead to a (normal and tangential) hyperstress in addition to the stress and the couple-stress of the present theory. One should note that the balance of linear (and angular) momentum and micro linear (and angular) momentum in this case are coupled in a way that makes transformation cloaking highly non-trivial. Moreover, one needs to derive a set of compatibility conditions for the normal and tangential components of the director gradient.

Interestingly, transformation elasticity has direct applications in developing the theory of perfectly matched layer[1] (PML) for modeling elastic waves. Accurate modeling of elastic waves often requires modeling unbounded domains. This, to be computationally feasible, necessitates that the physical domain be properly truncated to prevent unwanted wave reflections form the truncated boundaries. A perfectly matched layer (PML), which is an artificial layer (medium in 3D) attached to the physical domain can be used to truncate the computational domain properly as if an unbounded domain was considered in the simulation. One can use transformation elasticity to design and optimize PMLs and PMMs in the context of different theories of elasticity.

[1]Perfectly matched medium (PMM) in 3D.

Appendices

APPENDIX A

A NONLINEAR ELASTIC SOLID TORUS WITH A TOROIDAL INCLUSION

A.1 Analytical expressions for the functions $g_o(R)$ and $h_o(R)$

$$
\begin{aligned}
g_o^p(R) = &-\frac{1}{k^{\frac{1}{2}} R\left(R^2+\eta\right)^{\frac{1}{2}}}\left[c_{o_1}^p\left(R^2+\eta\right)^{\frac{1}{2}}+c_{o_2}^p\left(\left(R^2+\eta\right)^{\frac{1}{2}}\left(3R^2+2\eta\right)+R\eta\sinh^{-1}\frac{R}{\eta^{\frac{1}{2}}}\right)+Rc_{o_4}^p\right.\\
&\left.+c_{o_3}^p\left(R\left(3R^2+\eta\right)-\eta\left(R^2+\eta\right)^{\frac{1}{2}}\sinh^{-1}\frac{R}{\eta^{\frac{1}{2}}}\right)\right]+\frac{1}{16BR^3\left(R^2+\eta\right)^{\frac{1}{2}}}\left[k^{\frac{3}{2}}R\left(8R^4+2R^2\eta+\eta^2\right)\right.\\
&+2R^2\left(R^2+\eta\right)^{\frac{1}{2}}\left(11R^2+\eta\right)+\eta^2k^{\frac{3}{2}}\left(R^2+\eta\right)^{\frac{1}{2}}\sinh^{-1}\frac{R}{\eta^{\frac{1}{2}}}-k^{\frac{3}{2}}\eta R^3\ln(R^2\eta)\\
&+R^2\eta\left(R+k^{\frac{3}{2}}\left(R^2+\eta\right)^{\frac{1}{2}}\right)\sinh^{-1}\frac{R}{\eta^{\frac{1}{2}}}\ln\frac{R^2}{R^2+\eta}+R^2\left(\left(R^2+\eta\right)^{\frac{1}{2}}-k^{\frac{3}{2}}R\right)\left(3R^2\ln\frac{R^2}{R^2+\eta}\right.\\
&\left.+\eta\ln\eta\right)+R^2\eta\left(3k^{\frac{3}{2}}R-2\left(R^2+\eta\right)^{\frac{1}{2}}\right)\ln(R^2+\eta)+2R^2\eta^2\left\{\frac{R}{\eta}\ln\left(R+\left(R^2+\eta\right)^{\frac{1}{2}}\right)-RJ_1^p(R)\right.\\
&\left.\left.+\eta k^{\frac{3}{2}}\left(R^2+\eta\right)^{\frac{1}{2}}J_2^p(R)\right\}\right],
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
g_o^n(R) = &-\frac{1}{k^{\frac{1}{2}} R\left(R^2+\eta\right)^{\frac{1}{2}}}\left[c_{o_1}^n\left(R^2+\eta\right)^{\frac{1}{2}}+c_{o_2}^n\left(\left(R^2+\eta\right)^{\frac{1}{2}}\left(3R^2+2\eta\right)+R\eta\cosh^{-1}\frac{R}{(-\eta)^{\frac{1}{2}}}\right)+Rc_{o_4}^n\right.\\
&\left.+c_{o_3}^n\left(R\left(3R^2+\eta\right)-\eta\left(R^2+\eta\right)^{\frac{1}{2}}\cosh^{-1}\frac{R}{(-\eta)^{\frac{1}{2}}}\right)\right]+\frac{1}{16BR^3\left(R^2+\eta\right)^{\frac{1}{2}}}\left[k^{\frac{3}{2}}R\left(8R^4+2R^2\eta+\eta^2\right)\right.\\
&+2R^2\left(R^2+\eta\right)^{\frac{1}{2}}\left(11R^2+\eta\right)+\eta^2k^{\frac{3}{2}}\left(R^2+\eta\right)^{\frac{1}{2}}\cosh^{-1}\frac{R}{(-\eta)^{\frac{1}{2}}}-k^{\frac{3}{2}}\eta R^3\ln(-R^2\eta)\\
&+R^2\eta\left(R+k^{\frac{3}{2}}\left(R^2+\eta\right)^{\frac{1}{2}}\right)\cosh^{-1}\frac{R}{(-\eta)^{\frac{1}{2}}}\ln\frac{R^2}{R^2+\eta}+R^2\left(\left(R^2+\eta\right)^{\frac{1}{2}}-k^{\frac{3}{2}}R\right)\left(3R^2\ln\frac{R^2}{R^2+\eta}\right.\\
&\left.+\eta\ln(-\eta)\right)+R^2\eta\left(3k^{\frac{3}{2}}R-2\left(R^2+\eta\right)^{\frac{1}{2}}\right)\ln(R^2+\eta)+2R^2\eta^2\left\{\frac{R}{\eta}\ln\left(R+\left(R^2+\eta\right)^{\frac{1}{2}}\right)-RJ_1^n(R)\right.\\
&\left.\left.+\eta k^{\frac{3}{2}}\left(R^2+\eta\right)^{\frac{1}{2}}J_2^n(R)\right\}\right],
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
h_o^p(R) = \frac{1}{16Bk^{\frac{1}{2}}R^3\left(R^2+\eta\right)^2}\Bigg[&-\left(R^2+\eta\right)^{\frac{1}{2}}\left\{16R^3\left(R^2+\eta\right)^2+k^3R\left(2R^2+\eta\right)\left(16R^2\left(R^2+\eta\right)-\eta^2\right)\right\} \\
&+R^2k^{\frac{3}{2}}\Bigg\{2R\left(4R^4+7R^2\eta+2\eta^2\right)\left(k^{\frac{3}{2}}\left(R^2+\eta\right)^{\frac{1}{2}}-R\right)+k^{\frac{3}{2}}\eta^2\Bigg(3R\left(R^2+\eta\right)^{\frac{1}{2}} \\
&+\eta\sinh^{-1}\frac{R}{\eta^{\frac{1}{2}}}\Bigg)\Bigg\}\ln\frac{R^2}{R^2+\eta}-2k^{\frac{3}{2}}R^2\left(R^2+\eta\right)\left(16R^4+14R^2\eta-\eta^2\right)+2\eta^3k^{\frac{3}{2}}R^2\ln\frac{\eta^{\frac{1}{2}}}{R^2+\eta} \\
&+k^3\eta^4\left\{2\eta R^2J_2^p(R)+\sinh^{-1}\frac{R}{\eta^{\frac{1}{2}}}\right\}+16BkR^2\Bigg\{-c_{o_1}^p\eta^2+2c_{o_2}^p\left(R^2+\eta\right)^2\left(4R^2-\eta\right) \\
&+c_{o_3}^pR\left(R^2+\eta\right)^{\frac{1}{2}}\left(8R^4+14R^2\eta+7\eta^2\right)+c_{o_3}^p\eta^3\sinh^{-1}\frac{R}{\eta^{\frac{1}{2}}}\Bigg\}\Bigg],
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
h_o^n(R) = \frac{1}{16Bk^{\frac{1}{2}}R^3\left(R^2+\eta\right)^2}\Bigg[&-\left(R^2+\eta\right)^{\frac{1}{2}}\left\{16R^3\left(R^2+\eta\right)^2+k^3R\left(2R^2+\eta\right)\left(16R^2\left(R^2+\eta\right)-\eta^2\right)\right\} \\
&+R^2k^{\frac{3}{2}}\Bigg\{2R\left(4R^4+7R^2\eta+2\eta^2\right)\left(k^{\frac{3}{2}}\left(R^2+\eta\right)^{\frac{1}{2}}-R\right)+k^{\frac{3}{2}}\eta^2\Bigg(3R\left(R^2+\eta\right)^{\frac{1}{2}} \\
&+\eta\cosh^{-1}\frac{R}{(-\eta)^{\frac{1}{2}}}\Bigg)\Bigg\}\ln\frac{R^2}{R^2+\eta}-2k^{\frac{3}{2}}R^2\left(R^2+\eta\right)\left(16R^4+14R^2\eta-\eta^2\right)+2\eta^3k^{\frac{3}{2}}R^2\ln\frac{(-\eta)^{\frac{1}{2}}}{R^2+\eta} \\
&+k^3\eta^4\left\{2\eta R^2J_2^n(R)+\cosh^{-1}\frac{R}{(-\eta)^{\frac{1}{2}}}\right\}+16BkR^2\Bigg\{-c_{o_1}^n\eta^2+2c_{o_2}^n\left(R^2+\eta\right)^2\left(4R^2-\eta\right) \\
&+c_{o_3}^nR\left(R^2+\eta\right)^{\frac{1}{2}}\left(8R^4+14R^2\eta+7\eta^2\right)+c_{o_3}^n\eta^3\cosh^{-1}\frac{R}{(-\eta)^{\frac{1}{2}}}\Bigg\}\Bigg].
\end{aligned}
\tag{A.4}
$$

A.2 Proof of the separability of the first-order deformation and pressure fields in the form given in (4.41)

Let us represent $r^{(1)}$, $\phi^{(1)}$, and $\frac{p^{(1)}}{\mu}$ by appropriate Fourier series expansions, given that they are even, odd, and even 2π-periodic functions[1], respectively, as

$$\begin{aligned}
r^{(1)} &= \frac{r_0^{(1)}(R)}{2} + \sum_{n=1}^{\infty} r_n^{(1)}(R)\cos(n\Phi)\,,\\
\phi^{(1)} &= \sum_{n=1}^{\infty} \phi_n^{(1)}(R)\sin(n\Phi)\,,\\
\frac{p^{(1)}}{\mu} &= \frac{p_0^{(1)}(R)}{2\mu} + \sum_{n=1}^{\infty} \frac{p_n^{(1)}(R)}{\mu}\cos(n\Phi)\,,
\end{aligned} \tag{A.5}$$

where for $n \in \mathbb{N} \cup \{0\}$, $r_n^{(1)}$, $\phi_n^{(1)}$, and $\frac{p_n^{(1)}}{\mu}$ are the real-valued Fourier coefficients given by

$$\begin{aligned}
r_n^{(1)}(R) &= \frac{1}{\pi}\int_{-\pi}^{\pi} r^{(1)}(R,\zeta)\cos(n\zeta)d\zeta,\\
\phi_n^{(1)}(R) &= \frac{1}{\pi}\int_{-\pi}^{\pi} \phi^{(1)}(R,\zeta)\sin(n\zeta)d\zeta,\\
\frac{p_n^{(1)}}{\mu} &= \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{p^{(1)}}{\mu}(R,\zeta)\cos(n\zeta)d\zeta\,.
\end{aligned} \tag{A.6}$$

We now show that all the Fourier coefficients vanish except those with $n = 1$, giving us what we have in (4.41). Substituting (A.5) into (4.39), one obtains the following ODEs in the inclusion and matrix

$$\begin{aligned}
n\phi_n^{(1)} &= -k^{-\frac{1}{2}} e^{-\frac{3\Omega_o}{4}} \left(r_n^{(1)\prime} + \frac{r_n^{(1)}}{R} \right), && 0 \leq R \leq R_i\,, \quad n \geq 2\,,\\
n\phi_n^{(1)} &= -k^{-\frac{1}{2}} \left(1 + \frac{\eta}{R^2}\right)^{-\frac{1}{2}} \left[\left(1 + \frac{\eta}{R^2}\right) r_n^{(1)\prime} + \frac{r_n^{(1)}}{R} \right], && R_i \leq R \leq R_o\,, \quad n \geq 2\,,\\
r_0^{(1)\prime} + \frac{r_0^{(1)}}{R} &= 0\,, && 0 \leq R \leq R_i\,,\\
\left(1 + \frac{\eta}{R^2}\right) r_0^{(1)\prime} + \frac{r_0^{(1)}}{R} &= 0\,, && R_i \leq R \leq R_o\,,
\end{aligned} \tag{A.7}$$

[1] This immediately follows from the symmetry of the problem for radially-symmetric eigenstrain distributions.

where $\eta = R_i^2\gamma_o$. Similarly, we substitute (A.5) into (4.29) and (4.30) to find the following ODEs for $n \geq 2$ and $n = 0$

$$r_n^{(1)''} + \frac{r_n^{(1)'}}{R} - \frac{(2+n^2)\, r_n^{(1)}}{2R^2} - \frac{e^{\frac{\Omega_o}{4}}}{2k^{\frac{1}{2}}}\frac{p_n^{(1)'}}{\mu} + nk^{\frac{1}{2}}e^{\frac{3\Omega_o}{4}}\left(\frac{\phi_n^{(1)'}}{2} - \frac{\phi_n^{(1)}}{R}\right) = 0\,, \qquad 0 \leq R \leq R_i\,, \tag{A.8a}$$

$$\begin{aligned} &2R^3(R^2+\eta)r_n^{(1)''} + (2R^4-\eta^2)r_n^{(1)'} - R\left(\frac{R^4}{R^2+\eta} + \left(1+n^2\right)(R^2+\eta)\right)r_n^{(1)} \\ &- k^{-\frac{1}{2}}R^2(R^2+\eta)^{\frac{3}{2}}\frac{p_n^{(1)'}}{\mu} + nk^{\frac{1}{2}}R^4(R^2+\eta)^{\frac{1}{2}}\phi_n^{(1)'} - 2nk^{\frac{1}{2}}R(R^2+\eta)^{\frac{3}{2}}\phi_n^{(1)} = 0\,, \qquad R_i \leq R \leq R_o\,, \end{aligned} \tag{A.8b}$$

and for $n \geq 2$

$$r_n^{(1)'} + \frac{3r_n^{(1)}}{R} - \frac{e^{\frac{\Omega_o}{4}}}{k^{\frac{1}{2}}}\frac{p_n^{(1)}}{\mu} - \frac{k^{\frac{1}{2}}e^{\frac{3\Omega_o}{4}}}{n}\left(R^2\phi_n^{(1)''} + 3R\phi_n^{(1)'} - 2n^2\phi_n^{(1)}\right) = 0\,, \qquad 0 \leq R \leq R_i\,, \tag{A.9a}$$

$$\begin{aligned} &2n^2R(R^2+\eta)\phi_n^{(1)} - R^2(3R^2+\eta)\phi_n^{(1)'} - R^3(R^2+\eta)\phi_n^{(1)''} - \frac{nR^3}{k}\frac{p_n^{(1)}}{\mu} \\ &+ nk^{-\frac{1}{2}}(R^2+\eta)^{\frac{3}{2}}r_n^{(1)'} + nk^{-\frac{1}{2}}R\left(3(R^2+\eta)^{\frac{1}{2}} - \frac{\eta^2}{(R^2+\eta)^{\frac{3}{2}}}\right)r_n^{(1)} = 0\,, \qquad R_i \leq R \leq R_o\,. \end{aligned} \tag{A.9b}$$

Clearly, $r_n^{(1)} = \phi_n^{(1)} = \frac{p_n^{(1)}}{\mu} = 0$, $n = 0$ and $n \geq 2$ is a solution of the system of linear ordinary differential equations (A.7), (A.8), and (A.9), and hence it is the unique solution satisfying the required boundary conditions (4.18) and the continuity of the displacement and traction fields at the inclusion-matrix interface.

APPENDIX B

NONLINEAR ELASTIC INCLUSIONS IN ANISOTROPIC SOLIDS

B.1 Radial equilibrium equation for the compressible transversely isotropic case

$$
\begin{aligned}
&8e^{8\omega_\Theta} W_{I_5 I_5} r'^7 \omega_R' R^9 - e^{6\omega_R+4\omega_\Theta} \Big(e^{4\omega_\Theta} \left(W_{I_1} + W_{I_4}\right) r'' R^4 - 2e^{2(\omega_R+\omega_\Theta)} r W_{I_1} R^2 + 2e^{2\omega_\Theta} r^2 W_{I_2} r'' R^2 - 2e^{2\omega_R} r^3 W_{I_2} \\
&+ r^4 W_{I_3} r''\Big) R^5 - 8e^{4\omega_\Theta} r'^6 \left(e^{4\omega_\Theta} W_{I_5 I_5} r'' R^4 + e^{2\omega_R} r \left(e^{2\omega_\Theta} W_{I_2 I_5} R^2 + r^2 W_{I_3 I_5}\right)\right) R^5 + 8e^{2\omega_R+4\omega_\Theta} r'^5 \Big(e^{4\omega_\Theta} (W_{I_1 I_5} \\
&+ W_{I_4 I_5}) \omega_R' R^5 + e^{2\omega_\Theta} r^2 W_{I_2 I_5} \left(2R\omega_R' + R\omega_\Theta' + 1\right) R^2 + r^4 W_{I_3 I_5} \left(R\omega_R' + R\omega_\Theta' + 1\right)\Big) R^4 \\
&- 2e^{4\omega_R} r'^2 \Big[e^{8\omega_\Theta} \left(W_{I_1 I_1} + 2W_{I_1 I_4} + W_{I_4 I_4} + 3W_{I_5}\right) r'' R^8 + e^{2\omega_R+6\omega_\Theta} r \left(W_{I_2} + 2\left(W_{I_1 I_1} + W_{I_1 I_4}\right)\right) R^6 \\
&+ 4e^{6\omega_\Theta} r^2 \left(W_{I_1 I_2} + W_{I_2 I_4}\right) r'' R^6 + e^{2\omega_R+4\omega_\Theta} r^3 \left(6W_{I_1 I_2} + W_{I_3} + 2W_{I_2 I_4}\right) R^4 + 2e^{4\omega_\Theta} r^4 (2W_{I_2 I_2} + W_{I_1 I_3} \\
&+ W_{I_3 I_4}) r'' R^4 + 2e^{2(\omega_R+\omega_\Theta)} r^5 \left(2W_{I_2 I_2} + W_{I_1 I_3}\right) R^2 + 4e^{2\omega_\Theta} r^6 W_{I_2 I_3} r'' R^2 + 2e^{2\omega_R} r^7 W_{I_2 I_3} + r^8 W_{I_3 I_3} r''\Big] R \\
&- 4e^{2\omega_R} r'^4 \Big\{2e^{4\omega_\Theta} \left(e^{4\omega_\Theta} W_{I_1 I_5} R^4 + e^{4\omega_\Theta} W_{I_4 I_5} R^4 + 2e^{2\omega_\Theta} r^2 W_{I_2 I_5} R^2 + r^4 W_{I_3 I_5}\right) r'' R^4 + e^{2\omega_R} r \Big(e^{6\omega_\Theta} (W_{I_1 I_2} \\
&+ W_{I_2 I_4} + 2W_{I_1 I_5}) R^6 + e^{4\omega_\Theta} r^2 \left(2W_{I_2 I_2} + W_{I_1 I_3} + W_{I_3 I_4} + 2W_{I_2 I_5}\right) R^4 + 3e^{2\omega_\Theta} r^4 W_{I_2 I_3} R^2 + r^6 W_{I_3 I_3}\Big)\Big\} R \\
&+ 2e^{4\omega_R} r'^3 \Big[e^{8\omega_\Theta} W_{I_5} (3R\omega_R' - 2R\omega_\Theta' - 2) R^8 + \Big(e^{8\omega_\Theta} \left(W_{I_1 I_1} + 2W_{I_1 I_4}\right) R^8 + e^{8\omega_\Theta} W_{I_4 I_4} R^8 + 4e^{6\omega_\Theta} r^2 (W_{I_1 I_2} \\
&+ W_{I_2 I_4}) R^6 + 2e^{4\omega_\Theta} r^4 (2W_{I_2 I_2} + W_{I_1 I_3} + W_{I_3 I_4}) R^4 + 4e^{2\omega_\Theta} r^6 W_{I_2 I_3} R^2 + r^8 W_{I_3 I_3}\Big) \omega_R' R + 2r^2 \Big(e^{6\omega_\Theta} (W_{I_1 I_2} + W_{I_2 I_4} \\
&+ 2W_{I_1 I_5}) R^6 + e^{4\omega_\Theta} r^2 (2W_{I_2 I_2} + W_{I_1 I_3} + W_{I_3 I_4} + 2W_{I_2 I_5}) R^4 + 3e^{2\omega_\Theta} r^4 W_{I_2 I_3} R^2 + r^6 W_{I_3 I_3}\Big) \left(R\omega_\Theta' + 1\right)\Big] \\
&+ e^{6\omega_R} r' \Big\{e^{8\omega_\Theta} W_{I_1} \left(R\omega_R' - 2R\omega_\Theta' - 2\right) R^8 + e^{8\omega_\Theta} W_{I_4} (R\omega_R' - 2R\omega_\Theta' - 2) R^8 + 2e^{6\omega_\Theta} r^2 (R W_{I_2} \omega_R' \\
&+ 2\left(W_{I_1 I_1} + W_{I_1 I_4}\right) \left(R\omega_\Theta' + 1\right)) R^6 + e^{4\omega_\Theta} r^4 \Big[4\left(3W_{I_1 I_2} + W_{I_2 I_4}\right) \left(R\omega_\Theta' + 1\right) + W_{I_3} (R\omega_R' + 2R\omega_\Theta' + 2)\Big] R^4 \\
&+ 4e^{2\omega_\Theta} r^6 \left(2W_{I_2 I_2} + W_{I_1 I_3}\right) \left(R\omega_\Theta' + 1\right) R^2 + 4r^8 W_{I_2 I_3} \left(R\omega_\Theta' + 1\right)\Big\} = 0\,.
\end{aligned}
\tag{B.1}
$$

B.2 Analytical expression for $k(R)$

$$
\begin{aligned}
k(R) = &-\frac{2e^{-2(2\omega_Z+\omega_\Theta)}}{r^{10}R^3\alpha^9}\Big(2e^{4\omega_Z}\alpha^7\left(W_{I_2I_2}\left(R\omega'_\Theta+R\omega'_Z+1\right)\alpha^2+e^{2\omega_Z}W_{I_1I_2}\left(R\omega'_\Theta+1\right)\right)r^{12}\\
&+e^{2\omega_Z+\omega_\Theta}R^2\alpha^5\Big[2e^{\omega_\Theta}W_{I_2I_2}\alpha^6+2e^{\omega_\Theta}RW_{I_2I_2}\omega'_Z\alpha^6+4e^{\omega_\Theta}RW_{I_2I_7}\omega'_Z\alpha^6+2e^{\omega_\Theta}RW_{I_2I_2}\omega'_\Theta\alpha^6+4e^{2\omega_Z+\omega_\Theta}W_{I_1I_2}\alpha^4\\
&+4e^{2\omega_Z+\omega_\Theta}RW_{I_1I_2}\omega'_Z\alpha^4+2e^{2\omega_Z+\omega_\Theta}RW_{I_2I_6}\omega'_Z\alpha^4+4e^{2\omega_Z+\omega_\Theta}RW_{I_1I_2}\omega'_\Theta\alpha^4+2e^{2\omega_Z+\omega_\Theta}W_{I_2I_4}\left(R\omega'_Z+R\omega'_\Theta+1\right)\alpha^4\\
&+e^{\omega_R+3\omega_Z}W_{I_1}\alpha^3-2e^{\omega_R+3\omega_Z}W_{I_2I_2}\alpha^3+2e^{4\omega_Z+\omega_\Theta}W_{I_1I_1}\alpha^2+2e^{4\omega_Z+\omega_\Theta}W_{I_1I_4}\alpha^2+2e^{4\omega_Z+\omega_\Theta}RW_{I_1I_1}\omega'_\Theta\alpha^2\\
&+2e^{4\omega_Z+\omega_\Theta}RW_{I_1I_4}\omega'_\Theta\alpha^2-2e^{\omega_R+5\omega_Z}W_{I_1I_2}\alpha-2e^{6\omega_Z+\omega_\Theta}RW_{I_2I_2}\omega'_Z+W_{I_2}\left(e^{\omega_R+\omega_Z}\alpha^5-2e^{4\omega_Z+\omega_\Theta}R\alpha^2\omega'_Z\right)\Big]r^{10}\\
&-2e^{3\omega_\Theta}R^4\alpha^5\Big\{-2e^{\omega_\Theta}RW_{I_2I_7}\omega'_Z\alpha^8-e^{2\omega_Z+\omega_\Theta}RW_{I_1I_2}\omega'_Z\alpha^6-e^{2\omega_Z+\omega_\Theta}RW_{I_2I_6}\omega'_Z\alpha^6-2e^{2\omega_Z+\omega_\Theta}RW_{I_1I_7}\omega'_Z\alpha^6\\
&-2e^{2\omega_Z+\omega_\Theta}RW_{I_4I_7}\omega'_Z\alpha^6+e^{\omega_R+3\omega_Z}W_{I_2I_2}\alpha^5-e^{4\omega_Z+\omega_\Theta}RW_{I_1I_1}\omega'_Z\alpha^4-e^{4\omega_Z+\omega_\Theta}RW_{I_1I_4}\omega'_Z\alpha^4-e^{4\omega_Z+\omega_\Theta}RW_{I_1I_6}\omega'_Z\alpha^4\\
&-e^{4\omega_Z+\omega_\Theta}RW_{I_4I_6}\omega'_Z\alpha^4+e^{4\omega_Z+\omega_\Theta}W_{I_2}\left(R\omega'_\Theta+1\right)\alpha^4+2e^{\omega_R+5\omega_Z}W_{I_1I_2}\alpha^3+e^{\omega_R+5\omega_Z}W_{I_2I_4}\alpha^3+e^{6\omega_Z+\omega_\Theta}W_{I_1}\alpha^2\\
&+e^{6\omega_Z+\omega_\Theta}W_{I_2I_2}\alpha^2-2e^{6\omega_Z+\omega_\Theta}W_{I_2I_5}\alpha^2+e^{6\omega_Z+\omega_\Theta}RW_{I_1}\omega'_Z\alpha^2+e^{6\omega_Z+\omega_\Theta}RW_{I_2I_2}\omega'_Z\alpha^2-2e^{6\omega_Z+\omega_\Theta}RW_{I_2I_5}\omega'_Z\alpha^2\\
&+e^{6\omega_Z+\omega_\Theta}RW_{I_1}\omega'_\Theta\alpha^2+e^{6\omega_Z+\omega_\Theta}RW_{I_2I_2}\omega'_\Theta\alpha^2-2e^{6\omega_Z+\omega_\Theta}RW_{I_2I_5}\omega'_\Theta\alpha^2+e^{6\omega_Z+\omega_\Theta}W_{I_4}\left(R\omega'_Z+R\omega'_\Theta+1\right)\alpha^2\\
&+e^{\omega_R+7\omega_Z}W_{I_1I_1}\alpha+e^{\omega_R+7\omega_Z}W_{I_1I_4}\alpha+e^{8\omega_Z+\omega_\Theta}W_{I_1I_2}+e^{8\omega_Z+\omega_\Theta}W_{I_2I_4}-2e^{8\omega_Z+\omega_\Theta}W_{I_1I_5}+2e^{8\omega_Z+\omega_\Theta}RW_{I_1I_2}\omega'_Z\\
&+2e^{8\omega_Z+\omega_\Theta}RW_{I_2I_4}\omega'_Z+e^{8\omega_Z+\omega_\Theta}RW_{I_1I_2}\omega'_\Theta+e^{8\omega_Z+\omega_\Theta}RW_{I_2I_4}\omega'_\Theta-2e^{8\omega_Z+\omega_\Theta}RW_{I_1I_5}\omega'_\Theta\Big\}r^8\\
&-e^{4\omega_Z+5\omega_\Theta}R^6\alpha^3\Big[2e^{\omega_\Theta}W_{I_2I_2}\alpha^6-8e^{\omega_\Theta}RW_{I_5I_7}\omega'_Z\alpha^6+2e^{\omega_\Theta}RW_{I_2I_2}\omega'_\Theta\alpha^6-e^{\omega_R+\omega_Z}W_{I_2}\alpha^5+4e^{2\omega_Z+\omega_\Theta}W_{I_1I_2}\alpha^4\\
&+4e^{2\omega_Z+\omega_\Theta}W_{I_2I_4}\alpha^4+2e^{2\omega_Z+\omega_\Theta}RW_{I_1I_2}\omega'_Z\alpha^4+2e^{2\omega_Z+\omega_\Theta}RW_{I_2I_4}\omega'_Z\alpha^4-4e^{2\omega_Z+\omega_\Theta}RW_{I_1I_5}\omega'_Z\alpha^4\\
&-4e^{2\omega_Z+\omega_\Theta}RW_{I_5I_6}\omega'_Z\alpha^4+4e^{2\omega_Z+\omega_\Theta}RW_{I_1I_2}\omega'_\Theta\alpha^4+4e^{2\omega_Z+\omega_\Theta}RW_{I_2I_4}\omega'_\Theta\alpha^4-e^{\omega_R+3\omega_Z}W_{I_1}\alpha^3-2e^{\omega_R+3\omega_Z}W_{I_2I_2}\alpha^3\\
&-e^{\omega_R+3\omega_Z}W_{I_4}\alpha^3+4e^{\omega_R+3\omega_Z}W_{I_2I_5}\alpha^3+2e^{4\omega_Z+\omega_\Theta}W_{I_1I_1}\alpha^2+4e^{4\omega_Z+\omega_\Theta}W_{I_1I_4}\alpha^2+2e^{4\omega_Z+\omega_\Theta}W_{I_4I_4}\alpha^2\\
&+2e^{4\omega_Z+\omega_\Theta}RW_{I_1I_1}\omega'_Z\alpha^2+4e^{4\omega_Z+\omega_\Theta}RW_{I_1I_4}\omega'_Z\alpha^2+2e^{4\omega_Z+\omega_\Theta}RW_{I_4I_4}\omega'_Z\alpha^2+2e^{4\omega_Z+\omega_\Theta}RW_{I_1I_1}\omega'_\Theta\alpha^2\\
&+4e^{4\omega_Z+\omega_\Theta}RW_{I_1I_4}\omega'_\Theta\alpha^2+2e^{4\omega_Z+\omega_\Theta}RW_{I_4I_4}\omega'_\Theta\alpha^2+8e^{4\omega_Z+\omega_\Theta}W_{I_5}\left(R\omega'_Z+R\omega'_\Theta+1\right)\alpha^2-2e^{\omega_R+5\omega_Z}W_{I_1I_2}\alpha\\
&-2e^{\omega_R+5\omega_Z}W_{I_2I_4}\alpha+4e^{\omega_R+5\omega_Z}W_{I_1I_5}\alpha+4e^{6\omega_Z+\omega_\Theta}W_{I_2I_5}+8e^{6\omega_Z+\omega_\Theta}RW_{I_2I_5}\omega'_Z+4e^{6\omega_Z+\omega_\Theta}RW_{I_2I_5}\omega'_\Theta\Big]r^6\\
&+2e^{5\omega_Z+7\omega_\Theta}R^8\alpha^2\Big\{e^{\omega_R}W_{I_2I_2}\alpha^6+2e^{\omega_R+2\omega_Z}W_{I_1I_2}\alpha^4+2e^{\omega_R+2\omega_Z}W_{I_2I_4}\alpha^4-4e^{3\omega_Z+\omega_\Theta}W_{I_2I_5}\alpha^3\\
&-2e^{3\omega_Z+\omega_\Theta}RW_{I_2I_5}\omega'_Z\alpha^3-4e^{3\omega_Z+\omega_\Theta}RW_{I_2I_5}\omega'_\Theta\alpha^3+e^{\omega_R+4\omega_Z}W_{I_1I_1}\alpha^2+2e^{\omega_R+4\omega_Z}W_{I_1I_4}\alpha^2+e^{\omega_R+4\omega_Z}W_{I_4I_4}\alpha^2\\
&+3e^{\omega_R+4\omega_Z}W_{I_5}\alpha^2-4e^{5\omega_Z+\omega_\Theta}W_{I_1I_5}\alpha-4e^{5\omega_Z+\omega_\Theta}RW_{I_1I_5}\omega'_Z\alpha-4e^{5\omega_Z+\omega_\Theta}RW_{I_1I_5}\omega'_\Theta\alpha\\
&-4e^{5\omega_Z+\omega_\Theta}W_{I_4I_5}\left(R\omega'_Z+R\omega'_\Theta+1\right)\alpha+2e^{\omega_R+6\omega_Z}W_{I_2I_5}\Big\}r^4\\
&-8e^{9(\omega_Z+\omega_\Theta)}R^{10}\alpha\left(e^{3\omega_Z+\omega_\Theta}W_{I_5I_5}\left(R\omega'_Z+R\omega'_\Theta+1\right)-e^{\omega_R}\alpha\left(W_{I_2I_5}\alpha^2+e^{2\omega_Z}W_{I_1I_5}+e^{2\omega_Z}W_{I_4I_5}\right)\right)r^2\\
&+8e^{\omega_R+13\omega_Z+11\omega_\Theta}R^{12}W_{I_5I_5}\Big)\,.
\end{aligned}
\tag{B.2}
$$

B.3 Radial equilibrium equation for the compressible orthotropic case

$$
\begin{aligned}
&4e^{6\omega_R+4\omega_\Theta}R^5W_{I_2I_7}r'\omega_Z'\alpha^6+4e^{6\omega_R+2\omega_\Theta}r^2R^3W_{I_3I_7}r'\omega_Z'\alpha^6+8e^{4\omega_R+2\omega_Z+4\omega_\Theta}R^5W_{I_5I_7}r'^3\omega_Z'\alpha^4\\
&+2e^{2(3\omega_R+\omega_Z+\omega_\Theta)}r^2R^3W_{I_3I_6}r'\omega_Z'\alpha^4+4e^{6\omega_R+2\omega_Z+4\omega_\Theta}R^5W_{I_1I_7}r'\omega_Z'\alpha^4+4e^{2(3\omega_R+\omega_Z+\omega_\Theta)}r^2R^3W_{I_2I_7}r'\omega_Z'\alpha^4\\
&+4e^{6\omega_R+2\omega_Z+4\omega_\Theta}R^5W_{I_4I_7}r'\omega_Z'\alpha^4+2e^{6\omega_R+2\omega_Z}r^4W_{I_2I_3}r'\left(R\omega_Z'+R\omega_\Theta'+1\right)\alpha^4-2e^{4\omega_R+4\omega_Z+2\omega_\Theta}rR^3W_{I_1I_3}r'^4\alpha^2\\
&-2e^{4\omega_R+2\omega_Z}rR\left(2e^{2\omega_Z}r^2+e^{2\omega_\Theta}R^2\alpha^2\right)W_{I_2I_3}r'^4\alpha^2+2e^{4\omega_R+2\omega_Z}r^2R\left(e^{2\omega_Z}r^2+2e^{2\omega_\Theta}R^2\alpha^2\right)W_{I_2I_3}r'^3\omega_Z'\alpha^2\\
&+4e^{4(\omega_R+\omega_Z+\omega_\Theta)}R^5W_{I_5I_6}r'^3\omega_Z'\alpha^2+2e^{6\omega_R+4(\omega_Z+\omega_\Theta)}R^5\left(W_{I_1I_6}+W_{I_4I_6}\right)r'\omega_Z'\alpha^2\\
&+2e^{4\omega_R+2\omega_Z}r^2\left(2e^{2\omega_Z}r^2+e^{2\omega_\Theta}R^2\alpha^2\right)W_{I_2I_3}r'^3\left(R\omega_\Theta'+1\right)\alpha^2+4e^{4\omega_R+4\omega_Z+2\omega_\Theta}r^2R^2W_{I_2I_5}r'^3\left(R\omega_Z'+R\omega_\Theta'+1\right)\alpha^2\\
&+2e^{6\omega_R+4\omega_Z+2\omega_\Theta}r^2R^2W_{I_2I_4}r'\left(R\omega_Z'+R\omega_\Theta'+1\right)\alpha^2+2e^{4\omega_R+4\omega_Z+2\omega_\Theta}r^2R^2W_{I_1I_3}r'^3\left(2R\omega_R'+R\omega_Z'+R\omega_\Theta'+1\right)\alpha^2\\
&+e^{6\omega_R+4(\omega_Z+\omega_\Theta)}R^4W_{I_2}\left(r'\left(R\omega_R'+R\omega_Z'-R\omega_\Theta'-1\right)-Rr''\right)\alpha^2+2e^{2(\omega_R+2\omega_Z+\omega_\Theta)}rR^2r'^2\left(2W_{I_3I_5}r'^2+e^{2\omega_R}W_{I_3I_4}\right)\Big(-Rr'^2\\
&+r\left(2R\omega_R'+R\omega_Z'+R\omega_\Theta'+1\right)r'-2rRr''\Big)\alpha^2-4e^{2(\omega_R+3\omega_Z+\omega_\Theta)}rR^3W_{I_2I_5}r'^6-2e^{4\omega_R+6\omega_Z+2\omega_\Theta}rR^3W_{I_1I_2}r'^4\\
&-2e^{4\omega_R+6\omega_Z+2\omega_\Theta}rR^3W_{I_2I_4}r'^4-e^{6\omega_R+4\omega_Z+2\omega_\Theta}rR^3W_{I_2}\left(e^{2\omega_Z}r'^2-e^{2\omega_R}\alpha^2\right)+8e^{2\omega_R+6\omega_Z+4\omega_\Theta}R^5W_{I_1I_5}r'^5\omega_R'\\
&+8e^{2\omega_R+6\omega_Z+4\omega_\Theta}R^5W_{I_4I_5}r'^5\omega_R'+2e^{4\omega_R+6\omega_Z+4\omega_\Theta}R^5W_{I_1I_1}r'^3\omega_R'+4e^{4\omega_R+6\omega_Z+4\omega_\Theta}R^5W_{I_1I_4}r'^3\omega_R'\\
&+2e^{4\omega_R+2\omega_Z}r'\left[e^{2(\omega_R+\omega_Z+\omega_\Theta)}W_{I_1I_1}R^2+e^{2(\omega_Z+\omega_\Theta)}\left(2W_{I_1I_5}r'^2+e^{2\omega_R}W_{I_1I_4}\right)R^2+e^{2\omega_R}r^2\alpha^2W_{I_1I_3}\right]\Big\{e^{2\omega_\Theta}\alpha^2\omega_Z'R^3\\
&+e^{2\omega_Z}r^2\left(R\omega_\Theta'+1\right)\Big\}+2e^{2(\omega_R+2\omega_Z+\omega_\Theta)}R^2r'^3\left[2W_{I_2I_5}r'^2+e^{2\omega_R}W_{I_1I_2}+e^{2\omega_R}W_{I_2I_4}\right]\Big(e^{2\omega_\Theta}\alpha^2\omega_Z'R^3\\
&+2\left(e^{2\omega_Z}r^2+e^{2\omega_\Theta}R^2\alpha^2\right)\omega_R'R+e^{2\omega_Z}r^2\left(R\omega_\Theta'+1\right)\Big)+2e^{6\omega_R+2\omega_Z}W_{I_1I_2}r'\Big[e^{2\omega_Z}\left(e^{2\omega_Z}r^2+2e^{2\omega_\Theta}R^2\alpha^2\right)\left(R\omega_\Theta'+1\right)r^2\\
&+\left(e^{4\omega_\Theta}\alpha^4R^5+2e^{2(\omega_Z+\omega_\Theta)}r^2\alpha^2R^3\right)\omega_Z'\Big]+2e^{4\omega_R+2\omega_Z}\left(e^{2\omega_Z}r^2+e^{2\omega_\Theta}R^2\alpha^2\right)r'\Big[R\left(e^{2(\omega_R+\omega_\Theta)}W_{I_2I_6}\omega_Z'R^2+2r^2W_{I_2I_3}r'^2\omega_R'\right)\alpha^2\\
&+W_{I_2I_2}\Big(-e^{2\omega_Z}rRr'^3+\left(e^{2\omega_\Theta}\alpha^2\omega_Z'R^3+e^{2\omega_Z}r^2\left(R\omega_\Theta'+1\right)\right)r'^2-e^{2\omega_R}rR\alpha^2r'+e^{2\omega_R}r^2\alpha^2\left(R\omega_Z'+R\omega_\Theta'+1\right)\Big)\Big]\\
&+8e^{6\omega_Z+4\omega_\Theta}R^5W_{I_5I_5}r'^6\left(r'\omega_R'-r''\right)+2e^{4\omega_R+2\omega_Z}R\left(e^{2\omega_Z}r^2+e^{2\omega_\Theta}R^2\alpha^2\right)^2W_{I_2I_2}r'^2\left(r'\omega_R'-r''\right)\\
&+2e^{4\omega_R+6\omega_Z+4\omega_\Theta}R^5W_{I_4I_4}r'^2\left(r'\omega_R'-r''\right)-8e^{2\omega_R+6\omega_Z+4\omega_\Theta}R^5W_{I_4I_5}r'^4r''+2e^{4\omega_R+6\omega_Z+4\omega_\Theta}R^4W_{I_5}r'^2[r'(3R\omega_R'-R\omega_Z'\\
&-R\omega_\Theta'-1)-3Rr'']+e^{6\omega_R+6\omega_Z+4\omega_\Theta}R^4W_{I_4}\left(r'\left(R\omega_R'-R\omega_Z'-R\omega_\Theta'-1\right)-Rr''\right)+e^{6\omega_R+6\omega_Z+2\omega_\Theta}r^2R^2W_{I_2}[r'(R\omega_R'-R\omega_Z'\\
&+R\omega_\Theta'+1)-Rr'']+\left\{2e^{4\omega_R+2\omega_Z}r^3W_{I_3I_3}r'^2\alpha^4+e^{6\omega_R+4\omega_Z+2\omega_\Theta}rR^2W_{I_3}\alpha^2\right\}\left(-Rr'^2+r\left(R\omega_R'+R\omega_Z'+R\omega_\Theta'+1\right)r'-rRr''\right)\\
&+e^{6\omega_R+6\omega_Z+2\omega_\Theta}R^3W_{I_1}\left(-e^{2\omega_\Theta}r''R^2+e^{2\omega_\Theta}r'\left(R\omega_R'-R\omega_Z'-R\omega_\Theta'-1\right)R+e^{2\omega_R}r\right)-2e^{4\omega_R+6\omega_Z+2\omega_\Theta}R^3W_{I_1I_1}r'^2\Big(e^{2\omega_\Theta}r''R^2\\
&+e^{2\omega_R}r\Big)-2e^{2\omega_R+4\omega_Z}Rr'^2\left[e^{2(\omega_Z+\omega_\Theta)}\left(2W_{I_1I_5}r'^2+e^{2\omega_R}W_{I_1I_4}\right)R^2+e^{2\omega_R}r^2\alpha^2W_{I_1I_3}\right]\left(2e^{2\omega_\Theta}r''R^2+e^{2\omega_R}r\right)\\
&-2e^{4(\omega_R+\omega_Z)}RW_{I_1I_2}r'^2\left(2e^{4\omega_\Theta}\alpha^2r''R^4+2e^{2(\omega_R+\omega_\Theta)}r\alpha^2R^2+2e^{2(\omega_Z+\omega_\Theta)}r^2r''R^2+e^{2(\omega_R+\omega_Z)}r^3\right)\\
&-2e^{2(\omega_R+\omega_Z)}Rr'^2\left[e^{2(\omega_Z+\omega_\Theta)}\left(2W_{I_2I_5}r'^2+e^{2\omega_R}W_{I_2I_4}\right)R^2+e^{2\omega_R}r^2\alpha^2W_{I_2I_3}\right]\left\{e^{2\omega_R}r\alpha^2+2\left(e^{2\omega_Z}r^2+e^{2\omega_\Theta}R^2\alpha^2\right)r''\right\}=0\,.
\end{aligned}
\tag{B.3}
$$

APPENDIX C

ELASTODYNAMIC TRANSFORMATION CLOAKING

C.1 Riemannian Geometry

To make the chapter self-contained, in this appendix some basic concepts of Riemannian geometry are tersely reviewed. It should be emphasized that only what has been used in the chapter is discussed here.

For a smooth n-dimensional manifold $\mathcal{B}$, the tangent space of $\mathcal{B}$ at a point $X \in \mathcal{B}$ is denoted by $T_X\mathcal{B}$. Assume that $\mathcal{S}$ is another n-dimensional manifold and $\varphi : \mathcal{B} \to \mathcal{S}$ is a diffeomorphism (smooth and invertible map with a smooth inverse) between the two manifolds. A smooth vector field $\mathbf{W}$ on $\mathcal{B}$ assigns a vector $\mathbf{W}_X \in T_X\mathcal{B}$ for every $X \in \mathcal{B}$ such that the mapping $X \mapsto \mathbf{W}_X$ is smooth. If $\mathbf{W}$ is a vector field on $\mathcal{B}$, then the push-forward of $\mathbf{W}$ by φ is a vector field on $\varphi(\mathcal{B})$ defined as $\varphi_*\mathbf{W} = T\varphi \cdot \mathbf{W} \circ \varphi^{-1}$. Similarly, if $\mathbf{w}$ is a vector field on $\varphi(\mathcal{B}) \subset \mathcal{S}$, the pull-back of $\mathbf{w}$ by φ is a vector field on $\mathcal{B}$ defined as $\varphi^*\mathbf{w} = T(\varphi^{-1}) \cdot \mathbf{w} \circ \varphi$. Let us denote the tangent map of φ by $\mathbf{F}$, i.e., $\mathbf{F} = T\varphi$. Let $\{X^A\}$ and $\{x^a\}$ be the local charts for $\mathcal{B}$ and $\mathcal{S}$, respectively. More specifically, a local chart for $\mathcal{B}$ at $X \in \mathcal{B}$ is a homeomorphism from an open subset $\mathcal{U} \subset \mathcal{B}$ ($X \in \mathcal{U}$) to an open subset $\mathcal{V} \subset \mathbb{R}^n$. $\{X^A\}$ are components of this map. The derivative map $\mathbf{F}$ is a two-point tensor with the following representation in the local charts: $\mathbf{F} = F^a{}_A \frac{\partial}{\partial X^A} \otimes dx^a$, $F^a{}_A = \frac{\partial \varphi^a}{\partial X^A}$, where $\{\frac{\partial}{\partial X^A}\}$ and $\{dx^a\}$ are bases for $T_X\mathcal{B}$ and $T^*_{\varphi(X)}\varphi(\mathcal{B})$, respectively. Recall that $T^*_{\varphi(X)}\varphi(\mathcal{B})$ denotes the cotangent space (or the dual space) of $T_{\varphi(X)}\varphi(\mathcal{B})$. The push-forward and pull-back of vectors have the following coordinate representations: $(\varphi_*\mathbf{W})^a = F^a{}_A W^A$, and $(\varphi^*\mathbf{w})^A = (F^{-1})^A{}_a w^a$.

A type $\binom{0}{2}$-tensor at $X \in \mathcal{B}$ is a bilinear map $\mathbf{T} : T_X\mathcal{B} \times T_X\mathcal{B} \to \mathbb{R}$, where in a local coordinate chart $\{X^A\}$ for $\mathcal{B}$ reads $\mathbf{T}(\mathbf{U}, \mathbf{V}) = T_{AB}U^AV^B$, $\forall\, \mathbf{U}, \mathbf{V} \in T_X\mathcal{B}$. A Riemannian manifold $(\mathcal{B}, \mathbf{G})$ is a smooth manifold $\mathcal{B}$ endowed with an inner product $\mathbf{G}_X$ (a symmetric $\binom{0}{2}$-tensor field) on the tangent space $T_X\mathcal{B}$ that smoothly varies in the sense that if $\mathbf{U}$ and $\mathbf{V}$ are smooth vector fields on

$\mathcal{B}$, then $X \mapsto \mathbf{G}_X(\mathbf{U}_X, \mathbf{V}_X) =: \langle\!\langle \mathbf{U}_X, \mathbf{V}_X \rangle\!\rangle_{\mathbf{G}_X}$, is a smooth function. Let $(\mathcal{B}, \mathbf{G})$ and $(\mathcal{S}, \mathbf{g})$ be Riemannian manifolds and let $\varphi : \mathcal{B} \to \mathcal{S}$ be a diffeomorphism (smooth map with smooth inverse). The push-forward of the metric $\mathbf{G}$ is a metric on $\varphi(\mathcal{B}) \subset \mathcal{S}$, which is denoted by $\varphi_*\mathbf{G}$ defined as

$$(\varphi_*\mathbf{G})_{\varphi(X)} \left(\mathbf{u}_{\varphi(X)}, \mathbf{v}_{\varphi(X)}\right) := \mathbf{G}_X \left((\varphi^*\mathbf{u})_X, (\varphi^*\mathbf{v})_X\right). \tag{C.1}$$

In components, $(\varphi_*\mathbf{G})_{ab} = (F^{-1})^A{}_a (F^{-1})^B{}_b G_{AB}$. Similarly, the pull-back of the metric $\mathbf{g}$ is a metric in $\mathcal{B}$, which is denoted by $\varphi^*\mathbf{g}$ defined as

$$(\varphi^*\mathbf{g})_X(\mathbf{U}_X, \mathbf{V}_X) := \mathbf{g}_{\varphi(X)}((\varphi_*\mathbf{U})_{\varphi(X)}, (\varphi_*\mathbf{V})_{\varphi(X)}). \tag{C.2}$$

In components, $(\varphi^*\mathbf{g})_{AB} = F^a{}_A F^b{}_B g_{ab}$. The diffeomorphism φ is an isometry between two Riemannian manifolds $(\mathcal{B}, \mathbf{G})$ and $(\mathcal{S}, \mathbf{g})$ if $\mathbf{g} = \varphi_*\mathbf{G}$, or equivalently, $\mathbf{G} = \varphi^*\mathbf{g}$. An isometry, by definition, preserves distances.

Affine connections, and their torsion and curvature tensors. A linear (affine) connection on a manifold $\mathcal{B}$ is an operation $\nabla : \mathcal{X}(\mathcal{B}) \times \mathcal{X}(\mathcal{B}) \to \mathcal{X}(\mathcal{B})$, where $\mathcal{X}(\mathcal{B})$ is the set of vector fields on $\mathcal{B}$, such that $\forall\ \mathbf{X}, \mathbf{Y}, \mathbf{X}_1, \mathbf{X}_2, \mathbf{Y}_1, \mathbf{Y}_2 \in \mathcal{X}(\mathcal{B}), \forall\ f, f_1, f_2 \in C^\infty(\mathcal{B}), \forall\ a_1, a_2 \in \mathbb{R}$: i) $\nabla_{f_1\mathbf{X}_1 + f_2\mathbf{X}_2}\mathbf{Y} = f_1\nabla_{\mathbf{X}_1}\mathbf{Y} + f_2\nabla_{\mathbf{X}_2}\mathbf{Y}$, ii) $\nabla_{\mathbf{X}}(a_1\mathbf{Y}_1 + a_2\mathbf{Y}_2) = a_1\nabla_{\mathbf{X}}(\mathbf{Y}_1) + a_2\nabla_{\mathbf{X}}(\mathbf{Y}_2)$, iii) $\nabla_{\mathbf{X}}(f\mathbf{Y}) = f\nabla_{\mathbf{X}}\mathbf{Y} + (\mathbf{X}f)\mathbf{Y}$. $\nabla_{\mathbf{X}}\mathbf{Y}$ is called the covariant derivative of $\mathbf{Y}$ along $\mathbf{X}$. In a local coordinate chart $\{X^A\}$, $\nabla_{\partial_A}\partial_B = \Gamma^C{}_{AB}\partial_C$, where $\Gamma^C{}_{AB}$ are Christoffel symbols of the connection, and $\partial_A = \frac{\partial}{\partial x^A}$ are natural bases for the tangent space corresponding to a coordinate chart $\{x^A\}$. A linear connection is said to be compatible with a metric $\mathbf{G}$ on the manifold if

$$\nabla_{\mathbf{X}}\langle\!\langle \mathbf{Y}, \mathbf{Z} \rangle\!\rangle_{\mathbf{G}} = \langle\!\langle \nabla_{\mathbf{X}}\mathbf{Y}, \mathbf{Z} \rangle\!\rangle_{\mathbf{G}} + \langle\!\langle \mathbf{Y}, \nabla_{\mathbf{X}}\mathbf{Z} \rangle\!\rangle_{\mathbf{G}}, \tag{C.3}$$

where $\langle\!\langle .,. \rangle\!\rangle_{\mathbf{G}}$ is the inner product induced by the metric $\mathbf{G}$. It can be shown that ∇ is compatible with $\mathbf{G}$ if and only if $\nabla \mathbf{G} = \mathbf{0}$, or in components

$$G_{AB|C} = \frac{\partial G_{AB}}{\partial X^C} - \Gamma^S{}_{CA} G_{SB} - \Gamma^S{}_{CB} G_{AS} = 0. \tag{C.4}$$

Suppose $\mathbf{V}, \mathbf{W} \in \mathcal{X}(\mathcal{B})$ are vector fields and $\alpha : I \to \mathcal{B}$ is a smooth curve. The restriction of the vector fields to α, i.e., $\mathbf{V} \circ \alpha$ and $\mathbf{W} \circ \alpha$ are called vector fields along the curve α. The set of all vector fields along α is denoted by $\mathcal{X}(\alpha)$. Covariant derivative along the curve α is a map $D_t : \mathcal{X}(\alpha) \to \mathcal{X}(\alpha)$ with the following properties: $D_t(\mathbf{V} + \mathbf{W}) = D_t\mathbf{V} + D_t\mathbf{W}$, and $D_t(f\mathbf{W}) = \frac{df}{dt}\mathbf{W} + f D_t\mathbf{W}$. If $\mathbf{W} \in \mathcal{X}(\alpha)$ is the restriction of $\widetilde{\mathbf{W}} \in \mathcal{X}(\mathcal{B})$ to α, then, $D_t\mathbf{W} = \nabla_{\alpha'(t)}\widetilde{\mathbf{W}}$. If the connection ∇ is $\mathbf{G}$-compatible, then

$$\frac{d}{dt}\langle\!\langle \mathbf{X}, \mathbf{Y}(X,t) \rangle\!\rangle_{\mathbf{G}} = \langle\!\langle D_t\mathbf{X}, \mathbf{Y} \rangle\!\rangle_{\mathbf{G}} + \langle\!\langle \mathbf{X}, D_t\mathbf{Y} \rangle\!\rangle_{\mathbf{G}}. \tag{C.5}$$

The covariant derivative of a two-point tensor $\mathbf{T}$ is given by

$$\begin{aligned} T^{AB\cdots F}{}_{G\cdots Q}{}^{ab\cdots f}{}_{g\cdots q|K} =& \frac{\partial}{\partial X^k} T^{AB\cdots F}{}_{G\cdots Q}{}^{ab\cdots f}{}_{g\cdots q} \\ &+ T^{RB\cdots F}{}_{G\cdots Q}{}^{ab\cdots f}{}_{g\cdots q} \Gamma^A{}_{RK} + (\text{all upper referential indices}) \\ &- T^{AB\cdots F}{}_{R\cdots Q}{}^{ab\cdots f}{}_{g\cdots q} \Gamma^R{}_{GK} - (\text{all lower referential indices}) \\ &+ T^{RB\cdots F}{}_{G\cdots Q}{}^{lb\cdots f}{}_{g\cdots q} \gamma^a{}_{lr} F^r{}_K + (\text{all upper spatial indices}) \\ &- T^{AB\cdots F}{}_{G\cdots Q}{}^{ab\cdots f}{}_{l\cdots q} \gamma^l{}_{gr} F^r{}_K - (\text{all lower spatial indices})\,. \end{aligned} \tag{C.6}$$

The torsion of a connection is defined as $\boldsymbol{T}(\mathbf{X}, \mathbf{Y}) = \nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - [\mathbf{X}, \mathbf{Y}]$, where $[\mathbf{X}, \mathbf{Y}](F) = \mathbf{X}(\mathbf{Y}(F)) - \mathbf{Y}(\mathbf{X}(F))$, $\forall\, F \in C^\infty(\mathcal{S})$, is the commutator of $\mathbf{X}$ and $\mathbf{Y}$. In components, in a local chart $\{X^A\}$, $T^A{}_{BC} = \Gamma^A{}_{BC} - \Gamma^A{}_{CB}$, and $[\mathbf{X}, \mathbf{Y}]^a = \frac{\partial Y^a}{\partial x^b} X^b - \frac{\partial X^a}{\partial x^b} Y^b$. ∇ is symmetric if it is torsion-free, i.e., $\nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} = [\mathbf{X}, \mathbf{Y}]$. On any Riemannian manifold $(\mathcal{B}, \mathbf{G})$ there is a unique linear connection $\nabla^{\mathbf{G}}$ that is compatible with $\mathbf{G}$ and is torsion-free. This is the Levi-Civita connection. If the Levi-Civita connection $\nabla^{\mathbf{G}}$ is used, the covariant time derivative is denoted by $D_t^{\mathbf{G}}$. In a manifold with a connection the curvature is a map $\boldsymbol{\mathcal{R}} : \mathcal{X}(\mathcal{B}) \times \mathcal{X}(\mathcal{B}) \times \mathcal{X}(\mathcal{B}) \to \mathcal{X}(\mathcal{B})$ defined by $\boldsymbol{\mathcal{R}}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) = \nabla_{\mathbf{X}}\nabla_{\mathbf{Y}}\mathbf{Z} - \nabla_{\mathbf{Y}}\nabla_{\mathbf{X}}\mathbf{Z} - \nabla_{[\mathbf{X},\mathbf{Y}]}\mathbf{Z}$, or in components $\mathcal{R}^A{}_{BCD} = \frac{\partial \Gamma^A{}_{CD}}{\partial X^B} - \frac{\partial \Gamma^A{}_{BD}}{\partial X^C} + \Gamma^A{}_{BM}\Gamma^M{}_{CD} - \Gamma^A{}_{CM}\Gamma^M{}_{BD}$. The Riemannian curvature is the curvature tensor of the Levi-Civita connection $\nabla^{\mathbf{G}}$ and is denoted by $\boldsymbol{\mathcal{R}}_{\mathbf{G}}$. The Ricci identity for a vector field $\mathbf{U}$ with components W^A reads $U^A{}_{|BC} - U^A{}_{|CB} = \mathcal{R}^A{}_{BCD}U^D$. Ricci identity for a 1-form $\boldsymbol{\alpha}$ with components α_A reads $\alpha_{A|BC} - \alpha_{A|CB} = \mathcal{R}^D{}_{BCA}\alpha_D$. The Ricci curvature $\mathbf{Ric}$ is defined as $\mathrm{Ric}_{CD} = \mathcal{R}^A{}_{ACD}$, and is a symmetric tensor. The Ricci curvature of the Levi-Civita connection $\nabla^{\mathbf{G}}$ is denoted by $\mathbf{Ric}_{\mathbf{G}}$.

Vector bundles. Suppose $\mathcal{E}$ and $\mathcal{B}$ are sets and consider a map $\pi : \mathcal{E} \to \mathcal{B}$. The fiber over $X \in \mathcal{B}$ is the set $\mathcal{E}_X := \pi^{-1}(X) \subset \mathcal{E}$. For an onto map π fibers are non-empty and $\mathcal{E} = \sqcup_{X\in\mathcal{B}}\mathcal{E}_X$, where $\sqcup$ denoted disjoint union of sets. Now suppose $\mathcal{E}$ and $\mathcal{B}$ are manifolds and assume that for any $X \in \mathcal{B}$, there exists a neighborhood $\mathcal{U} \subset \mathcal{B}$ of X, a manifolds $\mathcal{F}$, and a diffeomorphism $\psi : \pi^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathcal{F}$ such that $\pi = \mathrm{pr}_1 \circ \psi$, where $\mathrm{pr}_1 : \mathcal{U} \times \mathcal{F} \to \mathcal{U}$ is projection onto the first factor. $(\mathcal{E}, \pi, \mathcal{B})$ is called a fiber bundle and $\mathcal{E}$, π, and $\mathcal{B}$ are called the total space, the projection, and the base space, respectively. If for any $X \in \mathcal{B}$, $\pi^{-1}(X)$ is a vector space, $(\mathcal{E}, \pi, \mathcal{B})$ is called a vector bundle. The set of sections of this bundle $\Gamma(\mathcal{E})$ is the set of all smooth maps $\sigma : \mathcal{B} \to \mathcal{E}$ such that $\sigma(X) \in \mathcal{E}_X,\ \forall\ X \in \mathcal{B}$. An important example of a vector bundle is the tangent bundle of a manifold for which $\mathcal{E} = T\mathcal{B}$.

Induced bundle and connection. Consider a map between Riemannian manifolds $\varphi : \mathcal{B} \to \mathcal{S}$. The tangent bundles of $\mathcal{B}$ and $\mathcal{S}$ are denoted by $T\mathcal{B} = \sqcup_{X\in\mathcal{B}}T_X\mathcal{B}$ and $T\mathcal{S} = \sqcup_{x\in\mathcal{S}}T_x\mathcal{S}$, respectively. We define an induced vector bundle $\varphi^{-1}T\mathcal{S}$, which is a vector bundle over $\mathcal{B}$ whose fiber over $X \in \mathcal{B}$ is $T_{\varphi(X)}\mathcal{S}$ [257]. The connection $\nabla^{\mathbf{g}}$ induces a unique connection ∇^{φ} on $\varphi^{-1}T\mathcal{S}$ defined as

$$\nabla^{\varphi}_{\mathbf{W}}\mathbf{w} \circ \varphi = \nabla^{\mathbf{g}}_{\varphi_*\mathbf{W}}\mathbf{w}, \quad \mathbf{W} \in T_X\mathcal{B},\ \mathbf{w} \in \Gamma(T\mathcal{S})\,. \tag{C.7}$$

∇^{φ} is called the induced connection. It can be shown that its connection coefficients with respect to the coordinate charts $\{X^A\}$ and $\{x^a\}$ of $\mathcal{B}$ and $\mathcal{S}$, respectively, are $\frac{\partial\varphi^b}{\partial X^A}\gamma^a{}_{bc}$. In particular, the variation field $\delta\varphi$ defined in §3 is a section of $\Gamma(\varphi^{-1}T\mathcal{S})$, i.e., $\delta\varphi$ defines a vector fields in $\mathcal{S}$ along the map φ. For a two-point tensor, e.g., deformation gradient, covariant derivative involves both $\nabla^{\mathbf{g}}$ and $\nabla^{\mathbf{G}}$: $F^a{}_{A|B} = \frac{\partial F^a{}_A}{\partial X^B} + (F^b{}_B \gamma^a{}_{bc}) F^c{}_A - \Gamma^C{}_{AB} F^a{}_C = \frac{\partial F^a{}_A}{\partial X^B} + \gamma^a{}_{bc} F^b{}_B F^c{}_A - \Gamma^C{}_{AB} F^a{}_C$. We denote the covariant derivative of the deformation gradient by $\nabla \mathbf{F} = F^a{}_{A|B} dX^B \otimes dX^A \otimes \frac{\partial}{\partial x^a}$. It is straightforward to show that [257]

$$\nabla^{\varphi} \mathbf{F}(\mathbf{X}, \mathbf{Y}) = \nabla^{\varphi}_{\mathbf{X}} \varphi_* \mathbf{Y} - \varphi_* \nabla^{\mathbf{G}}_{\mathbf{W}} \mathbf{Y}, \quad \nabla^{\varphi}_{\mathbf{X}} \varphi_* \mathbf{Y} - \nabla^{\varphi}_{\mathbf{Y}} \varphi_* \mathbf{X} = \varphi_* [\mathbf{X}, \mathbf{Y}]. \tag{C.8}$$

The metrics $\mathbf{G}$ and $\mathbf{g}$ induce an inner product $\langle , \rangle_X$ in $T_{\varphi(X)}\mathcal{S} \otimes T^*_X \mathcal{B}$. This is defined first for the basis $\left\{ \frac{\partial}{\partial x^a} \otimes dX^A, \ 1 \leq a \leq n, \ 1 \leq A \leq n \right\}$ as $\left\langle \frac{\partial}{\partial x^a} \otimes dX^A, \frac{\partial}{\partial x^b} \otimes dX^B \right\rangle_X = g_{ab} G^{AB}$, and then one extends it linearly to arbitrary elements in $T_{\varphi(X)}\mathcal{S} \otimes T^*_X \mathcal{B}$. $\varphi^{-1} T\mathcal{S} \otimes T^*\mathcal{B}$ is the vector bundle whose fiber at $X \in \mathcal{B}$ is $T_{\varphi(X)}\mathcal{S} \otimes T^*_X \mathcal{B}$. The two-point tensor $\mathbf{F} = T\varphi : \mathcal{B} \to \varphi^{-1} T\mathcal{S} \otimes T^*\mathcal{B}$, i.e., $\mathbf{F} \in \Gamma(\varphi^{-1} T\mathcal{S} \otimes T^*\mathcal{B})$. One can define a fiber metric $\langle\!\langle , \rangle\!\rangle$ on $\varphi^{-1} T\mathcal{S} \otimes T^*\mathcal{B}$ using the inner product $\langle , \rangle_X$ in $T_{\varphi(X)}\mathcal{S} \otimes T^*_X \mathcal{B}$ as follows. For $\sigma, \tau \in \Gamma(\varphi^{-1} T\mathcal{S} \otimes T^*\mathcal{B})$, define $\langle\!\langle \sigma, \tau \rangle\!\rangle(X) = \langle \sigma(X), \tau(X) \rangle_X, \ X \in \mathcal{B}$. One can define a connection ∇ in $\varphi^{-1} T\mathcal{S} \otimes T^*\mathcal{B}$ using the Levi-Civita connections $\nabla^{\mathbf{G}}$ and $\nabla^{\mathbf{g}}$: consider a section $\mathbf{W} \otimes \boldsymbol{\alpha} \in \Gamma(\varphi^{-1} T\mathcal{S} \otimes T^*\mathcal{B})$ and let $\nabla(\mathbf{W} \otimes \boldsymbol{\alpha}) = \nabla^{\varphi} \mathbf{W} \otimes \boldsymbol{\alpha} + \mathbf{W} \otimes \nabla^{\mathbf{G}} \boldsymbol{\alpha}$. This connection is compatible with the fiber metric $\langle\!\langle , \rangle\!\rangle$ in $\varphi^{-1} T\mathcal{S} \otimes T^*\mathcal{B}$.

The Piola transform. The Piola transform of a vector $\mathbf{w} \in T_{\varphi(X)}\mathcal{S}$ is a vector $\mathbf{W} \in T_X \mathcal{B}$ given by $\mathbf{W} = J \varphi^* \mathbf{w} = J \mathbf{F}^{-1} \mathbf{w}$. In coordinates, $W^A = J (F^{-1})^A{}_b w^b$, where $J = \sqrt{\frac{\det \mathbf{g}}{\det \mathbf{G}}} \det \mathbf{F}$ is the Jacobian of φ with $\mathbf{G}$ and $\mathbf{g}$ the Riemannian metrics of $\mathcal{B}$ and $\mathcal{S}$, respectively. It can be shown that $\operatorname{Div} \mathbf{W} = J (\operatorname{div} \mathbf{w}) \circ \varphi$. In coordinates, $W^A{}_{|A} = J w^a{}_{|a}$. This is also known as the Piola identity. Another way of writing the Piola identity is in terms of the unit normal vectors of a surface in $\mathcal{B}$ and its corresponding surface in $\mathcal{S}$ and the area elements. It is written as $\hat{\mathbf{n}} da = J \mathbf{F}^{-\star} \hat{\mathbf{N}} dA$, or in components, $n_a da = J (F^{-1})^A{}_a N_A dA$. In the literature of continuum mechanics, this is called Nanson's formula.

Lie derivative. Let $\mathbf{w} : \mathcal{U} \to T\mathcal{S}$ be a vector field, where $\mathcal{U} \subset \mathcal{S}$ is open. A curve $\alpha : I \to \mathcal{S}$, where I is an open interval, is an integral curve of $\mathbf{w}$ if $\frac{d\alpha(t)}{dt} = \mathbf{w}(\alpha(t)), \ \forall \ t \in I$. For a time-dependent vector field $\mathbf{w} : \mathcal{S} \times I \to T\mathcal{S}$, where I is some open interval, the collection of maps $\psi_{\tau,t}$ is the flow of $\mathbf{w}$ if for each t and x, $\tau \mapsto \psi_{\tau,t}(x)$ is an integral curve of $\mathbf{w}_t$, i.e., $\frac{d}{d\tau}\psi_{\tau,t}(x) = \mathbf{w}(\psi_{\tau,t}(x), \tau)$, and $\psi_{t,t}(x) = x$. Let $\mathbf{t}$ be a time-dependent tensor field on $\mathcal{S}$, i.e., $\mathbf{t}_t(x) = \mathbf{t}(x,t)$ is a tensor. The Lie derivative of $\mathbf{t}$ with respect to $\mathbf{w}$ is defined as $\mathbf{L}_{\mathbf{w}}\mathbf{t} = \frac{d}{d\tau}\psi^*_{\tau,t}\mathbf{t}_\tau\Big|_{\tau=t}$. Note that $\psi_{\tau,t}$ maps $\mathbf{t}_t$ to $\mathbf{t}_\tau$. Hence, to calculate the Lie derivative one drags $\mathbf{t}$ along the flow of $\mathbf{w}$ from τ to t and then differentiates the Lie dragged tensor with respect to τ. The autonomous Lie derivative of $\mathbf{t}$ with respect to $\mathbf{w}$ is defined as $\mathfrak{L}_{\mathbf{w}}\mathbf{t} = \frac{d}{d\tau}\psi^*_{\tau,t}\mathbf{t}_t\Big|_{\tau=t}$. Thus, $\mathbf{L}_{\mathbf{w}}\mathbf{t} = \partial\mathbf{t}/\partial t + \mathfrak{L}_{\mathbf{w}}\mathbf{t}$. For a scalar f, $\mathbf{L}_{\mathbf{w}}f = \partial f/\partial t + \mathbf{w}[f]$. In a coordinate chart $\{x^a\}$ this reads, $\mathbf{L}_{\mathbf{w}}f = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^a}w^a$. For a vector $\mathbf{u}$, one can show that $\mathbf{L}_{\mathbf{w}}\mathbf{u} = \frac{\partial \mathbf{w}}{\partial t} + [\mathbf{w}, \mathbf{u}]$. If ∇ is a torsion-free connection, then $[\mathbf{w}, \mathbf{u}] = \nabla_{\mathbf{w}}\mathbf{u} - \nabla_{\mathbf{u}}\mathbf{w}$. Thus, $\mathbf{L}_{\mathbf{w}}\mathbf{u} = \frac{\partial \mathbf{w}}{\partial t} + \nabla_{\mathbf{w}}\mathbf{u} - \nabla_{\mathbf{u}}\mathbf{w}$.

When linearizing nonlinear elasticity one starts with a one-parameter family of motions $\varphi_{t,\epsilon} : \mathcal{B} \to \mathcal{S}$. By definition of the variation field $\mathbf{U}_t = \delta\varphi_t$, $\varphi_{t,\epsilon}$ is the flow of the variation field. Given the tensor field $\mathbf{t}$ in $\mathcal{S}$, $\bar{\mathbf{T}}_\epsilon = \varphi^*_{t,\epsilon}\mathbf{t} \circ \varphi_{t,\epsilon}$ is a vector field on $\mathcal{B}$. Its linearization is defined as

$$\delta\bar{\mathbf{T}} = \frac{d}{d\epsilon}\bar{\mathbf{T}}_\epsilon\Big|_{\epsilon=0} = \left(\frac{d}{d\epsilon}\varphi^*_{t,\epsilon}\mathbf{t} \circ \varphi_{t,\epsilon}\right)\Big|_{\epsilon=0} = \left(\varphi^*_{t,\epsilon}\mathbf{L}_{\mathbf{U}_{t,\epsilon}}\mathbf{t} \circ \varphi_{t,\epsilon}\right)\Big|_{\epsilon=0} = \mathring{\varphi}^*_t\left(\mathbf{L}_{\mathbf{U}_t}\mathbf{t} \circ \mathring{\varphi}_t\right). \tag{C.9}$$

Thus, $\delta\mathbf{t} = \mathbf{L}_{\mathbf{u}_t}\mathbf{t}$, where $\mathbf{u}_t = \mathbf{U}_t \circ \mathring{\varphi}_t^{-1}$.

APPENDIX D

TRANSFORMATION CLOAKING IN ELASTIC PLATES

D.1 Variations of some geometric objects

In this appendix, we discuss the derivation of the variations of the right Cauchy-Green deformation tensor $\mathbf{C}$, the unit normal vector field (of the deformed shell) $\boldsymbol{\mathcal{N}}$, and $\boldsymbol{\Theta}$ used in obtaining the Euler-Lagrange equations in §8.3 (see also [267, 259, 268]).

Lie derivative. Let $\mathbf{w} : \mathcal{U} \to T\mathcal{S}$ be a (C^1) vector field, where $\mathcal{U} \subset \mathcal{S}$ is an open neighborhood. A curve $\alpha : I \to \mathcal{S}$, where I is an open interval, is an *integral curve* of $\mathbf{w}$ provided that $\frac{d\alpha(t)}{dt} = \mathbf{w}(\alpha(t)),\ \forall\, t \in I$. Consider a time-dependent vector field $\mathbf{w} : \mathcal{S} \times I \to T\mathcal{S}$, where I is some open interval. The collection of maps $\psi_{\tau,t}$ is the flow of $\mathbf{w}$ if for each t and x, $\tau \mapsto \psi_{\tau,t}(x)$ is an integral curve of $\mathbf{w}_t$, i.e., $\frac{d}{d\tau}\psi_{\tau,t}(x) = \mathbf{w}(\psi_{\tau,t}(x), \tau)$, and $\psi_{t,t}(x) = x$. Assume that $\mathbf{t}$ is a time-dependent tensor field on $\mathcal{S}$, i.e., $\mathbf{t}_t(x) = \mathbf{t}(x,t)$ is a tensor. The Lie derivative of $\mathbf{t}$ with respect to $\mathbf{w}$ is defined as $\mathbf{L}_{\mathbf{w}}\mathbf{t} = \frac{d}{d\tau}\psi^*_{\tau,t}\mathbf{t}_\tau\Big|_{\tau=t}$. Note that $\psi_{\tau,t}$ maps $\mathbf{t}_t$ to $\mathbf{t}_\tau$. Therefore, to calculate the Lie derivative one drags $\mathbf{t}$ along the flow of $\mathbf{w}$ from τ to t and then differentiates the Lie dragged tensor with respect to τ. The *autonomous* Lie derivative of $\mathbf{t}$ with respect to $\mathbf{w}$ is defined as $\mathfrak{L}_{\mathbf{w}}\mathbf{t} = \frac{d}{d\tau}\psi^*_{\tau,t}\mathbf{t}_t\Big|_{\tau=t}$. Hence, $\mathbf{L}_{\mathbf{w}}\mathbf{t} = \partial\mathbf{t}/\partial t + \mathfrak{L}_{\mathbf{w}}\mathbf{t}$. The Lie derivative for a scalar f is given by $\mathbf{L}_{\mathbf{w}}f = \partial f/\partial t + \mathbf{w}[f]$. In a coordinate chart $\{x^a\}$, this is written as, $\mathbf{L}_{\mathbf{w}}f = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x^a}w^a$. For a vector $\mathbf{u}$, it can be shown that $\mathbf{L}_{\mathbf{w}}\mathbf{u} = \frac{\partial \mathbf{w}}{\partial t} + [\mathbf{w}, \mathbf{u}]$. If ∇ is a torsion-free connection, then $[\mathbf{w}, \mathbf{u}] = \nabla_{\mathbf{w}}\mathbf{u} - \nabla_{\mathbf{u}}\mathbf{w}$, and thus, $\mathbf{L}_{\mathbf{w}}\mathbf{u} = \frac{\partial \mathbf{w}}{\partial t} + \nabla_{\mathbf{w}}\mathbf{u} - \nabla_{\mathbf{u}}\mathbf{w}$.

The rate of deformation tensor for shells is defined as [43]

$$2\mathbf{D}^\flat = \varphi_t^*\left((\nabla^{\mathbf{g}}\mathbf{v}^\top)^\flat + [(\nabla^{\mathbf{g}}\mathbf{v}^\top)^\flat]^\mathsf{T} - 2v^\perp\boldsymbol{\theta}\right) , \tag{D.1}$$

where the spatial velocity is decomposed into the normal and tangential components as $\mathbf{v} = \mathbf{v}^\top + v^\perp\mathbf{n}$.

In components

$$2D_{AB} = (v^{\top}_{a|b} + v^{\top}_{b|a}) F^a{}_A F^b{}_B - 2v^{\perp}\Theta_{AB}\,. \tag{D.2}$$

Note that

$$\mathbf{L}_{\mathbf{v}^{\top}}\mathbf{g} = (\nabla^{\mathbf{g}}\mathbf{v}^{\top})^{\flat} + [(\nabla^{\mathbf{g}}\mathbf{v}^{\top})^{\flat}]^{\mathsf{T}}\,. \tag{D.3}$$

Therefore

$$2\mathbf{D}^{\flat} = \varphi_t^*(\mathbf{L}_{\mathbf{v}^{\top}}\mathbf{g}) - 2v^{\perp}\boldsymbol{\Theta}\,. \tag{D.4}$$

Knowing that $\varphi_t^*(\mathbf{L}_{\mathbf{v}}\mathbf{g}) = 2\mathbf{D}^{\flat}$ (see, e.g., [43, 55]), one obtains

$$\varphi_t^*(\mathbf{L}_{\mathbf{v}}\mathbf{g}) = \varphi_t^*(\mathbf{L}_{\mathbf{v}^{\top}}\mathbf{g}) - 2v^{\perp}\boldsymbol{\Theta}\,. \tag{D.5}$$

Thus, $\mathbf{L}_{\delta\varphi}\mathbf{g} = \mathbf{L}_{\delta\varphi^{\top}}\mathbf{g} - 2\,\delta\varphi^{\perp}\,\boldsymbol{\theta}$, and hence, $\delta\mathbf{C}^{\flat} = \varphi_t^*(\mathbf{L}_{\delta\varphi}\mathbf{g}) = \varphi_t^*\,\mathbf{L}_{\delta\varphi^{\top}}\mathbf{g} - 2\,\delta\varphi^{\perp}\,\boldsymbol{\Theta}$. Note also

$$\mathbf{L}_{\delta\varphi^{\top}}\mathbf{g} = \left[g_{cb}(\delta\varphi^{\top})^c{}_{|a} + g_{ac}(\delta\varphi^{\top})^c{}_{|b}\right] dx^a \otimes dx^b\,. \tag{D.6}$$

Hence, in components, one obtains

$$\delta C_{AB} = F^a{}_A\,\delta\varphi^{\top}_{a|B} + F^b{}_B\,\delta\varphi^{\top}_{b|A} - 2\,\delta\varphi^{\perp}\,\Theta_{AB}\,. \tag{D.7}$$

Or

$$\delta C_{AB} = F^a{}_A\,g_{ac}(\delta\varphi^{\top})^c{}_{|B} + F^b{}_B\,g_{bc}(\delta\varphi^{\top})^c{}_{|A} - 2\,\delta\varphi^{\perp}\,F^a{}_A F^b{}_B \theta_{ab}\,. \tag{D.8}$$

Therefore, (8.51) is implied.

The covariant derivative of $\mathbf{v}$ is computed as

$$\bar{\nabla}^{\bar{\mathbf{g}}}\mathbf{v} = \bar{\nabla}^{\bar{\mathbf{g}}}(\mathbf{v}^{\top} + v^{\perp}\mathbf{n}) = \bar{\nabla}^{\bar{\mathbf{g}}}\mathbf{v}^{\top} + v^{\perp}\bar{\nabla}^{\bar{\mathbf{g}}}\mathbf{n} + \mathbf{d}v^{\perp}\otimes\mathbf{n}\,. \tag{D.9}$$

Using the relations (8.9) and (8.11) in the ambient space, one obtains

$$\bar{\nabla}^{\bar{\mathbf{g}}}\mathbf{v} = (\nabla^{\mathbf{g}}\mathbf{v}^\top - v^\perp\boldsymbol{\theta}) + (\mathbf{d}v^\perp + \boldsymbol{\theta}\cdot\mathbf{v}^\top)\otimes\mathbf{n}\,. \tag{D.10}$$

In components, the (surface) covariant derivative of the spatial velocity reads

$$\bar{\nabla}^{\bar{\mathbf{g}}}\mathbf{v} = \left[(v^\top)^a{}_{|b} - \theta^a{}_b v^\perp\right]\partial_a\otimes dx^b + \left[v^\perp{}_{,b} + \theta_{bc}(v^\top)^c\right]\mathbf{n}\otimes dx^b\,. \tag{D.11}$$

Similarly, one can write

$$\bar{\nabla}^{\bar{\mathbf{g}}}\delta\varphi = \left[(\delta\varphi^\top)^a{}_{|b} - \theta^a{}_b\delta\varphi^\perp\right]\partial_a\otimes dx^b + \left[\delta\varphi^\perp{}_{,b} + \theta_{bc}(\delta\varphi^\top)^c\right]\mathbf{n}\otimes dx^b\,. \tag{D.12}$$

Note that for an arbitrary vector field $\mathbf{W} = \mathbf{W}^\top + W^\perp\boldsymbol{\mathcal{N}}$ defined on a surface embedded in $\mathbb{R}^3$, the tangential and normal components of the covariant derivative with respect to the surface coordinates would be similarly given by

$$(\bar{\nabla}^{\bar{\mathbf{g}}}\mathbf{W})^\top = \nabla^{\mathbf{g}}\mathbf{W}^\top - W^\perp\boldsymbol{\theta}\,,\qquad (\bar{\nabla}^{\bar{\mathbf{g}}}\mathbf{W})^\perp = \mathbf{d}W^\perp + \boldsymbol{\theta}\cdot\mathbf{W}^\top\,. \tag{D.13}$$

Therefore, $\bar{\nabla}^{\bar{\mathbf{g}}}\mathbf{W} = (\bar{\nabla}^{\bar{\mathbf{g}}}\mathbf{W})^\top + (\bar{\nabla}^{\bar{\mathbf{g}}}\mathbf{W})^\perp$, in components, reads

$$(\bar{\nabla}^{\bar{\mathbf{g}}}\mathbf{W})^\top = \left[(W^\top)^a{}_{|b} - \theta^a{}_b W^\perp\right]\partial_a\otimes dx^b\,,\qquad (\bar{\nabla}^{\bar{\mathbf{g}}}\mathbf{W})^\perp = \left[W^\perp{}_{,b} + \theta_{bc}(W^\top)^c\right]\boldsymbol{\mathcal{N}}\otimes dx^b\,. \tag{D.14}$$

Thus, one can use $(\text{D.14})_1$ to write the variation of the deformation gradient in components as

$$\delta F^a{}_A = (\delta\varphi^\top)^a{}_{|A} - \theta^a{}_b F^b{}_A\delta\varphi^\perp\,. \tag{D.15}$$

Therefore, (8.100) follows.

At any time t, the deformation map $\varphi_t : \mathcal{H}\to\mathcal{S}$ is a smooth embedding of the (undeformed) shell into the ambient space. For each $X\in\mathcal{H}$ let $d\varphi_t(X) : T_X\mathcal{H}\to T_{\varphi_t(X)}\mathcal{S}$ be the tangent of φ_t at X.

The variation of the unit normal vector $\boldsymbol{\mathcal{N}}_\epsilon = \mathbf{n}_\epsilon \circ \varphi_{\epsilon,t}$ is defined as

$$\delta\boldsymbol{\mathcal{N}} = \frac{d}{d\epsilon}\boldsymbol{\mathcal{N}}_\epsilon\Big|_{\epsilon=0} = \bar{\nabla}^{\bar{\mathbf{g}}}_{\frac{\partial}{\partial\epsilon}}\boldsymbol{\mathcal{N}}_\epsilon\Big|_{\epsilon=0} = D_{\varphi_\epsilon(X,t)}\boldsymbol{\mathcal{N}}_\epsilon\Big|_{\epsilon=0} = \bar{\nabla}^{\bar{\mathbf{g}}}_{\delta\varphi}\boldsymbol{\mathcal{N}}. \tag{D.16}$$

In order to compute the variation, let $\mathbf{W}$ be a vector field in $\mathcal{S}$ tangent to $\varphi_t(\mathcal{H})$

$$\bar{\nabla}^{\bar{\mathbf{g}}}_{\delta\varphi}\mathbf{W} = [\delta\varphi, \mathbf{W}] + \bar{\nabla}^{\bar{\mathbf{g}}}_{\mathbf{W}}\delta\varphi\,. \tag{D.17}$$

From (D.10), one obtains

$$\bar{\nabla}^{\bar{\mathbf{g}}}_{\mathbf{W}}\delta\varphi = \left(\nabla^{\mathbf{g}}_{\mathbf{W}}\delta\varphi^\top - \delta\varphi^\perp\boldsymbol{\theta}\cdot\mathbf{W}\right) + \left((\mathbf{d}\,\delta\varphi^\perp)\cdot\mathbf{W} + \boldsymbol{\theta}\cdot\delta\varphi^\top\cdot\mathbf{W}\right)\mathbf{n}\,. \tag{D.18}$$

Therefore

$$\bar{\mathbf{g}}\left(\boldsymbol{\mathcal{N}}, \bar{\nabla}^{\bar{\mathbf{g}}}_{\mathbf{W}}\delta\varphi\right) = (\mathbf{d}\,\delta\varphi^\perp)\cdot\mathbf{W} + \boldsymbol{\theta}\cdot\delta\varphi^\top\cdot\mathbf{W} = -\bar{\mathbf{g}}\left(\bar{\nabla}^{\bar{\mathbf{g}}}_{\delta\varphi}\boldsymbol{\mathcal{N}}, \mathbf{W}\right), \tag{D.19}$$

where the second equality is a consequence of the metric compatibility of $\bar{\nabla}^{\bar{\mathbf{g}}}$ (8.10). By arbitrariness of $\mathbf{W}$, we have

$$\delta\boldsymbol{\mathcal{N}} = \bar{\nabla}^{\bar{\mathbf{g}}}_{\delta\varphi}\boldsymbol{\mathcal{N}} = -\boldsymbol{\theta}\cdot\delta\varphi^\top - \mathbf{d}\,\delta\varphi^\perp\,. \tag{D.20}$$

In components

$$\delta\boldsymbol{\mathcal{N}} = -\left[(\delta\varphi^\top)^b\theta^a{}_b + \delta\varphi^\perp{}_{,b}g^{ab}\right]\partial_a\,. \tag{D.21}$$

Hence, (8.59) and (8.60) are followed.

Note that $\delta\boldsymbol{\Theta}^\flat = \varphi_t^*(\mathbf{L}_{\delta\varphi}\boldsymbol{\theta})$, such that

$$\mathbf{L}_{\delta\varphi}\boldsymbol{\theta} = \mathbf{L}_{\delta\varphi^\top}\boldsymbol{\theta} - \delta\varphi^\perp\mathbf{III} + \mathrm{Hess}_{\delta\varphi^\perp}\,. \tag{D.22}$$

Thus

$$\delta\boldsymbol{\Theta}^\flat = \varphi_t^*\mathbf{L}_{\delta\varphi^\top}\boldsymbol{\theta} - \delta\varphi^\perp\,\varphi_t^*\mathbf{III} + \varphi_t^*\mathrm{Hess}_{\delta\varphi^\perp}, \tag{D.23}$$

where for $\boldsymbol{x},\boldsymbol{y}\in\mathcal{X}(\varphi(\mathcal{H}))$, the third fundamental form of the deformed hypersurface $\mathbf{III}$ and the

Hessian of $\delta\varphi^\perp$, i.e., $\mathrm{Hess}_{\delta\varphi^\perp}$ are given by

$$\begin{aligned} \mathbf{III}(\boldsymbol{x},\boldsymbol{y}) &= \mathbf{g}\left(\bar{\nabla}^{\bar{\mathbf{g}}}_{\boldsymbol{x}}\,\mathbf{n},\bar{\nabla}^{\bar{\mathbf{g}}}_{\boldsymbol{y}}\,\mathbf{n}\right), \\ \mathrm{Hess}_{\delta\varphi^\perp}(\boldsymbol{x},\boldsymbol{y}) &= \mathbf{g}\left(\bar{\nabla}^{\bar{\mathbf{g}}}_{\boldsymbol{x}}\,(\mathbf{d}\,\delta\varphi^\perp)^\sharp,\boldsymbol{y}\right)\,. \end{aligned} \tag{D.24}$$

The Lie derivative with respect to the tangential component of the variation field is given in components by (see, e.g., [43, p.97])

$$\mathbf{L}_{\delta\varphi^\top}\boldsymbol{\theta}^\flat = \left[\theta_{ab|c}\,(\delta\varphi^\top)^c + \theta_{ac}(\delta\varphi^\top)^c{}_{|b} + \theta_{cb}(\delta\varphi^\top)^c{}_{|a}\right]dx^a\otimes dx^b\,. \tag{D.25}$$

Therefore, in components, one can write the variation of $\boldsymbol{\Theta}^\flat$ as

$$\begin{aligned} \delta\Theta_{AB} =& F^a{}_A F^b{}_B\,\theta_{ab|c}\,(\delta\varphi^\top)^c + F^a{}_A\theta_{ac}(\delta\varphi^\top)^c{}_{|B} + F^b{}_B\theta_{bc}(\delta\varphi^\top)^c{}_{|A} \\ &- \delta\varphi^\perp F^a{}_A F^b{}_B\theta_{ac}\theta_{bd}\,g^{cd} + F^b{}_A\left(\frac{\partial\,\delta\varphi^\perp}{\partial\,x^b}\right)_{|B}\,. \end{aligned} \tag{D.26}$$

Using (D.15) and the fact that $\Theta_{AB} = F^a{}_A F^b{}_B\theta_{ab}$, one obtains the variation of $\boldsymbol{\theta}^\flat$ as

$$\delta\theta_{ab} = \theta_{ab|c}(\delta\varphi^\top)^c + \delta\varphi^\perp\theta_{ac}\theta_{bd}g^{cd} + (\delta\varphi^\perp{}_{,a})_{|b}\,. \tag{D.27}$$

Hence, (8.57) and (8.115) are followed.

Proof of the relation (D.23)**.** Here, we give a brief proof of the relation (D.23), see also [267, 269, 270]. For the sake of simplicity, we assume that the surface is embedded in three-dimensional Euclidean space. For this proof, we adopt different notations from the rest of the paper. Let us consider an embedded surface denoted by Σ in $\mathbb{R}^3$. The surface geometry is locally described by three functions $\mathbf{x}(x^1,x^2,x^3) = \mathbf{X}(\nu^\alpha)$ in the Cartesian coordinates $\{x^1,x^2,x^3\}$ such that $\{\nu^\alpha\}$, $\alpha = 1,2$, is a local coordinate chart on the surface. Let us define two tangent vectors $\mathbf{e}_\alpha = \partial\mathbf{X}/\partial\nu^\alpha$ on the surface. We note that the surface geometry is completely described by its induced metric $\eta_{\alpha\beta}$ and its induced second fundamental form $\Lambda_{\alpha\beta}$.[1] Note that $\eta_{\alpha\beta} = \mathbf{e}_\alpha \cdot \mathbf{e}_\beta$, where "$\cdot$" denotes the dot product in $\mathbb{R}^3$. Let us denote the surface covariant derivative with ∇_α. Then, one can write the Gauss-Weingarten equations as $\nabla_\alpha \mathbf{e}_\beta = \Lambda_{\alpha\beta}\mathbf{n}$, and $\nabla_\alpha \mathbf{n} = -\Lambda_{\alpha\beta}\mathbf{e}^\beta$, where $\mathbf{n}$ is the unit normal vector to the surface. Let us consider the deformation of the embedding functions of the surface $\mathbf{X}(\nu) \to \mathbf{X}(\nu) + \delta\mathbf{X}(\nu)$ such that the variation field $\delta\mathbf{X}$ is decomposed into the tangential and normal components as $\delta\mathbf{X} = (\psi^\top)^\alpha \mathbf{e}_\alpha + \psi^\perp \mathbf{n}$. Using the relation $\Lambda_{\alpha\beta} = \mathbf{n} \cdot \nabla_\alpha \mathbf{e}_\beta$, one may write

$$\delta\Lambda_{\alpha\beta} = \delta\mathbf{n} \cdot \nabla_\alpha \nabla_\beta \mathbf{X} + \mathbf{n} \cdot \nabla_\alpha \nabla_\beta \delta\mathbf{X}\,. \tag{D.28}$$

Knowing that the variation of the unit normal vector is purely tangential, the first term vanishes, i.e., $\delta\mathbf{n} \cdot \nabla_\alpha \nabla_\beta \mathbf{X} = \delta\mathbf{n} \cdot \nabla_\alpha \mathbf{e}_\beta = \Lambda_{\alpha\beta}(\delta\mathbf{n}) \cdot \mathbf{n} = 0$. After some simplification and through using the Codazzi-Mainardi equation $\nabla_\alpha \Lambda_{\beta\gamma} = \nabla_\beta \Lambda_{\alpha\gamma}$, one obtains

$$\delta\Lambda_{\alpha\beta} = \Lambda_{\beta\gamma}\nabla_\alpha(\psi^\top)^\gamma + \Lambda_{\alpha\gamma}\nabla_\beta(\psi^\top)^\gamma + (\psi^\top)^\gamma \nabla_\gamma \Lambda_{\alpha\beta} - \Lambda_{\alpha\gamma}\Lambda^\gamma{}_\beta \psi^\perp + \nabla_\alpha \nabla_\beta \psi^\perp\,. \tag{D.29}$$

Notice that the first three terms correspond to the Lie derivative of the induced second fundamental form with respect to the tangential component of the variation field, and thus, (D.29) can be rewritten as

$$\Lambda_{\alpha\beta} = (\mathbf{L}_{\psi^\top}\Lambda)_{\alpha\beta} - \Lambda_{\alpha\gamma}\Lambda^\gamma{}_\beta \psi^\perp + \nabla_\alpha \nabla_\beta \psi^\perp\,. \tag{D.30}$$

Therefore, (D.23) follows.

[1] Note that $\boldsymbol{\eta}$ and $\boldsymbol{\Lambda}$, respectively, correspond to $\mathbf{C}$ and $\boldsymbol{\Theta}$ defined previously.

D.2 The Euler-Lagrange equations of elastic shells

In this appendix, we discuss in detail the derivation of the Euler-Lagrange equations. Substituting (8.49), (8.52), (8.58), and (8.60) into (8.47), we obtain

$$\begin{aligned}\int_{t_0}^{t_1}\int_{\mathcal{H}}\Bigg(&\rho(\mathfrak{B}^\top)_a(\delta\varphi^\top)^a+\rho\mathfrak{B}^\perp\cdot\delta\varphi^\perp+\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\cdot\frac{D\,\delta\varphi}{dt}-\rho\mathfrak{L}_b\left((\delta\varphi^\top)^a\theta^b{}_a+\frac{\partial\,\delta\varphi^\perp}{\partial x^a}g^{ab}\right)\\&+\frac{\partial\mathcal{L}}{\partial C_{AB}}\left(F^a{}_A\delta\varphi^\top_{a|B}+F^b{}_B\delta\varphi^\top_{b|A}-2\delta\varphi^\perp F^a{}_AF^b{}_B\theta_{ab}\right)\\&+\frac{\partial\mathcal{L}}{\partial\Theta_{AB}}\Bigg[F^a{}_AF^b{}_B\,\theta_{ab|c}\,(\delta\varphi^\top)^c+F^a{}_A\theta_{ac}(\delta\varphi^\top)^c{}_{|B}+F^b{}_B\theta_{bc}(\delta\varphi^\top)^c{}_{|A}\\&-\delta\varphi^\perp F^a{}_AF^b{}_B\theta_{ac}\theta_{bd}\,g^{cd}+F^b{}_A\left(\frac{\partial\,\delta\varphi^\perp}{\partial\,x^b}\right)_{|B}\Bigg]\Bigg)dAdt=0\,.\end{aligned}\tag{D.31}$$

After some simplification, we have

$$\begin{aligned}\int_{t_0}^{t_1}\int_{\mathcal{H}}\Bigg(&\rho(\mathfrak{B}^\top)_a(\delta\varphi^\top)^a+\rho\mathfrak{B}^\perp\cdot\delta\varphi^\perp+\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\cdot\,\delta\varphi\right)-\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\right)\cdot\,\delta\varphi-\rho\mathfrak{L}_b(\delta\varphi^\top)^a\theta^b{}_a\\&-\left(\rho\mathfrak{L}_b\delta\varphi^\perp\,g^{ab}(F^{-1})^A{}_a\right)_{|A}+\left(\rho\mathfrak{L}_bg^{ab}(F^{-1})^A{}_a\right)_{|A}\delta\varphi^\perp\\&-2\frac{\partial\mathcal{L}}{\partial C_{AB}}F^a{}_AF^b{}_B\theta_{ab}\delta\varphi^\perp+2\left(\frac{\partial\mathcal{L}}{\partial C_{AB}}F^b{}_Bg_{cb}(\delta\varphi^\top)^c\right)_{|A}-2\left(\frac{\partial\mathcal{L}}{\partial C_{AB}}F^b{}_Bg_{cb}\right)_{|A}(\delta\varphi^\top)^c\\&+\frac{\partial\mathcal{L}}{\partial\Theta_{AB}}F^a{}_AF^b{}_B\,\theta_{ab|c}\,(\delta\varphi^\top)^c+2\left(\frac{\partial\mathcal{L}}{\partial\Theta_{AB}}F^a{}_A\theta_{ac}(\delta\varphi^\top)^c\right)_{|B}\\&-2\left(\frac{\partial\mathcal{L}}{\partial\Theta_{AB}}F^a{}_A\theta_{ac}\right)_{|B}(\delta\varphi^\top)^c-\frac{\partial\mathcal{L}}{\partial\Theta_{AB}}F^a{}_AF^b{}_B\theta_{ac}\theta_{bd}\,g^{cd}\delta\varphi^\perp\\&+\left(\frac{\partial\mathcal{L}}{\partial\Theta_{AB}}F^b{}_A\frac{\partial\,\delta\varphi^\perp}{\partial\,x^b}\right)_{|B}-\left[\left(\frac{\partial\mathcal{L}}{\partial\Theta_{AB}}F^b{}_A\right)_{|B}\delta\varphi^\perp(F^{-1})^D{}_b\right]_{|D}\\&+\left[\left(\frac{\partial\mathcal{L}}{\partial\Theta_{AB}}F^b{}_A\right)_{|B}(F^{-1})^D{}_b\right]_{|D}\delta\varphi^\perp\Bigg)dAdt=0\,.\end{aligned}\tag{D.32}$$

This can further be simplified to read

$$
\begin{aligned}
\int_{t_0}^{t_1}\int_{\mathcal{H}}\Bigg(\Bigg[&\rho(\mathfrak{B}^\top)_a-\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial(\dot\varphi^\top)^a}\right)-\rho\mathfrak{L}_b\theta^b{}_a-2\left(\frac{\partial\mathcal{L}}{\partial C_{AB}}F^b{}_Bg_{ab}\right)_{|A}+\frac{\partial\mathcal{L}}{\partial\Theta_{AB}}F^c{}_AF^b{}_B\,\theta_{cb|a}\\
&-2\left(\frac{\partial\mathcal{L}}{\partial\Theta_{AB}}F^b{}_A\theta_{ba}\right)_{|B}\Bigg](\delta\varphi^\top)^a+\Bigg[\rho\mathfrak{B}^\perp-\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\varphi^\perp}\right)+\big(\rho\mathfrak{L}_bg^{ab}(F^{-1})^A{}_a\big)_{|A}\\
&-2\frac{\partial\mathcal{L}}{\partial C_{AB}}F^a{}_AF^b{}_B\theta_{ab}-\frac{\partial\mathcal{L}}{\partial\Theta_{AB}}F^a{}_AF^b{}_B\theta_{ac}\theta_{bd}\,g^{cd}+\left(\left(\frac{\partial\mathcal{L}}{\partial\Theta_{AB}}F^b{}_A\right)_{|B}(F^{-1})^D{}_b\right)_{|D}\Bigg]\delta\varphi^\perp\\
&+\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\varphi}\cdot\delta\varphi\right)-\big(\rho\mathfrak{L}_b\delta\varphi^\perp g^{ab}(F^{-1})^A{}_a\big)_{|A}+2\left(\frac{\partial\mathcal{L}}{\partial C_{AB}}F^b{}_Bg_{cb}(\delta\varphi^\top)^c\right)_{|A}\\
&+2\left(\frac{\partial\mathcal{L}}{\partial\Theta_{AB}}F^a{}_A\theta_{ac}(\delta\varphi^\top)^c\right)_{|B}+\left(\frac{\partial\mathcal{L}}{\partial\Theta_{AB}}\frac{\partial\,\delta\varphi^\perp}{\partial X^A}\right)_{|B}\\
&-\left[\left(\frac{\partial\mathcal{L}}{\partial\Theta_{AB}}F^b{}_A\right)_{|B}\delta\varphi^\perp(F^{-1})^D{}_b\right]_{|D}\Bigg)dAdt=0\,.
\end{aligned}
\tag{D.33}
$$

We assume that $\delta\varphi(X,t_0)=\delta\varphi(X,t_1)=0$. Using Stokes' theorem, we have

$$
\begin{aligned}
\int_{t_0}^{t_1}\int_{\mathcal{H}}\Bigg(\Bigg[&\rho(\mathfrak{B}^\top)_a-\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial(\dot\varphi^\top)^a}\right)-\rho\mathfrak{L}_b\theta^b{}_a-2\left(\frac{\partial\mathcal{L}}{\partial C_{AB}}F^b{}_Bg_{ab}\right)_{|A}+\frac{\partial\mathcal{L}}{\partial\Theta_{AB}}F^c{}_AF^b{}_B\,\theta_{cb|a}\\
&-2\left(\frac{\partial\mathcal{L}}{\partial\Theta_{AB}}F^b{}_A\theta_{ba}\right)_{|B}\Bigg](\delta\varphi^\top)^a+\Bigg[\rho\mathfrak{B}^\perp-\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot\varphi^\perp}\right)+\big(\rho\mathfrak{L}_bg^{ab}(F^{-1})^A{}_a\big)_{|A}\\
&-2\frac{\partial\mathcal{L}}{\partial C_{AB}}F^a{}_AF^b{}_B\theta_{ab}-\frac{\partial\mathcal{L}}{\partial\Theta_{AB}}F^a{}_AF^b{}_B\theta_{ac}\theta_{bd}\,g^{cd}\\
&+\left(\left(\frac{\partial\mathcal{L}}{\partial\Theta_{AB}}F^b{}_A\right)_{|B}(F^{-1})^D{}_b\right)_{|D}\Bigg]\delta\varphi^\perp\Bigg)dAdt+\int_{t_0}^{t_1}\int_{\partial\mathcal{H}}\Bigg(\frac{\partial\mathcal{L}}{\partial\Theta_{AB}}\frac{\partial\,\delta\varphi^\perp}{\partial X^A}\mathsf{T}_B\\
&-\left[\rho\mathfrak{L}_bg^{ab}(F^{-1})^A{}_a+\left(\frac{\partial\mathcal{L}}{\partial\Theta_{CB}}F^a{}_C\right)_{|B}(F^{-1})^A{}_a\right]\mathsf{T}_A\delta\varphi^\perp\\
&+2\left[\frac{\partial\mathcal{L}}{\partial C_{AB}}F^a{}_Bg_{ac}+\frac{\partial\mathcal{L}}{\partial\Theta_{AB}}F^a{}_B\theta_{ac}\right]\mathsf{T}_A(\delta\varphi^\top)^c\Bigg)dL\,dt=0\,,
\end{aligned}
\tag{D.34}
$$

where $\mathbf{T}$ is the outward vector field normal to the boundary curve $\partial\mathcal{H}$. Knowing that $\delta\varphi^{\top}$, $\delta\varphi^{\perp}$, and $\mathbf{d}(\delta\varphi^{\perp})$ are arbitrary, the Euler-Lagrange equations (8.61), along with the boundary conditions (8.62) are obtained from (D.34).

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